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Research Article

Stability, Boundedness, and Existence of Periodic Solutions to Certain Third-Order Delay Differential Equations with Multiple Deviating Arguments

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The behaviour of solutions for certain third-order nonlinear differential equations with multiple deviating arguments is considered. By employing Lyapunov's second method, a complete Lyapunov functional is constructed and used to establish sufficient conditions that guarantee existence of unique solutions that are periodic, uniformly asymptotically stable, and uniformly ultimately bounded. Obtained results not only are new but also include many outstanding results in the literature. Finally, the correctness and effectiveness of the results are justified with examples.

1. Introduction

Differential equations of second and third order with and without delay are essential tools in scientific modeling of problems from many fields of sciences and technologies, such as biology, chemistry, physics, mechanics, electronics, engineering, economy, control theory, medicine, atomic energy, and information theory. Many authors have proposed different methods, in the literature, to discuss qualitative bahaviour of solutions to nonlinear differential equations. Here, we will single out two methods. In this direction, we can mention Lyapunov's second method which demands the construction of a suitable positive definite function (or functional) whose derivative is negative definite; that is, it involves finding the system of closed surfaces that contained the origin and are converging to it. The second method is the frequency domain technique which involves the study of position of the characteristics polynomial roots in the complex plane to obtain certain matrix inequalities which must be positive.

The qualitative behaviors of solutions of differential equations of third order have been discussed extensively and are still receiving attention of authors because of their practical applications. In this regard, we can mention the works of Burton [1, 2], Driver [3], Hale [4], and Yoshizawa [5, 6] which contain general results on the subject matters and expository papers of Abou-El-Ela et al. [7], Ademola et al. [8–10], Adesina [11], Afuwape and Omeike [12], Chukwu [13], Gui [14], Omeike [15, 16], Sadek [17], Tejumola and Tchegnani [18], Tunç et al. [19–28], Yao and Wang [29], and Zhu [30] and the references cited therein.

Recently, Tunç [27] employed Lyapunov's second method to prove two results on stability and boundedness of nonautonomous differential equations with constant delay

$$\begin{aligned} &\dddot{x} + f\left(t, x, x\left(t-\tau\right), \dot{x}, \dot{x}\left(t-\tau\right), \ddot{x}, \ddot{x}\left(t-\tau\right)\right) \\ &\quad + g\left(\dot{x}\left(t-\tau\right)\right) + h\left(x\left(t-\tau\right)\right) \\ &\quad = p\left(t, x, x\left(t-\tau\right), \dot{x}, \dot{x}\left(t-\tau\right), \ddot{x}, \ddot{x}\left(t-\tau\right)\right). \end{aligned} \tag{1}$$

Furthermore, Ademola [9] discussed existence and uniqueness of a periodic solution to the third-order differential equation

$$\begin{aligned} &\dddot{x} + f\left(t, x, \dot{x}, \ddot{x}\right)\ddot{x} \\ &\quad + \sum_{i=1}^{n} g_i\left(t, x\left(t-\tau_i\left(t\right)\right), \dot{x}\left(t-\tau_i\left(t\right)\right)\right) \end{aligned}$$

$$+\sum_{i=1}^{n} h_i\left(t, x\left(t-\tau_i(t)\right)\right)$$
$$= p\left(t, x, x\left(t-\tau_i(t)\right), \dot{x}, \dot{x}\left(t-\tau_i(t)\right), \ddot{x}, \ddot{x}\left(t-\tau_i\right)\right). \tag{2}$$

Unfortunately, the problem of uniform asymptotic stability, uniform ultimate boundedness, and existence and uniqueness of periodic solutions of the third-order delay differential equation (3), where all the nonlinear terms (specifically, the forcing term p_i and the function f_i) are sum of multiple deviating arguments, is yet to be investigated. This is not unconnected with the difficulties associated with the construction of suitable complete Lyapunov functional. The aim of this paper is to fill this gap. We will consider

$$\begin{aligned}\dddot{x} &+ \sum_{i=1}^{n} f_i\left(t, x, x\left(t-\tau_i(t)\right), \dot{x}, \dot{x}\left(t-\tau_i(t)\right), \ddot{x}, \ddot{x}\left(t-\tau_i(t)\right)\right) + \sum_{i=1}^{n} g_i\left(\dot{x}\left(t-\tau_i(t)\right)\right) \\ &+ \sum_{i=1}^{n} h_i\left(x\left(t-\tau_i(t)\right)\right) = \sum_{i=1}^{n} p_i\left(t, x, x\left(t-\tau_i(t)\right), \dot{x}, \dot{x}\left(t-\tau_i(t)\right), \ddot{x}, \ddot{x}\left(t-\tau_i(t)\right)\right),\end{aligned} \tag{3}$$

where f_i, g_i, h_i, and p_i are continuous functions in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^{3n+3}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^+ \times \mathbb{R}^{3n+3}$, respectively, with $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. The dots indicate differentiation with respect to the independent variable t. Equation (3) is equivalent to the system of first-order delay differential equation

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \sum_{i=1}^{n} p_i\left(t, x, x\left(t-\tau_i(t)\right), y, y\left(t-\tau_i(t)\right), z, z\left(t-\tau_i(t)\right)\right) - \sum_{i=1}^{n} f_i\left(t, x, x\left(t-\tau_i(t)\right), y, y\left(t-\tau_i(t)\right), z, z\left(t-\tau_i(t)\right)\right) - \sum_{i=1}^{n} g_i\left(y(t)\right) \\ &\quad - \sum_{i=1}^{n} h_i\left(x(t)\right) + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_i\left(y(s)\right) z(s)\, ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_i'\left(x(s)\right) y(s)\, ds,\end{aligned} \tag{4}$$

where $0 \le \tau_i(t) \le \gamma$, $\gamma > 0$ is a constant to be determined latter, and the derivatives g_i' and h_i' for all i $(i = 1, 2, 3, \ldots, n)$ exist and are continuous for all x and y with $h_i(0) = 0$. This work is motivated by the recent works in [9, 27]. Our results are new; in fact according to our observation from relevant literature, this is the first paper where both the functions f_i and the forcing term p_i contain sum of multiple deviating arguments. For the next section, for easy references, we recall the main mathematical tools that will be used in the sequel. Our main results are stated and proved in Section 3 while in the last section, examples are given.

2. Preliminary Results

Consider the following general nonlinear nonautonomous delay differential equation

$$\dot{X} = \frac{dX}{dt} = F\left(t, X_t\right), \qquad X_t = X(t+\theta), \quad -r \le \theta < 0, \; t \ge 0, \tag{5}$$

where $F : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous mapping and $F(t+\omega, \phi) = F(t, \phi)$ for all $\phi \in C$ and for some positive constant ω. We assume that F takes closed bounded sets into bounded sets in $\mathbb{R}^n$. $(C, \|\cdot\|)$ is the Banach space of continuous function $\varphi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; for $H > 0$, we define $C_H \subset C$ by $C_H = \{\varphi \in C : \|\varphi\| < H\}$, and C_H is the open H-ball in C, $C = C([-r, 0], \mathbb{R}^n)$.

Definition 1 (see [2]). A continuous function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$, $W(s) > 0$ if $s \ne 0$ and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer.)

