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Research Article

Power Geometry and Elliptic Expansions of Solutions to the Painlevé Equations

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We consider an ordinary differential equation (ODE) which can be written as a polynomial in variables and derivatives. Several types of asymptotic expansions of its solutions can be found by algorithms of 2D Power Geometry. They are power, power-logarithmic, exotic, and complicated expansions. Here we develop 3D Power Geometry and apply it for calculation power-elliptic expansions of solutions to an ODE. Among them we select regular power-elliptic expansions and give a survey of all such expansions in solutions of the Painlevé equations $P_1, \ldots, P_6$.

1. Universal Nonlinear Analysis

We develop a new calculus based on Power Geometry [1–4]. Now it allows to compute local and asymptotic expansions of solutions to nonlinear equations of three classes: (A) algebraic, (B) ordinary differential, and (C) partial differential, as well as to systems of such equations.

Principal ideas and algorithms are common for all classes of equations. Computation of asymptotic expansions of solutions consists of the 3 following steps (we describe them for one equation $f = 0$).

(1) Isolation of truncated equations $\hat{f}_j^{(d)} = 0$ by means of faces of the convex polyhedron $\Gamma(f)$ which is a generalization of the Newton polyhedron: the first term of the expansion of a solution to the initial equation $f = 0$ is a solution to the corresponding truncated equation $\hat{f}_j^{(d)} = 0$.

(2) Finding solutions to a truncated equation $\hat{f}_j^{(d)} = 0$ which is quasihomogenous: using power and logarithmic transformations of coordinates we can reduce the equation $\hat{f}_j^{(d)} = 0$ to such simple form that can be solved. Among the solutions found we must select appropriate ones which give the first terms of asymptotic expansions.

(3) Computation of the tail of the asymptotic expansion. Each term in the expansion is a solution to a linear equation which can be written down and solved.

Applications

Class A. (1) Sets of stability of multiparameter problems [5, 6].

Class B. (2) Asymptotic forms and expansions of solutions to the Painlevé equations [4, 7, 8].

(3) Periodic motions of a satellite around its mass center moving along an elliptic orbit [9].

(4) New properties of motion of a top [10].

(5) Families of periodic solutions of the restricted three-body problem and distribution of asteroids [11, 12].

(6) Integrability of ODE systems [13].

Class C. (7) Boundary layer on a needle [14].

(8) Evolution of the turbulent flow [15].

For a survey of these applications see [16].

2. Introduction

Let $w(u)$ be a formal elliptic asymptotic form of a solution to an ODE; that is, it is a solution of a corresponding truncated equation. The form $w(u)$ is suitable if it can be extended into power asymptotic expansion $v = w(u) + \sum_{j=1}^{\infty} b_j u^{-j}$, where $b_j = b_j(u)$ are some functions. The expansion is regular, if all b_j are not branching functions of $w(u)$ and its derivatives. If all functions $b_j(u) = \tilde{b}_j(w, \dot{w})$ have no branching, then they are elliptic functions with the same periods as $w(u)$. Selection of such cases is our aim. For given $w(u)$ and fixed point w^0 (including infinity), we can compute power-logarithmic expansions of functions $\tilde{b}_j(w, \dot{w})$ near $w = w^0$. In these expansions logarithmic branching can appear, only if w^0 is a singular point, and algebraic branching (of finite order) can be for subsingular points w^0. To each singular point w^0 and suitable asymptotic form $w(u)$, we assign unique regular expansion $v = \tilde{v}(w^0, w(u))$, so called basic, and we are looking for such basic expansions near singular point w^0, which have no branching.

We propose algorithms for (1) finding all formal elliptic asymptotic forms, (2) finding all suitable elliptic asymptotic forms, and (3) calculation of power-logarithmic expansions of functions $\tilde{b}_j(w, \dot{w})$ near a singular point w^0 and selection of basic expansions without branching. All algorithms are based on 3D Power Geometry.

Expansions are formal; their convergence is not considered. Application of these algorithms to the Painlevé equations $P_1, \ldots, P_6$ gives following.

(1) P_1, P_2, and P_4 have continuum of 2-parameter families of elliptic asymptotic forms each, P_3 has three, P_5 has two of them, and P_6 does not have.

(2) P_1, P_2, and P_4 have countable sets of families of suitable asymptotic forms each, and all 5 forms of P_3 and P_5 are suitable.

(3) Basic expansions for all suitable forms have no branching for P_1, for P_2 if the independent variable tends to infinity, for P_3 if condition C is fulfilled, and for P_5 if condition D is fulfilled and $a = b = 0$, and $d \neq 0$.

History of calculation of elliptic expansions of solutions to the Painlevé equations is as follows.

A hundred years ago, Boutroux [17] found 2 families of elliptic asymptotic forms of solutions to the Painlevé equations P_1 and P_2. During the last 5 years we found 6 additional families of elliptic asymptotic forms of solutions to P_3 (three) [18, 19], P_4 (one) [20], and P_5 (two) [21]. Moreover the Painlevé equations P_1, P_2, and P_4 have continuum of families of elliptic asymptotic forms each, and I proposed a criterion for selection suitable asymptotic forms, which can be extended as asymptotic expansions. All 8 known elliptic asymptotic forms are suitable. Solutions to the equation P_6 have no elliptic asymptotic forms at all.

Near infinity of the independent variable, the Painlevé equations P_1–P_5 have 12 families of suitable asymptotic forms and near zero of the independent variable equations P_1, P_2, and P_4 have countable sets of such families each. Next I extend these suitable elliptic asymptotic forms $w(u)$ into power-elliptic expansions $v = w(u) + \sum_{j=1}^{\infty} b_j u^{-j}$, where coefficients b_j are functions of the corresponding elliptic asymptotic forms and their derivatives. To each family of suitable elliptic asymptotic forms, I put in correspondence unique basic formal power-elliptic expansion near $w^0 = \infty$ for P_1–P_5, near $w^0 = 0$ for P_3–P_5, and near $w^0 = 1$ for P_5. Obstacles (logarithmic branching) in calculations of these basic expansions appeared only for P_2 if the independent variable tends to zero, for P_4 and for P_5 if $|a| + |b| \neq 0$ or $d = 0$.

Thus, near infinity of the independent variable, there are 10 families of regular (i.e., without branching) elliptic expansions of solutions to equations P_1–P_6: 4 for P_1, 2 for P_2, 3 for P_3, and 1 for P_5. Existence of these expansions for two Boutroux families of asymptotic forms was proven in [22], and this is all known up-to-date. Near zero of the independent variable there is a countable set of families of such expansions for P_1. The results were obtained by means of algorithms of 3D Power Geometry [18–24], realized in very cumbersome calculations.

Here I introduce the third variant of 3D Power Geometry. The first was in [18, 20, 24], and the second was in [19, 21–23].

In more precise form main results are as in Theorems 12, 14, 15, and 16 and Conditions C and D.

Equation P_6 cannot be studied by proposed approach.

3. 3D Power Geometry

Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A *differential monomial* $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$. The sum of differential monomials

$$f(x, y) = \sum a_i(x, y) \tag{1}$$

is called the *differential sum*. Let n be the maximal value of l in $f(x, y)$.

In [2–4] it was shown that as $x \to 0$ $(\omega = -1)$ or as $x \to \infty$ $(\omega = 1)$ solutions $y = \varphi(x)$ to the ODE $f(x, y) = 0$, where $f(x, y)$ is a differential sum, can be found by means of algorithms of Plane (2D) Power Geometry, if

$$p_\omega\left(\frac{d^l\varphi}{dx^l}\right) = p_\omega(\varphi(x)) - l, \quad l = 1, \ldots, n, \tag{2}$$

where the order

$$p_\omega(\varphi(x)) = \omega \limsup_{x^\omega \to \infty} \frac{\log|\varphi(x)|}{\omega \log|x|} \tag{3}$$

on a ray $\arg x = $ const and n is the maximal order of derivatives in $f(x, y)$. Order of the power function $\varphi(x) = x^\alpha$ with $\alpha \in \mathbb{C}$ is $p_\omega(x^\alpha) = \operatorname{Re}\alpha$.

Here we introduce algorithms, which allow to calculate solutions $y = \varphi(x)$ with the property

$$p_\omega\left(\frac{d^l\varphi}{dx^l}\right) = p_\omega(\varphi(x)) - l\gamma_\omega, \quad l = 1, \ldots, n, \tag{4}$$

where $\gamma_\omega \in \mathbb{R}$.

Theorem 1. $\omega - \omega\gamma_\omega \geqslant 0$.

For example, $\gamma_1 = 0$ for $\varphi = \sin x$ and $\gamma_{-1} = 2$ for $\varphi = \sin(1/x)$. Note that in Plane Power Geometry we had $\gamma_\omega = 1$; that is, $\omega - \omega\gamma_\omega = 0$. So, new interesting possibilities correspond to $\omega - \omega\gamma_\omega > 0$.

Problem 2. Select leading terms in the sum (1) after substitution $y = \varphi(x)$ with property (4).

Below we describe algorithms for solution of the problem. To each differential monomial $a_i(x, y)$, we assign its *(3D) power exponent* $\mathbf{Q}(a_i) = (q_1, q_2, q_3) \in \mathbb{R}^3$ by the following rules:

q_3 = sum of orders of all derivatives;

q_2 = order of y;

q_1 = difference of order of x and q_3.

