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Research Article

Nonlocal Boundary Value Problems for q-Difference Equations and Inclusions

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We study boundary value problems for q-difference equations and inclusions with nonlocal and integral boundary conditions which have different quantum numbers. Some new existence and uniqueness results are obtained by using fixed point theorems. Examples are given to illustrate the results.

1. Introduction

In this paper we introduce a new class of boundary value problems for q-difference equations with nonlocal boundary conditions given by

$$\begin{gathered} D_q^2 x(t) = f(t, x(t)), \quad t \in I_q^T, \\ x(\xi) = g(x), \qquad \alpha D_r x(\eta) + \beta \int_\eta^T x(s)\, d_p s = 0, \\ 0 < \xi < \eta < T, \end{gathered} \tag{1}$$

where $f \in C(I_q^T \times \mathbb{R}, \mathbb{R})$ is such that $f(t,x)$ is continuous at $t = 0, T$, $I_q^T = I_q \cap [0,T]$, $I_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, $q \in (0,1)$ is a fixed constant, $\xi, \eta \in I_q^T \setminus \{0, T\} := (0,T)_q$, $0 < p, r < 1$, and α, β are given constants such that $\xi\beta(T-\eta) \neq \alpha + \beta(T^2 - \eta^2)/(1+p)$.

The study of q-difference equations, initiated by Jackson [1, 2], Carmichael [3], Mason [4], and Adams [5] in the first quarter of 20th century, has been developed over the years; for instance, see [6–8]. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in the papers [9–21].

Nonlocal conditions were initiated by Bitsadze [22]. As remarked by Byszewski [23, 24], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \cdots < t_p \leq T$.

In Section 3 we give some sufficient conditions for the existence and uniqueness of solutions and for the existence of at least one solution of problem (1). The first result is based on Banach's contraction principle and the second on a fixed point theorem due to O'Regan [25]. Concrete examples are also provided to illustrate the possible applications of the established analytical results.

In Section 4, we extend the results to cover the multivalued case, considering the following boundary value problem for q-difference inclusions with nonlocal and integral boundary conditions:

$$\begin{gathered} D_q^2 x(t) \in F(t, x(t)), \quad t \in I_q^T, \\ x(\xi) = g(x), \qquad \alpha D_r x(\eta) + \beta \int_\eta^T x(s)\, d_p s = 0, \end{gathered} \tag{2}$$

where $F : I_q^T \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map and $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We give an existence result for the problem (2) in the case when the right hand side is convex valued by using the nonlinear alternative for contractive maps.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, in Section 3 we prove our main results for single-valued case, and in Section 4 we prove our main results for multivalued case.

2. Preliminaries

Let us recall some basic concepts of q-calculus [7, 8, 26].

Let $0 < q < 1$ and f a function defined on a q-geometric set A; that is, $qt \in A$ for all $t \in A$. The q-difference operator is defined by

$$D_q f(t) = \begin{cases} \dfrac{f(t) - f(qt)}{(1-q)t}, & t \in A \setminus \{0\}, \\ \lim\limits_{n\to\infty} \dfrac{f(tq^n) - f(0)}{tq^n}, & t = 0, \end{cases} \tag{3}$$

provided that the limit exists and does not depend on t. The higher order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{4}$$

The Jackson q-integration [1] is

$$\begin{gathered} \int_0^a f(t)\, d_q t = a(1-q) \sum_{n=0}^{\infty} q^n f(aq^n), \\ \int_a^b f(t)\, d_q(t) = \int_0^b f(t)\, d_q(t) - \int_0^a f(t)\, d_q(t), \end{gathered} \tag{5}$$

where $a, b \in A$, provided that the series converge. Here we remark that the integral $\int_a^b f(t) d_q(t)$ is understood as a right inverse of the q-derivative.

For $0 \in A$, f is called q-regular at zero if $\lim_{n\to\infty} f(tq^n) = f(0)$ for every $t \in A$, $t \neq 0$. It is important to note that continuity at zero implies q-regularity at zero but the converse is not true (see an example on page 7 in [26]).

Definition 1. Let f be a function defined on a q-geometric set A. Then f is q-integrable on A if and only if $\int_0^t f(\mu) d_q \mu$ exists for all $t \in A$.

The q-integration by parts rule is

$$\begin{aligned} &\int_a^b u(qt)\, D_q v(t)\, d_q t \\ &\quad = u(b)\,v(b) - u(a)\,v(a) + \int_a^b D_q u(t)\, v(t)\, d_q t, \end{aligned} \tag{6}$$

provided that u and v are q-regular at zero functions.

Let f be a q-regular at zero function defined on a q-geometric set A containing zero. Then

$$F(z) = \int_c^z f(s)\, d_q s, \quad z \in A \tag{7}$$

is q-regular at zero, where c is a fixed point in A. Furthermore, $D_q F(z)$ exists for every $z \in A$ and

$$D_q F(z) = f(z), \quad z \in A. \tag{8}$$

Conversely, if a and b are two points in A, then

$$\int_a^b D_q f(s)\, d_q s = f(b) - f(a). \tag{9}$$

We denote by $\mathscr{C} = C(I_q^T, \mathbb{R})$ the Banach space of all continuous functions from $I_q^T \to \mathbb{R}$ which are q-regular at zero.