Definition 2 (see [2]). The zero solution of (5) is asymptotically stable if it is stable and if for each $t_0 \ge 0$ there is $\eta > 0$ such that $\|\phi\| \le \eta$ implies that

$$X\left(t, t_0, \phi\right) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{6}$$

Definition 3 (see [1]). An element $\psi \in C_H$ is in the ω-limit set of ϕ, say $\Omega(\phi)$, if $X(t, 0, \phi)$ is defined on $\mathbb{R}^+$ and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to +\infty$, with $\|X_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$, where $X_{t_n}(\phi) = X(t_n + \theta, 0, \phi)$ for $-r \le \theta < 0$.

Definition 4 (see [30]). A set $Q \subset C_H$ is an invariant set if, for any $\phi \in Q$, the solution $X(t, 0, \phi)$ of (5) is defined on $\mathbb{R}^+$ and $X_t(\phi) \in Q$ for $t \in \mathbb{R}^+$.

Lemma 5 (see [6]). *Suppose that $F(t, \phi) \in \overline{C}_0(\phi)$ and $F(t, \phi)$ is periodic in t of period ω, $\omega \ge r$, and consequently for any $\alpha > 0$ there exists $L(\alpha) > 0$ such that $\phi \in C_\alpha$ implies $|F(t, \phi)| \le L(\alpha)$. Suppose that a continuous Lyapunov functional $V(t, \phi)$ exists, defined on $t \in \mathbb{R}^+$, $\phi \in S^*$, S^* is the set of $\phi \in C$ such that $|\phi(0)| \ge H$ (H may be large), and $V(t, \phi)$ satisfies the following conditions:*

(i) *$a(|\phi(0)|) \le V(t, \phi) \le b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing, and positive for $r \ge H$ and $a(r) \to \infty$ as $r \to \infty$;*

(ii) *$\dot{V}_{(5)}(t, \phi) \le -c(|\phi(0)|)$, where $c(r)$ is continuous and positive for $r \ge H$.*

Suppose that there exists $H_1 > 0$, $H_1 > H$, such that

$$hL\left(\gamma^*\right) < H_1 - H, \tag{7}$$

where $\gamma^* > 0$ is a constant which is determined in the following way: By the condition on $V(t,\phi)$ there exist $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ such that $b(H_1) \le a(\alpha)$, $b(\alpha) \le a(\beta)$, and $b(\beta) \le a(\gamma)$. γ^* is defined by $b(\gamma) \le a(\gamma^*)$. Under the above conditions, there exists a periodic solution of (5) of period ω. In particular, relation (7) can always be satisfied if h is sufficiently small.

Lemma 6 (see [6]). *Suppose that $F(t,\phi)$ is defined and continuous on $0 \le t \le c$, $\phi \in C_H$, and there exists a continuous Lyapunov functional $V(t,\phi,\varphi)$ defined on $0 \le t \le c$, ϕ, and $\varphi \in C_H$ which satisfy the following conditions:*

(i) *$V(t,\phi,\varphi) = 0$ if $\phi = \varphi$;*

(ii) *$V(t,\phi,\varphi) > 0$ if $\phi \ne \varphi$;*

(iii) *for the associated system*

$$\begin{aligned} \dot{x}(t) &= F(t, x_t), \\ \dot{y}(t) &= F(t, y_t), \end{aligned} \tag{8}$$

we have $V'_{(8)}(t,\phi,\varphi) \le 0$, where, for $\|\phi\| = H$ or $\|\varphi\| = H$, we understand that the condition $V'_{(8)}(t,\phi,\varphi) \le 0$ is satisfied in case V' can be defined.

Then, for given initial value $\phi \in C_{H_1}$, $H_1 < H$, there exists a unique solution of (5).

Lemma 7 (see [6]). *Suppose that a continuous Lyapunov functional $V(t,\phi)$ exists, defined on $t \in \mathbb{R}^+$, $\|\phi\| < H$, and $0 < H_1 < H$ which satisfies the following conditions:*

(i) *$a(\|\phi\|) \le V(t,\phi) \le b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing, and positive;*

(ii) *$\dot{V}_{(5)}(t,\phi) \le -c(\|\phi\|)$, where $c(r)$ is continuous and positive for $r \ge 0$;*

then the zero solution of (5) is uniformly asymptotically stable.

Lemma 8 (see [1]). *Let $V : \mathbb{R}^+ \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) *$W_0(|X_t|) \le V(t,X_t) \le W_1(|X_t|) + W_2(\int_{t-r(t)}^{t} W_3(X_t(s))ds)$;*

(ii) *$\dot{V}_{(5)}(t,X_t) \le -W_4(|X_t|) + N$, for some $N > 0$, where W_i $(i = 0,1,2,3,4)$ are wedges,*

then X_t of (5) is uniformly bounded and uniformly ultimately bounded for bound B.

3. Main Results

We will give the following notations before we state our main results. Let

$$\begin{aligned} &\sum_{i=1}^{n} f_i\left(t,x,x\left(t-\tau_i(t)\right),y,y\left(t-\tau_i(t)\right),z,z\left(t-\tau_i(t)\right)\right) \\ &\quad = \sum_{i=1}^{n} f_i(\cdot), \\ &\sum_{i=1}^{n} p_i\left(t,x,x\left(t-\tau_i(t)\right),y,y\left(t-\tau_i(t)\right),z,z\left(t-\tau_i(t)\right)\right) \\ &\quad = \sum_{i=1}^{n} p_i(\cdot). \end{aligned} \tag{9}$$

For the first case of consideration set $\sum_{i=1}^{n} p_i(\cdot) \equiv 0$, system (4) reduces to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\sum_{i=1}^{n} f_i(\cdot) - \sum_{i=1}^{n} g_i\left(y(t)\right) - \sum_{i=1}^{n} h_i\left(x(t)\right) \\ &\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_i\left(y(s)\right) z(s)\, ds \\ &\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_i'\left(x(s)\right) y(s)\, ds, \end{aligned} \tag{10}$$

where f_i, g_i, h_i, and τ_i are functions defined in Section 1. Let (x_t, y_t, z_t) be any solution of (10); the continuously differentiable functional employed in the proof of our results is $V = V(t, x_t, y_t, z_t)$ defined as

$$\begin{aligned} 2V &= 2\sum_{i=1}^{n}(\alpha + a_i)\int_0^x h_i(\xi)\, d\xi + 4\sum_{i=1}^{n}\int_0^y g_i(\eta)\, d\eta \\ &\quad + 4y\sum_{i=1}^{n} h_i(x) + 2\sum_{i=1}^{n}(\alpha + a_i)\, yz + 2z^2 + \sum_{i=1}^{n}\left(\alpha^2 + \beta\right. \\ &\quad \left. + a_i^2\right) y^2 + \sum_{i=1}^{n}\beta b_i x^2 + 2\sum_{i=1}^{n} a_i\beta xy + 2\beta xz \\ &\quad + \int_{-\tau_i(t)}^{0}\int_{t+s}^{t}\left[\lambda_1 x^2(\theta) + \lambda_2 y^2(\theta) + \lambda_3 z^2(\theta)\right] d\theta\, ds, \end{aligned} \tag{11}$$

where α and β are fixed positive constants satisfying

$$\begin{aligned} &b_i^{-1} c_i < \alpha < a_i \quad \forall i \ \ (i = 1,2,3,\ldots,n); \\ &\quad 0 < \beta \\ &\quad < \min\left\{ b_i, \sum_{i=1}^{n}(\alpha b_i - c_i) A_1^{-1}, \sum_{i=1}^{n}(a_i - \alpha) A_2^{-1} \right\}, \end{aligned} \tag{12}$$

with

$$A_1 := 2\left[1 + a_i + \delta_i^{-1}\left(y^{-1}g_i(y) - b_i\right)^2\right],$$
$$A_2 := 4\left[1 + \delta_i^{-1}\left(z^{-1}f_i(\cdot) - a_i\right)^2\right]. \tag{13}$$

λ_1, λ_2, and λ_3 are nonnegative constants which will be determined later.