Then the 2D vector $Q = (q_1, q_2)$ is the same as in 2D Power Geometry [2–4] and q_3 corresponds to the total order of derivatives. The power exponent of the product of differential monomials is the sum of power exponents of factors: $\mathbf{Q}(a_1 a_2) = \mathbf{Q}(a_1) + \mathbf{Q}(a_2)$.

The set $\widetilde{\mathbf{S}}(f)$ of power exponents $\mathbf{Q}(a_i)$ of all differential monomials $a_i(x, y)$ presented in the differential sum $f(x, y)$ is called the *3D support of the sum* $f(x, y)$. Obviously, $\widetilde{\mathbf{S}}(f) \subset \mathbb{R}^3$. The convex hull $\mathbf{\Gamma}(f)$ of the support $\widetilde{\mathbf{S}}(f)$ is called the *polyhedron* of the sum $f(x, y)$. The boundary $\partial\mathbf{\Gamma}(f)$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of the vertices $\mathbf{\Gamma}_j^{(0)}$, the edges $\mathbf{\Gamma}_j^{(1)}$, and the faces $\mathbf{\Gamma}_j^{(2)}$. They are called *(generalized) faces* $\mathbf{\Gamma}_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to the 3D *truncated sum*:

$$\check{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ over } \mathbf{Q}(a_i) \in \mathbf{\Gamma}_j^{(d)} \cap \widetilde{\mathbf{S}}(f). \quad (5)$$

All these definitions are applied to differential equation

$$f(x, y) = 0. \quad (6)$$

Thus, each generalized face $\mathbf{\Gamma}_j^{(d)}$ corresponds to the truncated equation

$$\check{f}_j^{(d)}(x, y) = 0. \quad (7)$$

Let $\mathbf{N}_j = (n_1, n_2, n_3)$ be the external normal to two-dimensional face $\mathbf{\Gamma}_j^{(2)}$. We will consider only normals with $n_1 \neq 0$.

Example 3. Consider the second Painlevé equation P_2:

$$f(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 + xy + a = 0, \quad (8)$$

where a is the complex parameter.

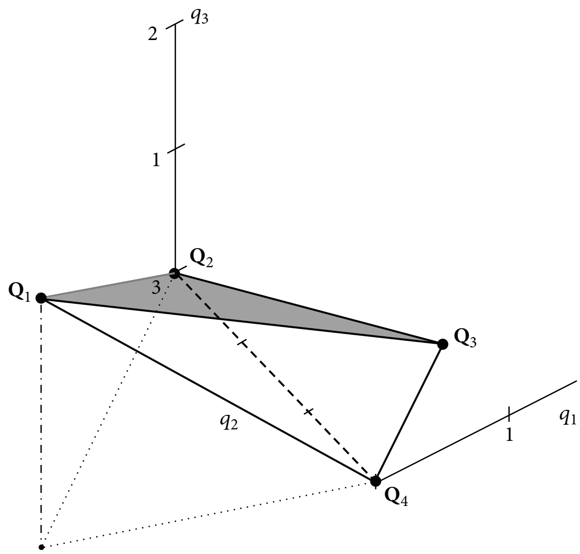

Figure 1: 3D support $\widetilde{\mathbf{S}}(f)$ and polyhedron $\mathbf{\Gamma}(f)$ of equation P_2 (8). The grey face is $\mathbf{\Gamma}_1^{(2)}$; the grey edge is $\mathbf{\Gamma}_1^{(1)}$. Projection on the plane (q_1, q_2) is shown by dotted lines. Dashed line is the invisible edge.

If $a \neq 0$, the 3D support $\widetilde{\mathbf{S}}(f)$ consists of 4 points

$$\begin{aligned} \mathbf{Q}_1 &= (-2, 1, 2), \\ \mathbf{Q}_2 &= (0, 3, 0), \\ \mathbf{Q}_3 &= (1, 1, 0), \\ \mathbf{Q}_4 &= 0. \end{aligned} \quad (9)$$

They are shown in Figure 1.

Their convex hull $\mathbf{\Gamma}(f)$ is a tetrahedron. It has 4 vertices $\mathbf{Q}_1$–$\mathbf{Q}_4$, 6 edges $\mathbf{\Gamma}_j^{(1)}$, and 4 faces $\mathbf{\Gamma}_j^{(2)}$. Face $\mathbf{\Gamma}_1^{(2)} = [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3]$ is distinguished in Figure 1; its external normal $\mathbf{N}_1 = (2, 1, 3)$ and its truncated equation

$$\check{f}_1^{(2)}(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 + xy = 0. \quad (10)$$

Edge $\mathbf{\Gamma}_1^{(1)} = [\mathbf{Q}_1, \mathbf{Q}_2]$ is also distinguished in Figure 1; its truncated equation

$$\check{f}_1^{(1)}(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 = 0. \quad (11)$$

Example 3 is finished.

Let $y = \varphi(x)$ be a solution to (6) with property (4) and $p = p_\omega(\varphi)$, $\gamma = \gamma_\omega(\varphi)$; then the order of a monomial $a(x, y)$ with $\mathbf{Q}(a) = (q_1, q_2, q_3)$ is

$$q_1 + q_2 p + q_3(1 - \gamma) = \langle P, \mathbf{Q} \rangle, \quad (12)$$

where $P = (1, p, 1 - \gamma)$ and $\langle \cdot, \cdot \rangle$ is the scalar product. Leading terms of the sum (1) after substitution $y = \varphi(x)$ are monomials $a(x, y)$, for which $\omega\langle P, \mathbf{Q} \rangle = \langle \omega P, \mathbf{Q} \rangle$ reaches the maximal value on the support $\widetilde{\mathbf{S}}(f)$. Here $\omega P = (\omega, \omega p_\omega, \omega(1 - \gamma_\omega))$ and $\omega(1 - \gamma_\omega) \geqslant 0$ according to Theorem 1.

On the support $\tilde{\mathbf{S}}(f) = \{\mathbf{Q}_i\}$ maximum of the scalar product $\langle \omega P, \mathbf{Q}_i\rangle$ is achieved on a generalized face $\mathbf{\Gamma}_j^{(d)}$ of the polyhedron $\mathbf{\Gamma}(f)$.

By $\mathbb{R}^3$ we denote the 3D real space, where we put power exponents $\mathbf{Q}$, and by $\mathbb{R}_*^3$ we denote the space dual (conjugate) to $\mathbb{R}^3$. We will denote points in $\mathbb{R}_*^3$ as $\mathbf{R} = (r_1, r_2, r_3)$. Then we have the scalar product

$$\langle \mathbf{Q}, \mathbf{R}\rangle = q_1 r_1 + q_1 r_2 + q_3 r_3. \tag{13}$$

Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to its *normal cone* [2]:

$$\mathbf{U}_j^{(d)} = \left\{ \mathbf{R} : \begin{array}{l} \langle \mathbf{Q}', \mathbf{R}\rangle = \langle \mathbf{Q}'', \mathbf{R}\rangle, \mathbf{Q}', \mathbf{Q}'' \in \mathbf{\Gamma}_j^{(d)}, \\ \langle \mathbf{Q}', \mathbf{R}\rangle > \langle \mathbf{Q}''', \mathbf{R}\rangle, \mathbf{Q}''' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_j^{(d)} \end{array} \right\}. \tag{14}$$

Thus, normal cone $\mathbf{U}_j^{(2)}$ of the face $\mathbf{\Gamma}_j^{(2)}$ is a ray spanned on the exterior normal $\mathbf{N}_j$ of the face $\mathbf{\Gamma}_j^{(2)}$, normal cone $\mathbf{U}_j^{(1)}$ of the edge $\mathbf{\Gamma}_j^{(1)}$ is 2D angle spanned on rays $\mathbf{U}_k^{(2)}$ and $\mathbf{U}_l^{(2)}$, where $\mathbf{\Gamma}_j^{(1)} = \mathbf{\Gamma}_k^{(2)} \cap \mathbf{\Gamma}_l^{(2)}$; normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\mathbf{\Gamma}_j^{(0)}$ is a 3D angle spanned on exterior normals $\mathbf{N}_k$ of all 2D faces $\mathbf{\Gamma}_k^{(2)}$ containing the vertex $\mathbf{\Gamma}_j^{(0)}$ (see [2]).

Thus, selection of the truncated sums $\check{f}_j^{(d)}(x, y)$ can be made by the following method. First we compute the support $\tilde{\mathbf{S}}(f)$ of the initial sum $f(x, y)$. Using support $\tilde{\mathbf{S}}(f)$, we compute the polyhedron $\mathbf{\Gamma}(f)$ of sum $f(x, y)$, that is, all its vertices $\mathbf{\Gamma}_j^{(0)}$, edges $\mathbf{\Gamma}_j^{(1)}$, and faces $\mathbf{\Gamma}_j^{(2)}$. Next we compute their normal cones $\mathbf{U}_j^{(d)}$ and select only such truncated equations $\check{f}_j^{(d)}(x, y) = 0$ for which the intersection $\mathbf{U}_j^{(d)} \cap \{p_3 \geqslant 0\} \neq \emptyset$. But truncated equations $\check{f}_j^{(d)}(x, y) = 0$ with $p_3 = 0$ can be studied by algorithms of 2D Power Geometry. So 3D Power Geometry studies truncated equations $\check{f}_j^{(d)}(x, y) = 0$ with nonempty intersection $\mathbf{U}_j^{(d)} \cap \{p_3 > 0\}$.

Example 4 (continuation of Example 3). Polyhedron $\mathbf{\Gamma}(f)$ for equation P_2 (8) has 4 following faces with exterior normal

$$\begin{aligned} \mathbf{\Gamma}_1^{(2)} &= [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3], \\ \mathbf{N}_1 &= (2, 1, 3), \\ \mathbf{\Gamma}_2^{(2)} &= [\mathbf{Q}_1, \mathbf{Q}_3, \mathbf{Q}_4], \\ \mathbf{N}_2 &= (2, -2, 3), \\ \mathbf{\Gamma}_3^{(2)} &= [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_4], \\ \mathbf{N}_3 &= (-1, 0, -1), \\ \mathbf{\Gamma}_4^{(2)} &= [\mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4], \\ \mathbf{N}_4 &= (0, 0, -1). \end{aligned} \tag{15}$$

Only two of them, $\mathbf{N}_1$ and $\mathbf{N}_2$, have $r_3 > 0$. Hence, all edges exept $\mathbf{\Gamma}_6^{(1)} = [\mathbf{Q}_2, \mathbf{Q}_4]$ and all vertices $\mathbf{\Gamma}_j^{(0)}$ have vectors $R = (r_1, r_2, r_3)$ with $r_3 > 0$ in their normal cones $\mathbf{U}_j^{(1)}$ and $\mathbf{U}_j^{(0)}$.

4. Power Transformations

If the face $\mathbf{\Gamma}_j^{(d)}$ has the normal $\mathbf{N}_j = (1, 0, 1)$ then the corresponding truncation $\check{f}_j^{(d)}(x, y) = x^q g(y)$, where the differential sum $g(y)$ contains y and its derivatives but does not contain x. In that case the full sum $f(x, y)$ can be written as $f(x, y) = x^q g(y) + x^{q-r} h(x, y)$, where $r > 0$ and $h(x, y)$ is a differential sum.

Remark 5. If $y(x)$ is a solution to the equation $g(y) = 0$ with the property

$$0 < \varepsilon < \left|y(x)\right|, \left|y'(x)\right|, \ldots, \left|y^{(n)}(x)\right| < \varepsilon^{-1}, \tag{16}$$

when $x \to 0$ or $x \to \infty$, then $y(x)$ can be the asymptotic form of the solutions to the full equation (6). Here ε is a small real number. We call $y(x)$ as *formal asymptotic form*.

Let the power transformation of variables $x, y \to u, v$

$$\begin{aligned} y &= x^{\alpha} v, \\ u &= \frac{1}{\beta} x^{\beta}, \end{aligned} \tag{17}$$

transform $f(x, y)$ into $f^*(u, v)$: $f^*(u, v) = f(x, y)$.

Theorem 6. *Let the face $\mathbf{\Gamma}_i^{(d)}$ of $\mathbf{\Gamma}(f)$ have the exterior normal $\mathbf{N}_i = (n_1, n_2, n_3)$ with*

$$\begin{aligned} n_1 &\neq 0, \\ n_3 &> 0; \end{aligned} \tag{18}$$

then the power transformation (17) with $\alpha = n_2/n_1$, $\beta = n_3/n_1$ transforms the truncation $\check{f}_i^{(d)}(x, y)$ of $f(x, y)$ into the truncation

$$\check{f}_i^{*(d)}(u, v) = u^q g(v) \tag{19}$$

of $f^*(u, v)$, corresponding to the face $\mathbf{\Gamma}_i^{*(d)}$ of $\mathbf{\Gamma}(f^*)$ with the exterior normal $\mathbf{N}_i^* = (1, 0, 1)$. Here $\check{f}_i^{*(d)}(u, v)$ equals $\check{f}_i^{(d)}(x, y)$ after substitution

$$\frac{u^{[\alpha + l(\beta - 1)]/\beta} d^l v}{du^l} \tag{20}$$

instead of $y^{(l)} = d^l y / dx^l$.

So, if $v = \varphi(u)$ is a solution to the equation $g(v) = 0$ and $|\varphi(u)|$ is bounded from zero and infinity as $|y|$ in (16), then the initial equation $f(x, y) = 0$ can have a solution with the asymptotic form:

$$y \sim x^{\alpha} \varphi\left(\frac{x^{\beta}}{\beta}\right), \quad x^{\omega} \longrightarrow \infty. \tag{21}$$

Herewith the power transformation (17) induces the following formulas for derivatives:

$$\begin{aligned} y' &= x^{\alpha+\beta-1}\dot{v} + \alpha x^{\alpha-1} v, \\ y'' &= x^{2\beta+\alpha-2}\ddot{v} + (2\alpha+\beta-1)\, x^{\beta+\alpha-2}\dot{v} \\ &\quad + \alpha(\alpha-1)\, x^{\alpha-2} v, \end{aligned} \tag{22}$$

where $\dot{v} = dv/du$.

Theorem 7. *Let an equation of order n*

$$g(v) + \sum_{j=1}^{m} h_j(v)\, u^{-j} = 0 \tag{23}$$

have a solution of the form

$$v = w + \sum_{j=1}^{\infty} b_j(w)\, u^{-j}, \tag{24}$$

where $w = w(u)$ is the solution to the truncated equation

$$g(w) = 0 \tag{25}$$

with the property

$$0 < \varepsilon < |w|, \left|\frac{dw}{du}\right|, \dots, \left|\frac{d^n w}{du^n}\right| < \frac{1}{\varepsilon} < \infty. \tag{26}$$

Then $b_j(w)$ satisfies the linear equation

$$\mathscr{L}(u)\, b_j(w) + \theta_j(w) = 0, \tag{27}$$

where $\mathscr{L}(u) = (\delta g/\delta v)|_{v=w}$, $\theta_j(w)$ is a polynomial on $w^{(l)}$ depending on $g(w)$, and $h_i(w)$ and $b_i^{(l)}(w)$ for $i < j$ and $l = 0, 1, 2, \dots, n$. $\delta g/\delta v$ is the first variation.

The first variation is the formal Frechet derivative (see [3]).

Solution $v = \psi(u)$ to the transformed equation $f^*(u, v) = 0$ is expanded into series (24) with integer j only if the transformed equation $f^*(u, v) = 0$ divided by u^q has form (23) with integer j. In that case, *solutions* $v = w(u)$ to the truncated equation $g(v) = 0$ are *suitable asymptotic forms* for continuation by power expansion (24) and corresponding *normal* $\mathbf{N}_i$ is also *suitable*.

External normal $\mathbf{N}_i = (n_1, n_2, n_3)$ to 2D face $\boldsymbol{\Gamma}_i^{(2)}$ is unique up to positive scalar factor. Hence, power transformation (17) of Theorem 6 is unique and we must only check that the transformed equation has form (23) with integer j. The external normal $\mathbf{N} = (n_1, n_2, n_3)$ to 1D edge $\boldsymbol{\Gamma}_i^{(1)}$ belongs to the normal cone $\mathbf{U}_i^{(1)}$. Hence, in the cone $\mathbf{U}_i^{(1)}$ we must select suitable vectors $\mathbf{N}$ with mentioned property of integer j. Things for a vertex $\boldsymbol{\Gamma}_j^{(0)}$ are the same, but usually solutions $v = w(u)$ to corresponding equation $g(v) = 0$ are so simple, which do not give interesting expansion.

Let $\tilde{\mathbf{S}}(f) = \{\mathbf{Q}_1, \dots, \mathbf{Q}_M\}$, $\tilde{\mathbf{S}}(\check{f}_j^{(d)}) = \{\mathbf{Q}_1, \dots, \mathbf{Q}_L\}$, $0 < L < M$, $\mathbf{N} = (n_1, n_2, n_3) \subset \mathbf{U}_i^{(d)}$, and $n_1 \neq 0$, $n_3 > 0$. Denote

$$\mathring{\mathbf{Q}}_l = \mathbf{Q}_{L+l} - \mathbf{Q}_1, \quad l = 1, \dots, M-L \tag{28}$$

and $\mathring{\mathbf{N}} = (n_1/n_3, n_2/n_3, 1)$.

Theorem 8. *The transformed equation (23) has the property of integer j if and only if all numbers*

$$-\left\langle \mathring{\mathbf{N}}, \mathring{\mathbf{Q}}_l \right\rangle, \quad l = 1, \dots, M-L \tag{29}$$

are natural.

There are 8 essentially different polyhedrons for Painlevé equations P_1–P_5 [19]. Each of them has exactly one 2D face in which truncated equation has elliptic solutions. It was shown [23] that all those elliptic asymptotic forms are suitable. Among 8 polyhedrons only 3 have an edge which truncated equation has elliptic solutions. These are P_1, P_2, and P_4. No truncated equations corresponding to vertices of these 8 polyhedrons have elliptic solutions.