Remark 9. The Lyapunov functional defined in (11) is an improvement on the one used in [9].

At last, we now state our main results and give their proofs.

Theorem 10. *Further to the assumptions on the functions f_i, g_i, h_i, and τ_i, suppose that, for all i $(i = 1, 2, 3, \ldots, n)$, a_i, δ_i, c_i, B_i, ρ and γ are positive constants and for all $t \geq 0$:*

(i) $f_i(\cdot)/z \geq a_i$ *for all* $z \neq 0$;

(ii) $b_i \leq g_i(y)/y \leq B_i$ *for all* $y \neq 0$;

(iii) $h_i(0) = 0, h_i(x)/x \geq \delta_i$ *for all* $x \neq 0$;

(iv) $h_i'(x) \leq c_i$ *for all* x *and* $a_ib_i - c_i > 0$;

(v) $\tau_i(t) \leq \gamma$, $\tau_i'(t) \leq \rho$, $\rho \in (0, 1)$; *and if*

$$\gamma < \min\left\{\sum_{i=1}^{n}\delta_i\left(B_i + c_i\right)^{-1}, \sum_{i=1}^{n}\left(a_ib_i - c_i\right)A_3^{-1}, \sum_{i=1}^{n}\left(a_i - \alpha\right)A_4^{-1}\right\}, \tag{14}$$

where

$$A_3 := \left(B_i + c_i\right)\left(\alpha + a_i\right) + c_i\left(1 - \rho\right)^{-1}\left(\alpha + \beta + a_i + 2\right), \tag{15}$$
$$A_4 := 22\left(B_i + c_i\right) + B_i\left(1 - \rho\right)^{-1}\left(\alpha + \beta + a_i + 2\right).$$

then the trivial solution of (10) is uniformly asymptotically stable.

Remark 11. (i) If $i = 1$, $f_1(\cdot) = \varphi(\dot{x})$, $g_1(\dot{x}(t - \tau_1(t))) = g(\dot{x}(t - \tau(t)))$, and $h_1(x(t - \tau_1(t))) = h(x(t - \tau(t)))$, (10) reduces to the system considered in [29] and some of our hypotheses agree with the hypotheses obtained therein.

(ii) When $i = 1$, the functions $f_1(\cdot) = g(x, \dot{x})\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = f(x(t - \tau), \dot{x}(t - \tau))$, and $h_1(x(t - \tau_1(t))) = h(x(t - \tau))$ the above result includes that discussed in [24].

(iii) Whenever $i = 1$, $f_1(\cdot) = h(\dot{x})\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = g(\dot{x}(t - \tau(t)))$, and $h_1(x(t - \tau_1(t))) = h(x(t - \tau(t)))$, (10) specializes to that studied in [12]. Thus, the result of Theorem 10 coincides with results in [12] if $i = 1$.

(iv) When $\sum_{i=1}^{n} f_i(\cdot) = a\ddot{x}$, $\sum_{i=1}^{n} g_i(\dot{x}(t - \tau_i(t))) = b\dot{x}$, $\sum_{i=1}^{n} h_i(x(t - \tau_i(t))) = cx$, and $\sum_{i=1}^{n} p_i(\cdot) = 0$, (3) reduces to linear constant coefficients differential equations and conditions (i) to (v) of Theorem 10 specialize to the corresponding Routh-Hurwitz criteria $a > 0$, $ab > c$, and $c > 0$.

(v) When $i = 1$, $f_1(\cdot) = a_1\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = f_2(\dot{x}(t - \tau_1(t)))$, and $h_i(x(t - \tau_1(t))) = a_3x(t)$, (10) specializes to that discussed in [28]. Theorem 10 coincides with the stability result in [28].

(vi) When $\sum_{i=1}^{n} f_i(\cdot) = f(\ddot{x})$, $\sum_{i=1}^{n} g_i(\dot{x}(t - \tau_i(t))) = g(\dot{x})$, $\sum_{i=1}^{n} h_i(x(t - \tau_i(t))) = h(x)$, and $\sum_{i=1}^{n} p_i(\cdot) = p(t, x, \dot{x}, \ddot{x})$, then (3) reduces to the ordinary differential equation studied in [31].

(vii) If $i = 1$ and $\tau_i(t) = \tau$ then (3) coincides with (2) discussed in [27]; hence our hypotheses coincide with that of Tunç in [27].

(viii) Whenever $i = 1$, $f_1(\cdot) = f(x, \dot{x})\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = g(x(t - \tau(t)), \dot{x}(t - \tau(t)))$, and $h_1(x(t - \tau_1(t))) = h(x(t - \tau(t)))$, (10) is a particular case of that studied in [7]. Our hypotheses coincide with that in [7] except for $\sup\{h'(x)\} = c > 0$ which is replaced by a more general condition.

(ix) Finally, the functions $f_i(\cdot)$ and $p_i(\cdot)$ used in this paper extend the works in [7–10, 12, 24, 27–29, 31].

In what follows, we will state and prove a result that would be useful in the proof of Theorem 10 and subsequent ones.

Lemma 12. *Under the hypotheses of Theorem 10 there exist positive constants $D_0 = D_0(a_i, b_i, c_i, \delta_i, \alpha, \beta)$, $D_1 = D_1(a_i, b_i, c_i, B_i, \alpha, \beta)$, and $D_2 = D_2(\lambda_1, \lambda_2, \lambda_3)$ such that for all $(x_t, y_t, z_t) \in \mathbb{R}^3$*

$$D_0\left(x^2(t) + y^2(t) + z^2(t)\right) \leq V\left(t, x_t, y_t, z_t\right) \leq D_1\left(x^2(t) + y^2(t) + z^2(t)\right) + D_2\int_{-\tau_i(t)}^{0}\int_{t+s}^{t}\left(x^2(\theta) + y^2(\theta) + z^2(\theta)\right)d\theta\, ds. \tag{16}$$

Furthermore, there exists a constant $D_3 = D_3(a_i, b_i, c_i, B_i, \delta_i, \alpha, \beta, \gamma, \rho, \lambda_1, \lambda_2, \lambda_3) > 0$ such that

$$\dot{V}_{(10)} = \left.\frac{d}{dt}V\left(t, x_t, y_t, z_t\right)\right|_{(10)} \leq -D_3\left(x^2(t) + y^2(t) + z^2(t)\right). \tag{17}$$