Example 9 (continuation of Examples 3 and 4). Polyhedron $\boldsymbol{\Gamma}(f)$ of equation P_2 (8) has edge $\boldsymbol{\Gamma}_1^{(1)} = [\mathbf{Q}_1, \mathbf{Q}_2]$ with truncated equation $f_1^{(1)}(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 = 0$. Its first integral is

$$y'^2 = y^4 + C_0 \stackrel{\text{def}}{=} P(y), \tag{30}$$

where C_0 is arbitrary constant. If $C_0 \neq 0$, solutions to (30) are elliptic functions. The same will be true after any power transformation (17). Let us apply Theorem 8 to the edge $\boldsymbol{\Gamma}_1^{(1)}$. The edge $\boldsymbol{\Gamma}_1^{(1)} = \boldsymbol{\Gamma}_1^{(2)} \cap \boldsymbol{\Gamma}_3^{(2)}$. So normal cone $\mathbf{U}_1^{(1)}$ is the conic hull of two normals $\mathbf{N}_1 = (2, 1, 3)$ and $\mathbf{N}_3 = (-1, 0, -1)$; that is, up to positive scalar factor, vectors $\mathbf{N} \in \mathbf{U}_1^{(1)}$ have the form

$$\begin{aligned} \mathbf{N} &= \varkappa \mathbf{N}_1 + (1-\varkappa)\, \mathbf{N}_3 \\ &= (3\varkappa - 1, \varkappa, 4\varkappa - 1), \quad 0 < \varkappa < 1. \end{aligned} \tag{31}$$

Here $M = 4$, $L = 2$, $\mathring{\mathbf{Q}}_1 = (3, 0, -2)$, $\mathring{\mathbf{Q}}_2 = -\mathbf{Q}_1 = (2, -1, -2)$, and $\mathring{\mathbf{N}} = ((3\varkappa-1)/(4\varkappa-1), \varkappa/(4\varkappa-1), 1)$. Conditions of Theorem 8 are

$$\begin{aligned} \left\langle \mathring{\mathbf{N}}, \mathring{\mathbf{Q}}_1 \right\rangle &= \frac{3(3\varkappa-1)}{4\varkappa-1} - 2 = \frac{\varkappa-1}{4\varkappa-1} = -k, \\ \left\langle \mathring{\mathbf{N}}, \mathring{\mathbf{Q}}_2 \right\rangle &= \frac{2(3\varkappa-1)}{4\varkappa-1} - \frac{\varkappa}{4\varkappa-1} - 2 = -\frac{3\varkappa}{4\varkappa-1} = -l, \end{aligned} \tag{32}$$

where k and l are natural numbers. Hence $\varkappa = (k+1)/(4k+1) = l/(4l-3)$; that is, $l = k+1$, $k = 1, 2, \dots$.

We can write $\mathbf{N}' = (2-k, k+1, 3)$. Condition (18) of Theorem 6 means that $k \neq 2$. If $k = 1$, then $n_1 > 0$; that is, $x \to \infty$; if $k > 2$, then $n_1 < 0$; that is, $x \to 0$.

So there is a countable set of suitable normals $\mathbf{N}'$ to edge $\mathbf{\Gamma}_1^{(1)}$. According to Theorem 6, here

$$\alpha = \frac{k+1}{2-k}, \qquad \beta = \frac{3}{2-k} = \alpha + 1. \tag{33}$$

5. Computation of Expansions

Below we consider the case when the truncated equation $g(w) = 0$ has the first integral of the form

$$\dot{w}^2 = P(w) \overset{\text{def}}{=} \sum_{k=0}^{\lambda} p_k w^k, \quad p_k = \text{const} \in \mathbb{C}. \tag{34}$$

Differentiating with respect to u and dividing by $2\dot{w}$, we obtain

$$\ddot{w} = \frac{1}{2} P'(w). \tag{35}$$

Here and below the prime denotes the derivative with respect to w.

Using (34) and (35), any power series R of w and its derivatives $d^l w/du^l$ can be written as the sum $R = R^*(w) + \dot{w}R^{**}(w)$, where $R^*(w)$ and $R^{**}(w)$ are power series only of w. Let $b_j(w) = F_j(w) + \dot{w}G_j(w)$, where F_j and G_j are functions only of w. Then, omitting the index j, by (34) and (35), we obtain

$$\begin{aligned} \dot{b} &= F'\dot{w} + PG' + \frac{1}{2}P'G, \\ \ddot{b} &= PF'' + \frac{1}{2}P'F' + \dot{w}\left(PG'' + \frac{3}{2}P'G' + \frac{1}{2}P''G\right). \end{aligned} \tag{36}$$

Further derivatives of b do not need us here, because we consider only (23) of the second order. In our case

$$\mathcal{L}b = \mathcal{F}(w) F(w) + \dot{w}\mathcal{G}(w) G(w). \tag{37}$$

Thus, (27) splits in two:

$$\begin{aligned} \mathcal{F}(w) F_j(w) + \theta_j^*(w) &= 0, \\ \mathcal{G}(w) G_j(w) + \theta_j^{**}(w) &= 0, \end{aligned} \tag{38}$$

where $\theta_j(w) = \theta_j^*(w) + \dot{w}\theta_j^{**}(w)$. Note that in (38) differential operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ are operators on w and do not depend on u. If polynomial $P(w)$ in (34) does not have multiple roots and its degree λ is greater than one, that is,

$$\lambda > 1, \quad \Delta(P) \neq 0, \tag{39}$$

where $\Delta(P)$ is discriminant of the polynomial $P(w)$, then solution $w(u)$ to the truncated equation (25) is periodic (if $\lambda = 2$), or elliptic (if $\lambda = 3$ or 4), or hyperelliptic (if $\lambda \geqslant 5$) function.

Near some point $w = w^0$ we will compute asymptotic expansions of fundations $F_j(w)$ and $G_j(w)$:

$$F_j = \sum_{i=-a_j}^{\infty} \varphi_{ji}\xi^i, \qquad G_j = \sum_{i=-b_j}^{\infty} \gamma_{ji}\xi^i, \tag{40}$$

where $\xi = w - w^0$ if $w^0 \neq \infty$ and $\xi = w^{-1}$ if $w^0 = \infty$. If initial equation (23) is a differential sum then according to [3, Theorem 3.1] coefficients φ_{ji} and γ_{ji} are either constants or polynomial of $\log\xi$; that is, expansions (40) are either *power* or *power-logarithmic* [3]. Moreover, according to [3, Theorem 3.4] (see proof of Theorem 1.7.2 in [4]) power expansions (40) converge for small $|\xi|$.

If the solutions $F_j(w)$ and $G_j(w)$ to the system (38) have no branching, then they are also periodic or (hyper)elliptic functions. Finally, if for the sequence of (38) with $j = 1, 2, \ldots$, there exist solutions $F_j(w)$ and $G_j(w)$ without branching, the solutions to (23) have a *regular asymptotic expansion* (24).

Let operators $\mathcal{F}^{-1}(w)$ and $\mathcal{G}^{-1}(w)$ be inverse to operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$, respectively. Then the solutions of (27) are of forms

$$\begin{aligned} F_j(w) &= -\mathcal{F}^{-1}(w)\,\theta_j^*(w), \\ G_j(w) &= -\mathcal{G}^{-1}(w)\,\theta_j^{**}(w). \end{aligned} \tag{41}$$

In our case the initial ODE (23) has order two. Hence operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ are of the second order. Moreover, in our case factors of F'' in $\mathcal{F}$ and of G'' in $\mathcal{G}$ are the same. Denote it as $R(w)$. Singular points w^0 of operators $\mathcal{F}$ and $\mathcal{G}$ are roots of $R(w)$. Indeed $R(w) = r(w)P(w)$, where $r(w)$ is a simple polynomial. So roots w^0 of $r(w)$ and $w^0 = \infty$ will be *singular points* of operators $\mathcal{F}$ and $\mathcal{G}$, but roots w^0 of polynomial $P(w)$ different from singular points will be their *subsingular points*.

Theorem 10. *If functions $\theta_j^*(w)$ and $\theta_j^{**}(w)$ are regular then the solutions to (41) can have logarithmic branching only at infinity $w = \infty$ and at singular points of the operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ but they can have algebraic branching and can be in singular and subsingular points only.*

For the existence of a regular expansion (24) we need to prove the existence of a sequence of functions $F_j(w)$ and $G_j(w)$ that do not have branching. From the other side, if it is shown that $F_j(w)$ or $G_j(w)$ have branching, then it proves the absence of regular expansion.

In [19, 23], for each polyhedron of the Painlevé equations, we selected suitable 2D faces, for each of them we wrote (23), operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$, and inverse ones $\mathcal{F}^{-1}(w)$ and $\mathcal{G}^{-1}(w)$. We found their singular points and the conditions on the parameters of the equation and on the solution $w(u)$ under which the functions $F_1(w)$ and $G_1(w)$ do not have logarithmic branching, as well as the conditions under which at least one of these functions has such branching.

It is a wonder that for each Painlevé equation P_l the operators $\mathcal{F}$ and $\mathcal{G}$ are expressed in the same way in terms of polynomial $P(w)$ and different cases distinguish only by this polynomial. At the same time, for all cases of faces $\Gamma_i^{(d)}$ of five Painlevé equations P_1–P_5, there are only four different pairs of operators $\mathcal{F}$ and $\mathcal{G}$.