Proof. Let (x_t, y_t, z_t) be any solution of (10); since $h_i(0) = 0$, (11) can be recast in the form

$$\begin{aligned} V = {} & \sum_{i=1}^{n} b_i^{-1}\int_0^x\left[\left(\alpha + a_i\right)b_i - 2h_i'(\xi)\right]h_i(\xi)\, d\xi + 2\sum_{i=1}^{n}\int_0^y\left(\eta^{-1}g_i(\eta) - b_i\right)\eta\, d\eta + \sum_{i=1}^{n} b_i^{-1}\left(h_i(x) + b_iy\right)^2 + \frac{1}{2}\beta y^2 + \frac{1}{2}\sum_{i=1}^{n}\left(\beta x + a_iy + z\right)^2 + \frac{1}{2}\left(\alpha y + z\right)^2 \\ & + \frac{1}{2}\sum_{i=1}^{n}\beta\left(b_i - \beta\right)x^2 + \frac{1}{2}\int_{-\tau_i(t)}^{0}\int_{t+s}^{t}\left(\lambda_1x^2(\theta) + \lambda_2y^2(\theta) + \lambda_3z^2(\theta)\right)d\theta\, ds. \end{aligned} \tag{18}$$

From hypotheses (ii), (iii), and (iv) and the fact that the double integrals appearing in inequality (18) are nonnegative, it follows that there exists a constant $K_0 > 0$ such that

$$V \geq K_0 \left(x^2 + y^2 + z^2\right) \tag{19}$$

for all x, y, and z, where

$$\begin{aligned} K_0 = \min \Bigg\{ &\sum_{i=1}^{n} b_i^{-1} \left[(\alpha + a_i) b_i - 2c_i\right] \delta_i \\ &+ \sum_{i=1}^{n} b_i^{-1} \min \left\{\sum_{i=1}^{n} \delta_i, \sum_{i=1}^{n} b_i\right\} + \frac{1}{2} \min \left\{\beta, \sum_{i=1}^{n} a_i, 1\right\} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \beta (b_i - \beta), \sum_{i=1}^{n} b_i^{-1} \min \left\{\sum_{i=1}^{n} \delta_i, \sum_{i=1}^{n} b_i\right\} \\ &+ \frac{1}{2} \min \left\{\beta, \sum_{i=1}^{n} a_i, 1\right\} + \frac{1}{2} \min \{\alpha, 1\} \\ &+ \frac{1}{2}\beta, \frac{1}{2} \min \left\{\beta, \sum_{i=1}^{n} a_i, 1\right\} + \frac{1}{2} \min \{\alpha, 1\} \Bigg\}. \end{aligned} \tag{20}$$

Estimate (19) establishes the lower inequality in (16) with $K_0 = D_0$, respectively. Moreover, from inequality (19) we find that $V(t, x_t, y_t, z_t) = 0$ if and only if $(x^2 + y^2 + z^2) = 0$ and $V(t, x_t, y_t, z_t) > 0$ if and only if $(x^2 + y^2 + z^2) \neq 0$, and it follows that for all x, y, z

$$V(t, x_t, y_t, z_t) \longrightarrow +\infty \quad \text{as } x^2 + y^2 + z^2 \longrightarrow \infty. \tag{21}$$

Furthermore, since $h_i(0) = 0$, $h_i(x) \leq c_i x$ for all $x \neq 0$, $g_i(y) \leq B_i y$ for all $y \neq 0$, and inequality $2x_1x_2 \leq x_1^2 + x_2^2$, there exists positive constants K_1, K_2 such that

$$\begin{aligned} V \leq{} & K_1 \left(x^2 + y^2 + z^2\right) \\ & + K_2 \int_{-\tau_i(t)}^{0} \int_{t+s}^{t} \left[x^2(\theta) + y^2(\theta) + z^2(\theta)\right] d\theta\, ds, \end{aligned} \tag{22}$$

where

$$\begin{aligned} K_1 = \frac{1}{2} \max \Bigg\{ &\sum_{i=1}^{n} \left[(\alpha + a_i + 2) c_i + \beta (a_i + b_i + 1)\right], \\ &\sum_{i=1}^{n} \left[2 (B_i + c_i) + a_i (\alpha + \beta + a_i) + \alpha^2 + \beta\right], \\ &\sum_{i=1}^{n} \left[\alpha + \beta + a_i + 2\right] \Bigg\}, \\ K_2 = \max & \left\{\lambda_1, \lambda_2, \lambda_3\right\}. \end{aligned} \tag{23}$$

Estimate (22) establishes the upper inequality in (16) with $K_1 = D_1$ and $K_2 = D_2$, respectively. Hence, from inequalities (19) and (22) estimate (16) of Lemma 12 is established.

Next, the time derivative of the functional defined in inequality (11) with respect to the independent variable t, along a solution of system (10), is simplified to give

$$\begin{aligned} \dot{V}_{(10)} = {} & U_1 - U_2 - \beta \sum_{i=1}^{n} \left(\frac{g_i(y)}{y} - b_i\right) xy \\ & - \beta \sum_{i=1}^{n} \left(\frac{f_i(\cdot)}{z} - a_i\right) xz + 2\beta yz \\ & - \sum_{i=1}^{n} \left[(\alpha + a_i) \frac{f_i(\cdot)}{z} - \left(\alpha^2 + a_i^2\right)\right] yz \\ & + \sum_{i=1}^{n} a_i \beta y^2 + \frac{1}{2} \left(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2\right) \tau_i(t) \\ & - \frac{1}{2} \left(1 - \tau_i'(t)\right) \\ & \cdot \int_{t+s}^{t} \left(\lambda_1 x^2(s) + \lambda_2 y^2(s) + \lambda_3 z^2(s)\right) ds, \end{aligned} \tag{24}$$

where

$$\begin{aligned} U_1 := {} & \left(\beta x + \sum_{i=1}^{n} (\alpha + a_i) y + 2z\right) \\ & \cdot \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(g_i'(y(s)) z(s) + h_i'(x(s)) y(s)\right) ds, \\ U_2 := {} & \sum_{i=1}^{n} \left(\beta \frac{h_i(x)}{x}\right) x^2 \\ & + \sum_{i=1}^{n} \left[(\alpha + a_i) \frac{g_i(y)}{y} - 2h_i'(x)\right] y^2 \\ & + \sum_{i=1}^{n} \left[2 \frac{f_i(\cdot)}{z} - (\alpha + a_i)\right] z^2. \end{aligned} \tag{25}$$

Now from the assumptions of Theorem 10 we find that

$$\begin{aligned} U_1 \leq {} & \frac{1}{2} \sum_{i=1}^{n} \left(\beta x^2 + (\alpha + a_i) y^2 + 2z^2\right) (B_i + c_i) \tau_i(t) \\ & + \frac{1}{2} \sum_{i=1}^{n} (\alpha + \beta + a_i + 2) \\ & \cdot \int_{t-\tau_i(t)}^{t} \left(c_i y^2(s) + B_i z^2(s)\right) ds \end{aligned} \tag{26}$$

for all x, y, z and $t \geq 0$ and

$$\begin{aligned} U_2 \geq {} & \sum_{i=1}^{n} \beta \delta_i x^2 + \sum_{i=1}^{n} \left[(\alpha + a_i) b_i - 2c_i\right] y^2 \\ & - \frac{1}{2} \sum_{i=1}^{n} (a_i - \alpha) z^2 \end{aligned} \tag{27}$$

for all x, y, z. Using estimates U_1 and U_2 in (24), we find that

$$\begin{aligned}\dot{V}_{(10)} \le{}& -\frac{1}{2}\sum_{i=1}^{n}\beta\delta_i x^2 - \frac{1}{2}\sum_{i=1}^{n}\left[(\alpha+a_i)b_i - 2c_i\right]y^2 - \frac{1}{2}\\ &\cdot\sum_{i=1}^{n}(a_i-\alpha)z^2 - \sum_{j=3}^{4}U_j - \frac{1}{2}\left(1-\tau_i'(t)\right)\\ &\cdot\lambda_1\int_{t-\tau_i(t)}^{t}x^2(s)\,ds + \frac{1}{2}\left[\sum_{i=1}^{n}\beta(B_i+c_i)+\lambda_1\right]\tau_i(t)\\ &\cdot x^2 - \frac{1}{2}\left[\left(1-\tau_i'(t)\right)\lambda_2 - \sum_{i=1}^{n}(\alpha+\beta+a_i+2)c_i\right]\\ &\cdot\int_{t-\tau_i(t)}^{t}y^2(s)\,ds + \beta z^2\\ &-\frac{1}{2}\left[\left(1-\tau_i'(t)\right)\lambda_3 - \sum_{i=1}^{n}(\alpha+\beta+a_i+2)B_i\right]\\ &\cdot\int_{t-\tau_i(t)}^{t}z^2(s)\,ds + \sum_{i=1}^{n}\beta(a_i+1)y^2\\ &+\frac{1}{2}\left[\sum_{i=1}^{n}(B_i+c_i)(\alpha+a_i)+\lambda_2\right]\tau_i(t)y^2\\ &+\frac{1}{2}\left[2\sum_{i=1}^{n}(B_i+c_i)+\lambda_3\right]\tau_i(t)z^2\end{aligned} \tag{28}$$

for all x, y, z, where

$$\begin{aligned}U_3 :={}& \frac{1}{4}\sum_{i=1}^{n}\beta\delta_i x^2 + \beta\sum_{i=1}^{n}\left(y^{-1}g_i(y)-b_i\right)xy + \frac{1}{4}\\ &\cdot\sum_{i=1}^{n}\beta\delta_i x^2 + \beta\sum_{i=1}^{n}\left(z^{-1}f_i(\cdot)-a_i\right)xz,\\ U_4 :={}& \frac{1}{2}\left[\sum_{i=1}^{n}\left[(\alpha+a_i)b_i-2c_i\right]y^2\right.\\ &+2\sum_{i=1}^{n}\left[(\alpha+a_i)z^{-1}f_i(\cdot)-\left(\alpha^2+a_i^2\right)\right]yz\\ &\left.+\sum_{i=1}^{n}(a_i-\alpha)z^2\right].\end{aligned} \tag{29}$$

Since $\beta>0$, $\delta_i>0$, and

$$\begin{aligned}&\sum_{i=1}^{n}\left[x+2\delta_i^{-1}\left(y^{-1}g_i(y)-b_i\right)y\right]^2 \ge 0,\\ &\sum_{i=1}^{n}\left[x+2\delta_i^{-1}\left(z^{-1}f_i(\cdot)-a_i\right)z\right]^2 \ge 0\end{aligned} \tag{30}$$

for all x, y, z, we find

$$\begin{aligned}U_3 \ge{}& -\beta\sum_{i=1}^{n}\delta_i^{-1}\left(y^{-1}g_i(y)-b_i\right)^2y^2\\ &-\beta\sum_{i=1}^{n}\delta_i^{-1}\left(z^{-1}f_i(\cdot)-a_i\right)^2z^2\end{aligned} \tag{31}$$

for all y and z. Moreover, using the estimate

$$\begin{aligned}&\sum_{i=1}^{n}\left[(\alpha+a_i)z^{-1}f_i(\cdot)-\left(\alpha^2+a_i^2\right)\right]^2\\ &\quad<\sum_{i=1}^{n}\left[(\alpha+a_i)b_i-2c_i\right](a_i-\alpha)\end{aligned} \tag{32}$$

in U_4 we obtain

$$\begin{aligned}U_4 \ge{}& \frac{1}{2}\left[\sqrt{\sum_{i=1}^{n}\left[(\alpha+a_i)b_i-2c_i\right]}\,|y|\right.\\ &\left.-\sqrt{\sum_{i=1}^{n}(a_i-\alpha)}\,|z|\right]^2 \ge 0\end{aligned} \tag{33}$$

for all y and z. Inserting estimates U_3 and U_4 in inequality (28) with hypothesis (v) of Theorem 10, choosing $\lambda_1=0$, $\lambda_2=\sum_{i=1}^{n}(1-\rho)^{-1}(\alpha+\beta+a_i+2)c_i>0$, and $\lambda_3=\sum_{i=1}^{n}(1-\rho)^{-1}(\alpha+\beta+a_i+2)B_i>0$, we have

$$\begin{aligned}\dot{V}_{(10)} \le{}& -\frac{1}{2}\sum_{i=1}^{n}\left[a_ib_i-c_i-\left[(B_i+c_i)(\alpha+a_i)\right.\right.\\ &\left.\left.+c_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)\right]\gamma\right]y^2 - \frac{1}{4}\sum_{i=1}^{n}\left[(a_i\right.\\ &-\alpha)-2\left[2(B_i+c_i)\right.\\ &\left.\left.+B_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)\right]\gamma\right]z^2 - \left\{\frac{1}{2}\sum_{i=1}^{n}(\alpha b_i\right.\\ &\left.-c_i)-\beta\sum_{i=1}^{n}\left[1+a_i+\delta_i^{-1}\left(y^{-1}g_i(y)-b_i\right)^2\right]\right\}y^2\\ &-\left\{\frac{1}{4}\sum_{i=1}^{n}(a_i-\alpha)-\beta\sum_{i=1}^{n}\left[1\right.\right.\\ &\left.\left.+\delta_i^{-1}\left(z^{-1}f_i(\cdot)-a_i\right)^2\right]\right\}z^2 - \frac{1}{2}\sum_{i=1}^{n}\beta\left[\delta_i-(B_i+c_i)\right.\\ &\left.\cdot\gamma\right]x^2\end{aligned} \tag{34}$$

for all x, y, z.

Now in view of the inequalities in (12) there exists a positive constant K_3 such that

$$\dot{V}_{(10)} \le -K_3\left(x^2+y^2+z^2\right) \quad \forall (x,y,z)\in\mathbb{R}^3, \tag{35}$$

where

$$\begin{aligned}K_3 = \min&\left\{\frac{1}{2}\sum_{i=1}^{n}\beta\left[\delta_i-(B_i+c_i)\gamma\right], \frac{1}{2}\sum_{i=1}^{n}\left[a_ib_i-c_i\right.\right.\\ &-\left[(B_i+c_i)(\alpha+a_i)+c_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)\right]\\ &\left.\cdot\gamma\right], \frac{1}{4}\sum_{i=1}^{n}\left[(a_i-\alpha)\right.\\ &\left.\left.-2\left[2(B_i+c_i)+B_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)\right]\gamma\right]\right\}.\end{aligned} \tag{36}$$

Inequality (35) establishes (17) with $K_3 = D_3$, respectively. This completes the proof of Lemma 12. □

Proof of Theorem 10. Let (x_t, y_t, z_t) be any solution of system (10), in view of the inequalities in (19), (22), and (35); all the assumptions of Lemma 7 hold. Thus by Lemma 7 the trivial solution of system (10) or (3) for $p_i(\cdot) = 0$ is uniformly asymptotically stable. This completes the proof of Theorem 10. □

Next, we will consider the case of $p_i(\cdot) \neq 0$, and we have the following result.