Singular point of operators $\mathcal{F}$ and $\mathcal{G}$ are $w^0 = \infty$ for P_1–P_5 and $w^0 = 1$ for P_3–P_5 and $w = 1$ for P_5. To each suitable elliptic asymptotic form and to each singular point w^0 we assign one basic formal asymptotic expansion (24). Our aim is to show existence or nonexistence of regular basic expansions by means of calculation of expansions (40) near the singular points.

6. Expansions for P_2

Details of calculation of expansions (24) will be explained for equation P_2

$$f(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 + xy + a = 0 \quad (42)$$

and its truncated equation

$$\breve{f}_1^{(1)}(x, y) \stackrel{\text{def}}{=} -y'' + 2y^3 = 0. \quad (43)$$

First, according to (33) and Theorem 6, we make power transformation $y = x^{\alpha}v$, $u = x^{\beta}/\beta$ (17) using formulas (22) and obtain equation P_2 (42) in the form (23):

$$\begin{aligned} &g(v) + h_1(v)u^{-1} + h_2(v)u^{-2} + h_k(v)u^{-k} \\ &\quad + h_{k+1}(v)u^{-k-1} = 0, \end{aligned} \quad (44)$$

where

$$\begin{aligned} g(v) &= -\ddot{v} + 2v^3, \\ h_1(v) &= -\frac{3\alpha}{\beta}\dot{v}, \\ h_2 &= -\frac{\alpha(\alpha-1)}{\beta^2}v, \\ h_k(v) &= \beta^{-k}v, \\ h_{k+1}(v) &= a\beta^{-k-1}, \end{aligned} \quad (45)$$

$$P(w) = w^4 + C_0, \quad C_0 \neq 0. \quad (46)$$

Here $\dot{v} = dv/du$, and C_0 is arbitrary complex constant.

Operators $-\mathcal{F}^{-1}$ and $-\mathcal{G}^{-1}$ (41) are

$$\begin{aligned} F_j &= P^{1/2}\int \frac{1}{P^{3/2}}\int \theta_j^* \, dw\, dw, \\ G_j &= \int \frac{1}{P^{3/2}}\int P^{1/2}\theta_j^{**} \, dw\, dw. \end{aligned} \quad (47)$$

Here $r(w) \equiv 1$ [23] and singular points of operators (47) are only infinity. Let us introduce a function

$$H(w) = \int P^{-3/2} dw = \text{const}\cdot w^{-5} + \text{const}\cdot w^{-6} + \cdots. \quad (48)$$

Here the integral is determined by mentioned asymptotic expansion near $w = \infty$. Solutions of system (38) or (47) have 4 arbitrary constants C_1–C_4:

$$\begin{aligned} F &= C_1P^{1/2} + C_2P^{1/2}H + F^0, \\ G &= C_3 + C_4H + G^0, \end{aligned} \quad (49)$$

where F^0 and G^0 are fixed solutions. Here expansions near $w = \infty$ are

$$\begin{aligned} P^{1/2} &= \text{const}\cdot w^2 + \cdots, \\ P^{1/2}H &= \text{const}\cdot w^{-3} + \cdots. \end{aligned} \quad (50)$$

So we will assume that power expansion for F^0 does not contain terms $\text{const}\cdot w^2$ and $\text{const}\cdot w^{-3}$ but expansion for G^0 does not contain terms const and $\text{const}\cdot w^{-5}$. If it is necessary we can change constants C_1–C_4. Now the functions F_j^0 and G_j^0 are unique and *expansion* (24) is called *basic* if there all $b_j = F_j^0 + \dot{w}G_j^0$. Below we compute this basic expansion only.

Lemma 11. *If $C_1 = C_4 = 0$, then solutions (49) to (47) for P_2 are regular in subsingular points (if θ_j^* and θ_j^{**} are also regular in them).*

Let $\theta_j^*(w)$ and $\theta_j^{**}(w)$ be power series on decreasing power exponents of w and $A_jw^{\sigma_j}$ and let $B_jw^{\tau_j}$ be their terms with maximal power exponents σ_j and τ_j correspondingly, $0 \neq A_j, B_j \in \mathbb{C}$, $\sigma_j, \tau_j \in \mathbb{R}$. F_j and G_j contain $\log w$, if

$$\begin{aligned} \sigma_j &= -1 \text{ or } 4, \\ \tau_j &= -3 \text{ or } 2. \end{aligned} \quad (51)$$

So these *numbers* are *critical* for operators $\mathcal{F}^{-1}$ and $\mathcal{G}^{-1}$.

We will compute $\theta_j(w)$, θ_j^*, and θ_j^{**} as functions of $b_i = F_i + w'G_i$, h_i for $i < j$, and also will compute leading terms of F_j and G_j, that is, power exponents σ_j and τ_j and constants A_j and B_j.

For that we will use following expansions:

$$\begin{aligned} v &= w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_3}{u^3} + \frac{b_4}{u^4} + \cdots, \\ \dot{v} &= \dot{w} + \frac{\dot{b}_1}{u} + \frac{\dot{b}_2 - b_1}{u^2} + \frac{\dot{b}_3 - 2b_2}{u^3} + \frac{\dot{b}_4 - 3b_3}{u^4} + \cdots, \\ \ddot{v} &= \ddot{w} + \frac{\ddot{b}_1}{u} + \frac{\ddot{b}_2 - 2\dot{b}_1}{u^2} + \frac{\ddot{b}_3 - 4\dot{b}_2 + 2b_1}{u^3} \\ &\quad + \frac{\ddot{b}_4 - 6\dot{b}_3 + 6b_2}{u^4} + \cdots, \\ v^3 &= w^3 + \frac{3w^2b_1}{u} + \frac{3wb_1^2 + 3w^2b_2}{u^2} + \frac{3w^2b_3 + 6wb_1b_2 + b_1^3}{u^3} \\ &\quad + \frac{3w^2b_4 + 6wb_1b_3 + 3wb_2^2 + 3b_1^2b_2}{u^4} + \cdots. \end{aligned} \quad (52)$$

Case $k > 4$. According to (45), $h_1(v) = -(3\alpha/\beta)\dot{v}$; hence, $\theta_1^* = 0$, $\theta_1^{**} = -3\alpha/\beta$. According to (46) and (47) we obtain $F_1 = 0$, $G_1 = (\alpha/2\beta)w^{-2} + \cdots$. Next,

$$\theta_2 = 2\dot{b}_1 + 6wb_1^2 - \frac{3\alpha}{\beta}\dot{b}_1 - \frac{\alpha(\alpha-1)}{\beta^2}w. \tag{53}$$

Hence, according to (36),

$$\begin{aligned}\theta_2^* &= \left(2 - \frac{3\alpha}{\beta}\right)\left(\frac{1}{2}P'G_1 + PG_1'\right) + 6wG_1^2P - \frac{\alpha(\alpha-1)}{\beta^2}w \\ &= -\frac{\alpha(\alpha+2)}{2\beta^2}w + \cdots, \\ \theta_2^{**} &= 0.\end{aligned} \tag{54}$$

According to (47), $F_2 = -(\alpha(\alpha+2)/12\beta^2)w^{-1} + \cdots$, $G_2 = 0$. Next,

$$\begin{aligned}\theta_3 &= 4\dot{b}_2 - 2b_1 + 2\left(6wb_1b_2 + b_1^3\right) \\ &\quad - \frac{3\alpha}{\beta}\left(\dot{b}_2 - b_1\right) - \frac{\alpha(\alpha-1)}{\beta^2}b_1.\end{aligned} \tag{55}$$

Hence, $\theta_3^* = 0$, according to (36),

$$\begin{aligned}\theta_3^{**} &= \frac{\alpha+4}{\beta}F_2' - \frac{2(\alpha+1)^2 - 3\alpha(\alpha+1) + \alpha(\alpha-1)}{\beta^2}G_1 \\ &\quad + 12wG_1F_2 + 2PG_1^3 = -\frac{\alpha(\alpha+2)}{6\beta^2}w^{-2} + \cdots.\end{aligned} \tag{56}$$

According to (47), $F_3 = 0$, $G_3 = (\alpha(\alpha+2)/24\beta^2)w^{-4} + \cdots$.

Next,

$$\begin{aligned}\theta_4 &= 6\dot{b}_3 - 6b_2 + 2\left(3wb_2^2 + 6wb_1b_3 + 3b_1^2b_2\right) \\ &\quad - \frac{3\alpha}{\beta}\left(\dot{b}_3 - 2b_2\right) - \frac{\alpha(\alpha-1)}{\beta^2}b_2.\end{aligned} \tag{57}$$

Hence, according to (36),

$$\begin{aligned}\theta_4^* &= \frac{3(\alpha+2)}{\beta}\left(\frac{1}{2}P'G_3 + PG_3'\right) \\ &\quad + 12wPG_1G_3 - \frac{(\alpha+2)(\alpha+3)}{\beta^2}F_2 \\ &\quad + 6wF_2^2 + 6PF_2G_1^2 = 0w^{-1} + \cdots \overset{\text{def}}{=} A_4w^{-1} + \cdots.\end{aligned} \tag{58}$$

Here power exponent -1 of leading term in θ_4^* is critical for operator $\mathcal{F}^{-1}$ but $A_4 = 0$. Hence F_4 has no logarithmic branching.

Now we take into account terms $h_k(v)$ and $h_{k+1}(v)$ from (45). For $j = 4, \ldots, k-1$ power exponents σ_j and τ_j for F_j and G_j are small enough to neglect them. So

$$v = w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_k}{u^k} + \frac{b_{k+1}}{u^{k+1}} + \frac{b_{k+2}}{u^{k+2}} + \cdots. \tag{59}$$

We can write corresponding expansions for $\dot{v}$, $\ddot{v}$, and v^3. Then

$$\begin{aligned}\theta_k^* &= \beta^{-k}w + \cdots, \\ \theta_k^{**} &= 0, \text{hence} \\ F_k &= -\frac{1}{6\beta^k}w^{-1} + \cdots, \\ G_k &= 0. \\ \theta_{k+1} &= (k-1)\dot{b}_k + 12wb_1b_k + \frac{a}{\beta^{k+1}} + \frac{b_1}{\beta^k}, \text{hence} \\ \theta_{k+1}^* &= \frac{a}{\beta^{k+1}} + \cdots, \\ F_{k+1} &= -\frac{a}{4\beta^{k+1}}w^{-2} + \cdots, \\ \theta_{k+1}^{**} &= (k-1)F_k' + 12wG_1F_k + \frac{1}{\beta^k}G_1 \\ &= -\frac{1}{3\beta^k}w^{-2} + \cdots, \\ G_{k+1} &= \frac{1}{12\beta^k}w^{-4} + \cdots; \\ \theta_{k+2} &= 2(k+1)\dot{b}_{k+1} - k(k+1)b_k + 12wb_1b_{k+1} \\ &\quad + 12wb_2b_k + 6b_1^2b_k - (k+1)\left(\dot{b}_{k+1} - kb_k\right) \\ &\quad - \frac{\alpha(\alpha+1)}{\beta^2}b_k + \frac{1}{\beta^k}b_2.\end{aligned} \tag{60}$$

Hence

$$\begin{aligned}\theta_{k+2}^* &= (k+1)\left(\frac{1}{2}P'G_{k+1} + PG_{k+1}'\right) - \frac{\alpha(\alpha-1)}{\beta^2}F_k + \frac{1}{\beta^k}F_2 \\ &\quad + 12wPG_1G_{k+1} + 12wF_2F_k + 6PG_1^2F_k \\ &= 0 \cdot w^{-1} + \cdots, \\ \theta_{k+2}^{**} &= (k+1)F_{k+1}' - \frac{\alpha(\alpha-1)}{\beta^2}G_k + 12wG_1F_{k+1} \\ &\quad + 12wF_2G_k + 6PG_1^2G_k \\ &= 0 \cdot w^{-3} + \cdots.\end{aligned} \tag{61}$$

It means that F_{k+2} and G_{k+2} have no branching at $w = \infty$ and $\sigma_j < -1$ and $\tau_j < -3$ for $k+2 < j < 2k$.

So we neglect b_j for $j = k+2, \ldots, 2k-1$ and consider

$$v = w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_k}{u^k} + \frac{b_{k+1}}{u^{k+1}} + \frac{b_{k+2}}{u^{k+2}} + \frac{b_{2k}}{u^{2k}} + \cdots. \tag{62}$$

We have

$$\theta_{2k} = 6wb_k^2 + \cdots . \quad (63)$$

Hence, according to results after (59),

$$\theta_{2k}^* = 6wF_k^2 + \cdots = \frac{6}{36\beta^{2k}} w^{-1} + \cdots = A_{2k} w^{-1} + \cdots , \quad (64)$$

where $A_{2k} = 1/6\beta^{2k} \neq 0$ and F_{2k} has the logarithmic branching; that is, the regular expansion does not exist.

For $k = 4$, we must add $\beta^{-4}w$ to the computed value of θ_4^*, but it does not change result on existence of logarithmic branching in F_8.

Case $k = 3$ is close to the case $k \geqslant 4$ and it has branching in F_6.

Case $k = 1$ was calculated separately. It has no branching.

Case $k = 0$ corresponds to 2D face $\mathbf{\Gamma}_1^{(2)}$. It has no branching.

Thus, for equation P_2 (42), basic formal expansions are regular for two suitable asymptotic forms with $k = 0$ and $k = 1$ when $x \to \infty$.

Theorem 12. *For P_2, the regular basic families of formal power-elliptic expansions exist only for two suitable elliptic asymptotic forms with $k = 0$ and $k = 1$, that is, when $x \to \infty$.*

It is possible to prescribe power exponents σ_j and τ_j of leading terms in θ_j^* and θ_j^{**}. So we can compute such numbers j^* and j^{**}, where $\sigma_j < -1$ for $j > j^*$ and $\tau_j < -3$ for $j > j^{**}$. Here -1 and -3 are smaller critical values (51) of operators $\mathscr{F}^{-1}$ and $\mathscr{G}^{-1}$. And it is enough to calculate F_j and G_j up to $j = \max(j^*, j^{**})$.

7. Nonbasic Expansions for P_2

Basic expansions (24) were defined by formulas (47), (49) with $C_1 = C_2 = C_3 = C_4 = 0$. According to Lemma 11, condition $C_1 = C_4 = 0$ guarantees regularity of F_j and G_j in subsingular points. Now we want to study cases with nonzero C_3.

Example 13. Let us show that $C_3 \neq 0$ in G_j gives the logarithmic branching in $w = \infty$ for G_{j+2}. For $j = 1$, we put $C_3 = A \neq 0$. According to formulas for case $k \geqslant 4$, we obtain

$$\begin{aligned} F_1 &= 0, \\ G_1 &= A + \frac{\alpha}{2\beta} w^{-2} + \cdots , \\ \theta_2^* &= \frac{2-\alpha}{2\beta}\left(\frac{1}{2}P'G_1 + PG_1'\right) + 6wG_1^2 P - \frac{\alpha(\alpha-1)}{\beta^2} w + \cdots \\ &= \frac{2-\alpha}{2\beta} 2w^3 A + 6w^5\left(A + \frac{\alpha}{2\beta}w^{-2}\right)^2 \\ &\quad - \frac{\alpha(\alpha-1)}{\beta^2} w + \cdots \\ &= 6A^2 w^5 + \frac{5\alpha+2}{\beta} A w^3 + \frac{\alpha^2+2\alpha}{2\beta^2} w + \cdots . \end{aligned} \quad (65)$$

Hence,

$$\begin{aligned} F_2 &= A^2 w^3 - \frac{5\alpha+2}{4\beta} A w + \cdots , \\ G_2 &= F_3 = 0. \end{aligned} \quad (66)$$

Next,

$$\begin{aligned} \theta_3^{**} &= \frac{\alpha+4}{\beta}\left(3A^2w^2 - \frac{5\alpha+2}{4\beta}\right) - \frac{2}{\beta^2}\left(A + \frac{\alpha}{2\beta}w^{-2}\right) \\ &\quad + 12wG_1F_2 + 2PG_1^3 + \cdots \\ &= \frac{\alpha+4}{\beta} 3A^2w^2 - \frac{(\alpha+4)(5\alpha+2)}{4\beta} A - \frac{2}{\beta^2} A \\ &\quad + 12w\left(A + \frac{\alpha}{2\beta}w^{-2}\right)\left(A^2w^3 - \frac{5\alpha+2}{4\beta}Aw\right) \\ &\quad + 2w^4\left(A + \frac{\alpha}{2\beta}w^{-2}\right)^3 + \cdots . \end{aligned} \quad (67)$$

Power exponent 2 is critical for $\mathscr{G}^{-1}$ (see (51)). Coefficient for w^2 in θ_3^{**} is $-(3(\alpha-2)/\beta)A^2$. It is equal to zero only for $\alpha = 2$, but $\alpha = (k+1)/(2-k)$; that is, $k = 1$. But $k \geqslant 4$, then G_3 has logarithmic branching.

8. Equation P_1

Equation P_1 is

$$f(x, y) \overset{\text{def}}{=} -y'' + 3y^2 + x = 0. \quad (68)$$

Support $\tilde{\mathbf{S}}(f)$ consists of 3 points $\mathbf{Q}_1 = (-2, 1, 2)$, $\mathbf{Q}_2 = (0, 2, 0)$, and $\mathbf{Q}_3 = (1, 0, 0)$. Its polyhedron $\mathbf{\Gamma}(f)$ is a triangle with normal $\mathbf{N} = (4, 2, 5)$. So the equation is its own truncation. The edge $\mathbf{\Gamma}_1^{(1)} = [\mathbf{Q}_1, \mathbf{Q}_2]$ of the triangle $\mathbf{\Gamma}$ corresponds to the truncated equation

$$\check{f}_1^{(1)}(x, y) \overset{\text{def}}{=} -y'' + 3y^2 = 0, \quad (69)$$

which has the first integral

$$y'^2 = 2\left(y^3 + C_0\right) \tag{70}$$

with elliptic solutions.