Theorem 13. *If hypotheses (i)–(v) and the inequality in (14) of Theorem 10 hold and*

$$\left|p_i(\cdot)\right| \le \phi_i(t) + \varphi_i(t)\left(|x| + |y| + |z|\right), \tag{37}$$

for all i $(i = 1, 2, \ldots, n)$, $(x, y, z) \in \mathbb{R}^3$, and $t \ge 0$, where $\phi_i(t)$ and $\varphi_i(t)$ are continuous functions satisfying

$$\phi_i(t) \le d_i, \quad 0 < d_i < \infty \tag{38}$$

and there exists an $\epsilon_i > 0$ such that

$$0 \le \varphi_i(t) \le \epsilon_i, \tag{39}$$

then

(i) *the solutions of system (4) are uniformly bounded and uniformly ultimately bounded;*

(ii) *equation (4) has a unique periodic solution of period ω.*

Remark 14. (i) Whenever $i = 1$, $f_1(\cdot) = f(x, \dot{x})\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = g(x(t-\tau(t)), \dot{x}(t-\tau(t)))$, $h_1(x(t-\tau_1(t))) = h(x(t-\tau(t)))$, and $p_1(\cdot) = p(t, x, \dot{x}, \ddot{x}, x(t-\tau(t)), \dot{x}(t-\tau(t)))$, system (4) is a particular case of that studied in [7]. Our hypotheses coincide with that in [7] except for $\sup\{h'(x)\} = c > 0$ which is replaced by a more general condition in ours.

(ii) If $i = 1$, $f_1(\cdot) = \varphi(x, \dot{x})\ddot{x}$, $g_1(\dot{x}(t-\tau_1(t))) = g(\dot{x}(t-\tau(t)))$, $h_1(x(t-\tau_1(t))) = f(x(t-\tau(t)))$, and $p_1(\cdot) = p(t, x, \dot{x}, \ddot{x})$, system (4) reduces to that considered in [16].

(iii) When $i = 1$, the functions $f_1(\cdot) = g(x, \dot{x})\ddot{x}$, $g_1(\dot{x}(t - \tau_1(t))) = f(x(t-\tau), \dot{x}(t-\tau))$, $h_1(x(t-\tau_1(t))) = h(x(t-\tau))$, and $p_1(\cdot) = p(t, x, \dot{x}, x(t-\tau), \dot{x}(t-\tau), \ddot{x})$, the above result includes that discussed in [24].

(iv) Whenever $i = 1$, $f_1(\cdot) = h(\dot{x})\ddot{x}$, $g_1(\dot{x}(t-\tau_1(t))) = g(\dot{x}(t-\tau(t)))$, $h_1(x(t-\tau_1(t))) = h(x(t-\tau(t)))$, and $p_1(\cdot) = p(t, x, \dot{x}, x(t-\tau), \dot{x}(t-\tau), \ddot{x})$, (4) specializes to that studied in [12].

(v) When $i = 1$, $f_1(\cdot) = a_1\ddot{x}$, $g_1(\dot{x}(t-\tau_1(t))) = f_2(\dot{x}(t-\tau_1(t)))$, $h_i(x(t-\tau_1(t))) = a_3 x(t)$, $p_1(\cdot) = p(t, x, \dot{x}, x(t-\tau_1(t)), \dot{x}(t-\tau_1(t)), \ddot{x})$, and $\varphi_i(t) = 0$, system (4) reduces to that considered in [28]. Theorem 13 coincides with the boundedness result in [28].

(vi) If $i = 1$, and $\varphi_i(t) \equiv 0$ in inequality (37), our result specializes to that studied in [9, 27].

(vii) Whenever $i = 1$, in inequality (37) the result in Theorem 13 reduces to that discussed in [8].

Hence, Theorem 13 includes and improves the results in [7–9, 12, 16, 24, 27, 28].

Proof of Theorem 13. (i) Let (x_t, y_t, z_t) be any solution of system (4); the time derivative of the functional $V \equiv V(t, x_t, y_t, z_t)$ defined in system (11) along a solution of system (4) is

$$\dot{V}_{(4)} = \dot{V}_{(10)} + \sum_{i=1}^{n}\left[\beta x + (\alpha + a_i)\, y + 2z\right] p_i(\cdot). \tag{40}$$

Using inequality (35), the above relation becomes

$$\begin{aligned}\dot{V}_{(4)} \le{}& -K_3\left(x^2 + y^2 + z^2\right) \\ &+ K_4\left(|x| + |y| + |z|\right)\sum_{i=1}^{n}\left|p_i(\cdot)\right|,\end{aligned} \tag{41}$$

where $K_4 = \max\{\beta, \sum_{i=1}^{n}(\alpha + a_i), 2\}$. Applying inequality (37), we find that

$$\begin{aligned}\dot{V}_{(4)} \le{}& -K_3\left(x^2 + y^2 + z^2\right) \\ &+ K_4\left(|x| + |y| + |z|\right)\sum_{i=1}^{n}\phi_i(t) \\ &+ K_4\left(|x| + |y| + |z|\right)^2\sum_{i=1}^{n}\varphi_i(t).\end{aligned} \tag{42}$$

From estimates (38) and (39) and on choosing $K_3 > K_4\sum_{i=1}^{n}(d_i + \epsilon_i)$, there exist constants $K_5 > 0$ and $K_6 > 0$ such that

$$\dot{V}_{(4)} \le -K_5\left(x^2 + y^2 + z^2\right) + K_6, \quad \forall\left(x, y, z\right) \in \mathbb{R}^3, \tag{43}$$

where $K_5 := K_3 - K_4\sum_{i=1}^{n}(d_i + \epsilon_i)$ and $K_6 := 3K_4\sum_{i=1}^{n} d_i$.

The inequalities in (19), (22), and (43) establish the hypotheses of Lemma 8. Hence by Lemma 8, the solution (x_t, y_t, z_t) of system (4) is uniformly bounded and uniformly ultimately bounded.

(ii) From estimate (42), using the inequalities in (38) and (39), we have

$$\begin{aligned}\dot{V}_{(4)} \le{}& -\left(K_3 - 3K_4\sum_{i=1}^{n}\epsilon_i\right)\left(x^2 + y^2 + z^2\right) \\ &+ 3^{1/2}K_4\sum_{i=1}^{n}d_i\left(x^2 + y^2 + z^2\right)^{1/2}.\end{aligned} \tag{44}$$

Choosing ϵ_i $(i = 1, 2, \ldots, n)$ sufficiently small such that $K_7 := K_3 - 3K_4\sum_{i=1}^{n}\epsilon_i > 0$ and $K_8 := 3^{1/2}K_4\sum_{i=1}^{n} d_i$ we have

$$\dot{V}_{(4)} \le -K_9\left(x^2 + y^2 + z^2\right) \quad \forall\left(x, y, z\right) \in \mathbb{R}^3, \tag{45}$$

provided that $(x^2+y^2+z^2)^{1/2} \geq K_{10} := 2K_8K_7^{-1} > 0$ where $K_9 := (1/2)K_7 > 0$. In view of (19), (21), (22), and (45) all assumptions of Lemmas 5 and 6 are met. Hence by Lemmas 5 and 6, system (4) has a unique periodic solution of period ω. This completes the proof of Theorem 13. □

Next, if $p_i(\cdot)$ in system (4) is replaced by $p_i(t) \neq 0$, we have

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \sum_{i=1}^{n} p_i(t) - \sum_{i=1}^{n} f_i(\cdot) - \sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} h_i(x) \\ &\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_i'(y(s)) z(s)\, ds \\ &\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_i'(x(s)) y(s)\, ds, \end{aligned} \tag{46}$$

where f_i, g_i, and h_i are the functions defined in Section 1, and $p_i : \mathbb{R}^+ \to \mathbb{R}$, we have the following result.