Suitable normals $\mathbf{N}$ to the edge $\mathbf{\Gamma}_1^{(1)}$ are $\mathbf{N}_k = (4-k, 2(k+1), 5)$, $k = 1, 2, \ldots$, and $n_1 \neq 0$ if $k \neq 4$. Here $\alpha = 2(k+1)/(4-k)$, $\beta = 5/(4-k)$, and $\alpha = 2(\beta - 1)$, $\gamma = 2\beta = \alpha + 2$; the transformed equation is

$$-\ddot{v} + 3v^2 - \frac{5\alpha}{\gamma}\dot{v}u^{-1} - \frac{4\alpha(\alpha-1)}{\gamma^2}vu^{-2} + 2^k\gamma^{-k}u^{-k} = 0, \tag{71}$$

$P = 2(w^3 + C_0)$, operators $\mathcal{F}^{-1}$ and $\mathcal{G}^{-1}$ are again (47) and $r(w) \equiv 1$ [23]. Hence there is only one singular point $w^0 = \infty$ and Lemma 11 is true for P_1. Here $H(w) = \text{const} \cdot w^{-7/2} + \cdots$ and integral critical numbers are $\sigma_j = -1$ and $\tau_j = 1$. Formulas (47)–(49) again define basic expansions. If $k > 6$ then

$$\begin{gathered}
F_1 = 0, \\
G_1 = \frac{\alpha}{\gamma}w^{-1} + \cdots, \\
F_2 = \frac{\alpha(\alpha-8)}{6\gamma^2} + \cdots, \\
G_2 = F_3 = 0, \\
G_3 = \frac{\alpha(\alpha+4)}{3\gamma^3}w^{-2} + \cdots, \\
F_4 = -\frac{\alpha(\alpha+4)\left(\alpha^2 + 24\alpha + 48\right)}{60\gamma^4}w^{-1} + \cdots, \\
G_4 = F_5 = 0, \\
G_5 = \frac{\alpha(\alpha+4)\left(3\alpha^3 + 56\alpha^2 + 200\alpha + 192\right)}{180\gamma^5}w^{-3} + \cdots, \\
\theta_6^* = 0 \cdot w^{-1} + \cdots \stackrel{\text{def}}{=} A_6 w^{-1} + \cdots, \\
A_6 = 0.
\end{gathered} \tag{72}$$

Hence, F_6 has no logarithmic branching, if $k > 6$.

Similarly to the end of Section 6 (see (59)), we obtain

$$\begin{gathered}
\theta_k^* = \frac{2^k}{\gamma^k} + \cdots, \\
F_k = -\frac{2^k}{5\gamma^k}w^{-1} + \cdots, \\
G_k = F_{k+1} = 0, \\
G_{k+1} = \frac{(k+11)2^k}{75\gamma^k}w^{-3} + \cdots, \\
\theta_{k+2}^* = 0 \cdot w^{-1} + \cdots,
\end{gathered} \tag{73}$$

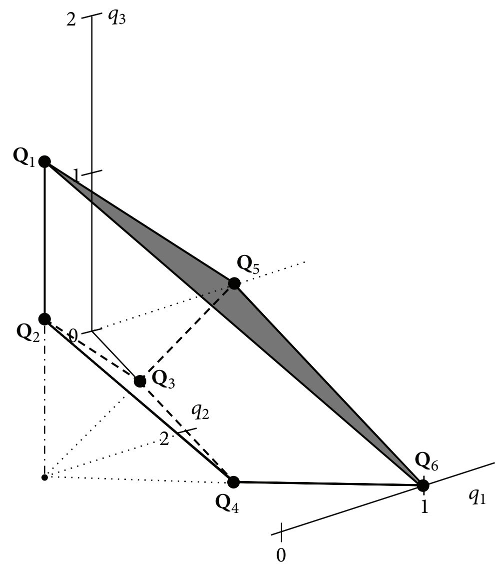

FIGURE 2: 3D support $\widetilde{\mathbf{S}}(f)$ and polyhedron $\mathbf{\Gamma}(f)$ of equation P_3 (74) with all $a, b, c, d \neq 0$. The grey face is $\mathbf{\Gamma}_1^{(2)}$. All dotted lines are in the plane q_1, q_2; they show projections of $\mathbf{\Gamma}(f)$ on the plane (q_1, q_2). Dashed lines are invisible edges.

$\sigma_j < -1$, $\tau_j < 1$ for $j > k+2$ and the regular expansion exists. If $4 < k < 7$, then the regular expansion exists; the same is true for $k = 1, 2, 3$. Case $k = 0$ corresponds to 2D face and to other $P = 2(w^3 + w + C_0)$, but $A_6 = 0$. Thus, equation P_1 has regular basic families of elliptic expansions corresponding to all suitable asymptotic forms. Thus we have the following.

Theorem 14. *To each suitable elliptic asymptotic form of P_1 there corresponds the basic family of formal power-elliptic expansions, which is regular.*

9. Equation P_3

Equation P_3 is

$$f(x, y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0 \tag{74}$$

which has 3 different polyhedrons depending on values of coefficients a, b, c, d [19, 23].

Case $cd \neq 0$. See Figure 2.

Here only one truncated equation

$$-xyy'' + xy'^2 + cxy^4 + dx = 0 \tag{75}$$

corresponding to the distinguished 2D face in Figure 2 has elliptic solutions. Here the power transformation (17) is identical.

Equation (74) with $cd \neq 0$ is of the form (23) with $m = 1$, where

$$\begin{aligned} g(v) &\stackrel{\text{def}}{=} -v\ddot{v} + \dot{v}^2 + cv^4 + d \\ &= 0, \\ h_1 &= -v\dot{v} + av^3 + bv, \\ P(w) &= cw^4 + C_0 w^2 - d, \\ \Delta(P) &= \frac{-cd\left(C_0^2 + 4cd\right)^2}{16} \neq 0. \end{aligned} \tag{76}$$

Solutions to (38) are of the form

$$\begin{aligned} F_j &= P^{1/2} \int \frac{w^2}{P^{3/2}} \int \frac{\theta_j^*}{w^3} dw\, dw, \\ G_j &= \int \frac{w^2}{P^{3/2}} \int \frac{P^{1/2}\theta_j^{**}}{w^3} dw\, dw. \end{aligned} \tag{77}$$

Here $r(w) = w^2$ [23], so there are 2 singular points $w^0 = \infty$ and $w^0 = 0$. This is true for all cases of P_3. Near the singular point $w^0 = \infty$, $H(w) = \int (w^2/P^{3/2}) dw = \text{const} \cdot w^{-3} + \cdots$. So $P^{1/2} = \text{const} \cdot w^2 + \cdots$, $P^{1/2}H = \text{const} \cdot w^{-1} + \cdots$ and expansions for F_j^0 and G_j^0 do not contain terms $\text{const} \cdot w^2$, $\text{const} \cdot w^{-1}$ and $\text{const} \cdot w^0$, $\text{const} \cdot w^{-3}$ correspondingly. Critical numbers for θ_j^* and θ_j^{**} are 2, 5 and 0, 3 correspondingly. Moreover, $\theta_2^* = 0 \cdot w^2 + \cdots$, $\theta_2^{**} = 0 \cdot w + \cdots$, and $\sigma_j < 2$, $\tau_j < 0$ for $j > 2$. So expansion has no logarithmic branching at $w = \infty$.

Near the singular point $w^0 = 0$ we have $H^0(w) = \int (w^2/P^{3/2}) dw = \text{const} \cdot w^3 + O(w^4)$. Here we have 4 constants $C_1^0, \dots, C_4^0$ and basic expansion if all $C_i^0 = 0$. Here Lemma 11 is correct for P_3.

Condition C. Condition C is $\int_0^\infty (w^2/P^{3/2}) dw = 0$.

Theorem 15. *If the Condition C is satisfied then basic expansions for P_3 are regular.*

Case $c = 0$, $ad \neq 0$. After the power transformation $y = x^{1/3} v$, $u = (3/2) x^{2/3}$, (74) with $c = 0$ takes the form (23) with $m = 1$, where

$$\begin{aligned} g(v) &= -v\ddot{v} + \dot{v}^2 - av^3 + d, \\ h_1 &= \frac{3}{2} bv - v\dot{v}, \\ P(w) &= 2aw^3 + C_0 w^2 - d, \\ \Delta(P) &= 4d\left(C_0^3 - 27a^2 d\right) \neq 0. \end{aligned} \tag{78}$$

Formula (77) is valid here. At $w = \infty$, θ_j^* and θ_j^{**} have critical number 2. $\theta_2^* = 0 \cdot w^2 + \cdots$ and orders of $\theta_j^*, \theta_j^{**}$ are less than 2 for $j > 2$.

The same is at $w^0 = 0$. Thus, here formal basic expansion is regular. Lemma 11 and Theorem 15 are true.

Case $c = d = 0$, $ab \neq 0$. After the power transformation $y = v$, $u = 2x^{1/2}$, (74) with $c = d = 0$ takes the form (23) with $m = 1$, where

$$\begin{aligned} g(v) &= -v\ddot{v} + \dot{v}^2 + av^3 + bv, \\ h_1 &= -\frac{v\dot{v}}{2}, \\ P(w) &= 2\left(aw^3 + C_0 w^2 - bw\right), \\ \Delta(P) &= 2^4 b^2 \left(C_0^2 + 4ab\right) \neq 0. \end{aligned} \tag{79}$$

At $w^0 = \infty$ critical values for θ_j^* and θ_j^{**} are 2, $\theta_2^* = 0 \cdot w^2 + \cdots$, $\sigma_j, \tau_j < 2$ for $j > 2$. So here basic expansion has no branching.

The same is at $w^0 = 0$. Lemma 11 and Theorem 15 are true.

Each of 3 polyhedrons has exactly one 2D face corresponding to a truncated equation with elliptic solutions [18, 19, 23]. They have different first integrals $(\dot{w})^2 = P(w)$, but common operators $\mathcal{F}^{-1}$ and $\mathcal{G}^{-1}$ with singularities in two points $w = 0$ and $w = \infty$.

10. Equation P_4

Equation P_4 is

$$\begin{aligned} f(x, y) &\stackrel{\text{def}}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4\left(x^2 - a\right) y^2 + 2b \\ &= 0. \end{aligned} \tag{80}$$

If complex parameters $a, b \neq 0$, its support $\tilde{\mathbf{S}}(f)$ consists of 6 points; polyhedron $\mathbf{\Gamma}(f)$ is a tetrahedron and has one 2D face $\mathbf{\Gamma}_1^{(2)}$ and one edge $\mathbf{\Gamma}_1^{(1)}$ with truncated equations

$$\begin{aligned} \check{f}_1^{(2)} &\stackrel{\text{def}}{=} -2yy'' + \left(y'\right)^2 + 3y^4 + 8xy^3 + 4x^2 y^2 = 0, \\ \check{f}_1^{(1)} &\stackrel{\text{def}}{=} -2yy'' + \left(y'\right)^2 + 3y^4 = 0, \end{aligned} \tag{81}$$

having elliptic solutions [19, 20, 23]. Normal to $\mathbf{\Gamma}_1^{(2)}$ is $\mathbf{N}_0 = (1, 1, 2)$, and suitable normals to $\mathbf{\Gamma}_1^{(1)}$ are $\mathbf{N}_k = (1-k, k+1, 2)$, $k = 2, 3, \dots$. After power transformation (17) with $\alpha = (k+1)/(1-k)$, $\beta = 2/(1-k) = \alpha + 1$, we obtain (23) with $m = 6$

$$\begin{aligned} & -2v\ddot{v} + \dot{v}^2 + 3v^4 - \frac{4\alpha}{\beta} v\dot{v}u^{-1} + \frac{\alpha(2-\alpha)}{\beta^2} v^2 u^{-2} \\ & + \frac{8}{\beta^k} v^3 u^{-k} - \frac{4a}{\beta^{k+1}} v^2 u^{-(k+1)} \\ & + \frac{4}{\beta^{2k}} v^2 u^{-2k} + \frac{2b}{\beta^{2(k+1)}} u^{-2(k+1)} = 0, \\ & P(w) = w^4 + C_0 w, \\ & C_0 \neq 0, \quad k = 2, 3, \dots. \end{aligned} \tag{82}$$

Here solutions to (38) are

$$F_j = \frac{1}{2}P^{1/2}\int \frac{w}{P^{3/2}}\int \frac{\theta_j^*}{w^2}dw\,dw,$$
$$G_j = \frac{1}{2}\int \frac{w}{P^{3/2}}\int \frac{P^{1/2}\theta_j^{**}}{w^2}dw\,dw, \tag{83}$$

$r(w) = w$ [23], so there are two singular points $w^0 = \infty$ and $w^0 = 0$. Near $w^0 = \infty$ $H = \int (w/P^{3/2})dw = \text{const}\cdot w^{-4} + \cdots$. Critical numbers for θ^* and θ^{**} are $1, 5$ and $-1, 3$ correspondingly. If $k > 3$, $F_1 = 0$, $G_1 = (\alpha/2\beta)w^{-2} + \cdots$, $F_2 = -(\alpha(\alpha+2)/12\beta^2)w^{-1} + \cdots$, $G_2 = 0$, $F_3 = 0$, and $\theta_3^{**} = 0\cdot w^{-1} + \cdots$.

Now we compute expansion of the form (59). Then $F_k = -1/\beta^k + \cdots$, $G_k = 0$, $F_{k+1} = (2a/3\beta^{k+1})w^{-1} + \cdots$, $G_{k+1} = (1/3\beta^{k+1})w^{-3} + \cdots$, $\theta_{k+2}^* = (4\alpha(2\alpha-1)/\beta^{k+2})w + \cdots$, and $\theta_{k+2}^{**} = 0\cdot w^{-1} + \cdots$. Thus, $A_{k+2} = 4\alpha(2\alpha-1)/\beta^{k+2} = 0$ only if $2\alpha - 1 = 0$; that is, $k = -1/3$, which is impossible. Thus, F_{k+2} has logarithmic branching and the regular basic expansion is absent. The same is true for $k = 3, 2$ and for $k = 0$, when $P = w^4 + 4w^3 + 4w^2 + C_0 w$.

11. Equation P_5

Equation P_5 is

$$f(x,y) \overset{\text{def}}{=} -x^2 y(y-1)y'' + x^2\frac{3y-1}{2}y'^2 - xy(y-1)y' + (y-1)^3\left(ay^2+b\right) + cxy^2(y-1) + dx^2y^2(y+1) = 0, \tag{84}$$

where a, b, c, d are complex parameters, having two different polyhedrons depending on values of parameter d [21, 23]. Each of the polyhedrons has only one 2D face with elliptic solutions.

Case $d \neq 0$. Here transformation (17) is identical $y = v$, $x = u$. So, in (23) $m = 2$,

$$g(v) = -v(v-1)\ddot{v} + \frac{(3v-1)\dot{v}^2}{2} + dv^2(v+1),$$
$$h_1 = -v(v-1)\dot{v} + cv^2(v-1),$$
$$h_2 = (v-1)^3\left(av^2+b\right), \tag{85}$$
$$P = -2dw\left[C_0(w-1)^2 + w\right],$$
$$\Delta(P) = (2d)^4 C_0^2(1-4C_0) \neq 0.$$

Solutions to (38) are

$$F_j = P^{1/2}\int \frac{w(w-1)^2}{P^{3/2}}\int \frac{\theta_j^*}{w^2(w-1)^3}dw\,dw,$$
$$G_j = \int \frac{w(w-1)^2}{P^{3/2}}\int \frac{P^{1/2}\theta_j^{**}}{w^2(w-1)^3}dw\,dw. \tag{86}$$

Here $r(w) = w(w-1)^2$ [23], so singular points are $w^0 = \infty, 0, 1$. Near the singular point $w^0 = \infty$

$$H = \int \frac{w(w-1)^2}{P^{3/2}}dw = \text{const}\cdot w^{-1/2} + \cdots \tag{87}$$

critical numbers for θ_j^* and θ_j^{**} are 4 and 3 correspondingly. If $a \neq 0$, then θ_2^* contains the term $-3aw^4$ and F_2 has logarithmic branching. If $a = 0$, then $\sigma_j < 4$ and $\tau_j < 3$ for all $j > 0$. Thus, the basic expansion is regular. Similarly basic expansions are regular near $w^0 = 0$ if and only if $b = 0$ and near $w^0 = 1$ without restrictions.

Condition D. Condition D is $\int_0^1 (w(w-1)^2/P^{3/2})dw = \int_1^\infty (w(w-1)^2/P^{3/2})dw = 0$.

Theorem 16. *If in equation P_5 with $d \neq 0$ and with $a = b = 0$, Condition D is fulfilled then basic expansions are regular. If one of these conditions is violated then all basic expansions are nonregular.*

Case $d = 0$, $c \neq 0$. After the change $y = v$, $u = 2x^{1/2}$, equation P_5 takes the form (23) with $m = 2$, where

$$g(v) = -v(v-1)\ddot{v} + \frac{3v-1}{2}\dot{v}^2 + cv^2(v-1),$$
$$h_1 = -v(v-1)\dot{v},$$
$$h_2 = (v-1)^3\left(av^2+b\right), \tag{88}$$
$$P = -2cw(w-1)\left[C_0(w-1)+1\right],$$
$$\Delta(P) = \left(C_0-1\right)^2 \neq 0, \quad C_0 \neq 0.$$

Formulas (86) are again valid. Here basic expansions near $w^0 = \infty$ are regular if and only if $a = 0$, near $w^0 = 0$ if and only if $b = 0$, and near $w = 1$ are always nonregular.

12. Equation P_6

In generic case has polyhedron $\mathbf{\Gamma}$ with ten 2D faces $\mathbf{\Gamma}_i^{(2)}$, but all external normal to them $\mathbf{N} = (n_1, n_2, n_3)$ does not satisfy conditions (18) $n_1 \neq 0$, $n_3 > 0$. Moreover, all edges $\mathbf{\Gamma}_i^{(1)}$ have no suitable normal. The same is true for degenerate cases.

13. Summary

Thus, all basic expansions are regular for P_1 without additional restrictions (Theorem 14), for P_2 if $x \to \infty$ (Theorem 12), for P_3 under Condition C (Theorem 15), and for P_5 with $a = b = 0$ and $d \neq 0$ under Condition D (Theorem 16).

As next step it is necessary to study convergence of found regular formal power-elliptic expansions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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