Theorem 15. *If hypotheses (i)–(v) and estimate (14) of Theorem 10 hold, and*

$$\int_0^t |p_i(s)|\, ds \leq q_i, \quad 0 \leq q_i < \infty, \quad (i = 1, 2, \ldots, n), \tag{47}$$

then for any given finite constants x_0, y_0, z_0 there exists a constant $D_4 = D_4(x_0, y_0, z_0, \alpha, \beta, a_i, b_i, c_i, \delta_i)$ such that any solution (x_t, y_t, z_t) of system (46) determined by $x_0 = x_0$, $y_0 = y_0$, $z_0 = z_0$, for $t = 0$, satisfies

$$\begin{aligned} |x_t| &\leq D_4, \\ |y_t| &\leq D_4, \\ |z_t| &\leq D_4 \end{aligned} \tag{48}$$

for all $t > 0$.

Remark 16. if $i = 1$, $f_1(\cdot) = f_1(t,x)\ddot{x}$, $g_1(\dot{x}(t-\tau_1(t))) = f_2(t,x)\dot{x} + g_0(t,x)$, $h_i(x(t-\tau_1(t))) = g_i(x(t-\tau(t)))$, and $p_1(\cdot) = p(t)$, (46) reduces to that considered in [19]. Our results are quite different from this because of the non-Liapunov approach used in [19].

Proof of Theorem 15. Let (x_t, y_t, z_t) be any solution of system (46). In view of the hypotheses (i)–(v) and estimate (14), inequality (19) holds. The derivative of the functional V defined in system (11) with respect to the independent variable t along a solution of system (46) is

$$\dot{V}_{(48)} = \dot{V}_{(10)} + \sum_{i=1}^{n} \left[\beta x + (\alpha + a_i) y + 2z\right] p_i(t). \tag{49}$$

By inequality (35), $\dot{V}_{(10)} \leq 0$ for all $(x, y, z) \in \mathbb{R}^3$, and from the fact that $|x_1| \leq 1 + x_1^2$, it follows that

$$\dot{V}_{(48)} \leq 3K_4 \sum_{i=1}^{n} |p_i(t)| + K_4 \left(x^2 + y^2 + z^2\right) \sum_{i=1}^{n} |p_i(t)|, \tag{50}$$

for all x, y, z and $t \geq 0$. Also from inequality (19), the above inequality becomes

$$\dot{V}_{(48)} - K_0^{-1} K_4 V \sum_{i=1}^{n} |p_i(t)| \leq 3K_4 \sum_{i=1}^{n} |p_i(t)|. \tag{51}$$

Solving this first-order differential inequality by multiplying each side by

$$\exp\left[-K_0^{-1} K_4 \sum_{i=1}^{n} \int_0^t |p_i(s)|\, ds\right], \tag{52}$$

integrating from 0 to t, and employing inequality (47), we find that

$$V = V(t, x_t, y_t, z_t) \leq [V(0) + 1]\, e^{K_0^{-1} K_4 \sum_{i=1}^{n} q_i} - 1, \tag{53}$$

where $V(0) = V(0, x_0, y_0, z_0)$.

Engaging inequality (19), we have

$$\begin{aligned} |x_t| &\leq K_{11}, \\ |y_t| &\leq K_{11}, \\ |z_t| &\leq K_{11} \end{aligned} \tag{54}$$

for all $t > 0$, where

$$K_{11} := \left\{\left[[V(0) + 1]\, e^{K_0^{-1} K_4 \sum_{i=1}^{n} q_i} - 1\right] K_0^{-1}\right\}^{1/2}. \tag{55}$$

Equating $K_{11} = D_4$, the inequalities in (48) are satisfied. This completes the proof of Theorem 15. □

4. Examples

Example 1. Consider the homogeneous third-order scalar delay differential equation

$$\begin{aligned} &\dddot{x} + \sum_{i=1}^{n} \left[3\ddot{x} + \frac{\ddot{x}}{5 + \sin t + |x(t-\tau_i(t))\dot{x}(t-\tau_i(t))| + |\dot{x}(t-\tau_i(t))\ddot{x}(t-\tau_i(t))|}\right] \\ &+ \sum_{i=1}^{n} \left[3\dot{x}(t-\tau_i(t)) + \frac{\dot{x}(t-\tau_i(t))}{4 + \sin 2t + |x(t-\tau_i(t))| + |\dot{x}(t-\tau_i(t))|}\right] + \sum_{i=1}^{n} \left[3x(t-\tau_i(t)) + \frac{x(t-\tau_i(t))}{3 + \sin 2t}\right] = 0. \end{aligned} \tag{56}$$

Reducing (56) to system of first-order delay differential equations by setting $\dot{x} = y$ and $\ddot{x} = z$ we obtain

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left[3 + \frac{4 + \sin\mu + |x|}{[4 + \sin\mu + |x| + |y|]^2} \right] d\mu + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left[3 + \frac{1}{[3 + \sin 2\mu]^2} \right] d\mu - \left[3x + \frac{x}{3 + \sin 2t} \right] \\
&\quad - \left[3y + \frac{y}{4 + \sin t + |x| + |y|} \right] - \sum_{i=1}^{n} \left[3z + \frac{z}{5 + \sin t + |x(t-\tau_i(t))\, y(t-\tau_i(t))| + |y(t-\tau_i(t))\, z(t-\tau_i(t))|} \right].
\end{aligned} \tag{57}$$

Comparing system (10) with system (57), we have the following relations:

(1) The function $\sum_{i=1}^{n} f_i(\cdot) = 3z + z/(5 + \sin t + |x(t - \tau_i(t))y(t - \tau_i(t))| + |y(t - \tau_i(t))z(t - \tau_i(t))|)$. It is clear from the above equation that

$$0 < 3 = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{f_i(\cdot)}{z}, \quad z \neq 0. \tag{58}$$

(2) The function $\sum_{i=1}^{n} g_i(y) = 3y + y/(4 + \sin t + |x| + |y|)$. It is not difficult to show that

$$3 = \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{n} \frac{g_i(y)}{y} \leq \sum_{i=1}^{n} B_i = 4, \quad y \neq 0. \tag{59}$$

(3) The function $\sum_{i=1}^{n} h_i(x) = 3x + x/(3 + \sin 2t)$, from where we obtain the following estimates:

(a) $3 = \sum_{i=1}^{n} \delta_i \leq \sum_{i=1}^{n} h_i(x)/x$, $x \neq 0$;
(b) $\sum_{i=1}^{n} h_i'(x) \leq \sum_{i=1}^{n} c_i$, $\sum_{i=1}^{n}(a_i b_i - c_i) = 5 > 0$.

(4) The calculation of the following constants also follows:

(a) $4/3 < \alpha < 3$; we choose $\alpha = 2$;
(b) $0 < \beta < \min\{4, 1/4, 1/4\} = 1/4$ or $0 < \beta < 1/4$, and we choose $\beta = 1/8$;
(c) $0 < \rho < 1$, and we choose $\rho = 1/2$;
(d) $\gamma < \min\{3/8, 1/21, 1/146\} = 1/146$.

All the assumptions of Theorem 10 are satisfied. Hence by Theorem 10 trivial solution of system (57) is uniformly asymptotically stable.

Example 2. Consider the nonhomogeneous third-order delay differential equation:

$$\begin{aligned}
&\dddot{x} + \sum_{i=1}^{n} \left[3\ddot{x} + \frac{\ddot{x}}{5 + \sin t + |x(t-\tau_i(t))\, \dot{x}(t-\tau_i(t))| + |\dot{x}(t-\tau_i(t))\, \ddot{x}(t-\tau_i(t))|} \right] \\
&\quad + \sum_{i=1}^{n} \left[3\dot{x}(t-\tau_i(t)) + \frac{\dot{x}(t-\tau_i(t))}{4 + \sin 2t + |x(t-\tau_i(t))| + |\dot{x}(t-\tau_i(t))|} \right] \\
&\quad + \sum_{i=1}^{n} \left[3x(t-\tau_i(t)) + \frac{x(t-\tau_i(t))}{3 + \sin 2t} \right] = \sum_{i=1}^{n} \left[\frac{1 + \cos 2t + M(t) N_i(*)}{(1 + \sin 2t + \cos 2t + \cos 2t \sin 2t) N_i(*)} \right],
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
M(t) &:= 2 + 2\sin 2t + \cos 2t + \cos 2t \sin 2t, \\
N_i(*) &:= 1 + x + \dot{x} + \ddot{x} + x(t-\tau_i(t)) + \dot{x}(t-\tau_i(t)) \\
&\quad + \ddot{x}(t-\tau_i(t)).
\end{aligned} \tag{61}$$

As usual, system (60) is equivalent to

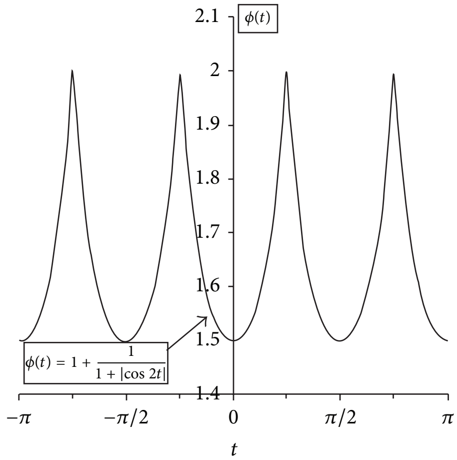

FIGURE 1: Periodic function $\phi(t) = \phi_i(t)$.

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left[3 + \frac{4 + \sin\mu + |x|}{\left[4 + \sin\mu + |x| + |y|\right]^2} \right] d\mu + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left[3 + \frac{1}{\left[3 + \sin 2\mu\right]^2} \right] d\mu - \left[3x + \frac{x}{3 + \sin 2t} \right] \\
&\quad - \left[3y + \frac{y}{4 + \sin t + |x| + |y|} \right] - \sum_{i=1}^{n} \left[3z + \frac{z}{5 + \sin t + |x(t-\tau_i(t))\, y(t-\tau_i(t))| + |y(t-\tau_i(t))\, z(t-\tau_i(t))|} \right] \\
&\quad + \sum_{i=1}^{n} \left[\frac{1 + \cos 2t + M(t)\, Q_i(*)}{(1 + \sin 2t + \cos 2t + \cos 2t \sin 2t)\, Q_i(*)} \right],
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
Q_i(*) :&= 1 + x + y + z + x(t - \tau_i(t)) + y(t - \tau_i(t)) \\
&\quad + z(t - \tau_i(t)).
\end{aligned} \tag{63}$$

Comparing systems (4) and (62) we observe that the function

$$\begin{aligned}
&\sum_{i=1}^{n} p_i(\cdot) \\
&\quad = \sum_{i=1}^{n} \left[\frac{1 + \cos 2t + M(t)\, Q_i(*)}{(1 + \sin 2t + \cos 2t + \cos 2t \sin 2t)\, Q_i(*)} \right].
\end{aligned} \tag{64}$$

Substituting appropriately for $Q_i(*)$ and $M(t)$, the above equation can be recast in the form

$$\sum_{i=1}^{n} p_i(\cdot) = \sum_{i=1}^{n} \left[1 + \frac{1}{1 + \cos 2t} + \frac{1}{(1 + \sin 2t)\left(1 + x + y + z + x(t-\tau_i(t)) + y(t-\tau_i(t)) + z(t-\tau_i(t))\right)} \right] \tag{65}$$

or

$$\begin{aligned}
\sum_{i=1}^{n} |p_i(\cdot)| \le \sum_{i=1}^{n} &\left[1 + \frac{1}{1 + |\cos 2t|} \right. \\
&\left. + \frac{1}{(1 + |\sin 2t|)(1 + |x| + |y| + |z|)} \right].
\end{aligned} \tag{66}$$

From inequalities (37) and (66) we obtain the following periodic functions (see Figures 1 and 2):

$$\begin{aligned}
\sum_{i=1}^{n} \phi_i(t) &= 1 + \frac{1}{1 + |\cos 2t|}, \\
\sum_{i=1}^{n} \varphi_i(t) &= \frac{1}{1 + |\sin 2t|}.
\end{aligned} \tag{67}$$

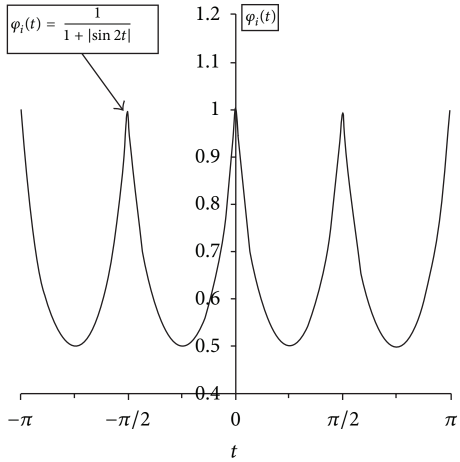

FIGURE 2: Periodic function $\varphi(t) = \varphi_i(t)$.

It is not difficult to show that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} \phi_i(t) \leq \sum_{i=1}^{n} d_i = 2, \\ 0 &\leq \sum_{i=1}^{n} \varphi_i(t) \leq \sum_{i=1}^{n} \epsilon_i = 1. \end{aligned} \tag{68}$$

The estimates in (68) and that of Example 1 satisfy the hypotheses of Theorem 13. Hence by Theorem 13,

(i) solutions of system (62) are uniformly bounded and uniformly ultimately bounded;

(ii) system (62) has a unique periodic solution of period $\omega = \pi/2$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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