To define the solution for the problem (1), we find the solution for its associated linear problem.

Lemma 2. *Let $y \in C(I_q^T, \mathbb{R})$, be a continuous function such that it is continuous at 0 and T. The solution of the q-difference equation*

$$D_q^2 x(t) = y(t), \quad t \in I_q^T \tag{10}$$

subject to the boundary conditions,

$$x(\xi) = g(x), \qquad \alpha D_r x(\eta) + \beta \int_\eta^T x(s)\, d_p s = 0, \tag{11}$$

is given by

$$\begin{aligned} x(t) = {} & \int_0^t (t - qs)\, y(s)\, d_q s + \frac{t - \xi}{D} \\ & \cdot \left(\beta \int_\eta^T \int_0^s (s - q\tau)\, y(\tau)\, d_q \tau\, d_p s \right. \\ & \left. + \alpha \int_0^{r\eta} y(s)\, d_q s + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs)\, y(s)\, d_q s \right) \\ & + \frac{1}{D} \left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta)\, t \right) \\ & \cdot \int_0^\xi (\xi - qs)\, y(s)\, d_q s \\ & - \frac{1}{D} \left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta)\, t \right) g(x), \\ & \qquad t \in I_q^T, \end{aligned} \tag{12}$$

where

$$D = \xi \beta (T - \eta) - \alpha - \frac{\beta (T^2 - \eta^2)}{1+p} \neq 0. \tag{13}$$

Proof. Integrating twice the given equation and changing the order of integration, we get

$$x(t) = \int_0^t (t - qs)\, y(s)\, d_q s + c_1 t + c_2, \tag{14}$$

for some constants $c_1, c_2 \in \mathbb{R}$ (for functions not necessarily continuous at zero, the constants c_0, c_1 are q-periodic functions [26]).

In particular, for $t = \xi$, we get

$$x(\xi) = \int_0^{\xi} (\xi - qs) y(s) d_q s + c_1 \xi + c_2. \tag{15}$$

Taking r-derivative for (14), for $t \neq 0$, we obtain

$$\begin{aligned} D_r x(t) &= D_r \left[\int_0^t (t - qs) y(s) d_q s + c_1 t + c_2 \right] \\ &= \frac{1}{(1-r)t} \left[\int_0^t (t - qs) y(s) d_q s \right. \\ &\qquad \left. - \int_0^{rt} (rt - qs) y(s) d_q s \right] + c_1 \\ &= \int_0^{rt} y(s) d_q s + \int_{rt}^t \frac{t - qs}{(1-r)t} y(s) d_q s + c_1. \end{aligned} \tag{16}$$

Therefore,

$$D_r x(\eta) = \int_0^{r\eta} y(s) d_q s + \int_{r\eta}^{\eta} \frac{\eta - qs}{(1-r)\eta} y(s) d_q s + c_1. \tag{17}$$

Taking the p-integral for (14) from 0 to t, we obtain

$$\int_0^t x(s) d_p s = \int_0^t \int_0^s (s - qv) y(v) d_q v d_p s + \frac{t^2}{1+p} c_1 + t c_2. \tag{18}$$

Then we have

$$\begin{aligned} \int_{\eta}^T x(s) d_p s &= \int_0^T x(s) d_p s - \int_0^{\eta} x(s) d_p s \\ &= \int_0^T \int_0^s (s - qv) y(v) d_q v d_p s + \frac{T^2}{1+p} c_1 + T c_2 \\ &\quad - \int_0^{\eta} \int_0^s (s - qv) y(v) d_q v d_p s - \frac{\eta^2}{1+p} c_1 - \eta c_2. \end{aligned} \tag{19}$$

By applying the boundary conditions we get to the system

$$\begin{aligned} &\left(\alpha + \frac{\beta(T^2 - \eta^2)}{1+p} \right) c_1 + \beta(T - \eta) c_2 \\ &\quad = -\beta \int_{\eta}^T \int_0^s (s - q\tau) y(\tau) d_q \tau d_p s \\ &\qquad - \alpha \int_0^{r\eta} y(s) d_q s - \alpha \int_{r\eta}^{\eta} \frac{\eta - qs}{(1-r)\eta} y(s) d_q s \\ &c_1 \xi + c_2 = g(x) - \int_0^{\xi} (\xi - qs) y(s) d_q s, \end{aligned} \tag{20}$$

from which we have

$$\begin{aligned} c_1 &= \frac{1}{D} \left\{ \beta \int_{\eta}^T \int_0^s (s - q\tau) y(\tau) d_q \tau d_p s \right. \\ &\quad + \alpha \int_0^{r\eta} y(s) d_q s + \alpha \int_{r\eta}^{\eta} \frac{(\eta - qs)}{(1-r)\eta} y(s) d_p s \\ &\quad \left. + \beta(T - \eta) \left(g(x) - \int_0^{\xi} (\xi - qs) y(s) d_q s \right) \right\}, \\ c_2 &= -\frac{1}{D} \left\{ \xi \left[\beta \int_{\eta}^T \int_0^s (s - q\tau) y(\tau) d_q \tau d_p s + \alpha \int_0^{r\eta} y(s) d_q s \right. \right. \\ &\quad \left. + \alpha \int_{r\eta}^{\eta} \frac{(\eta - qs)}{(1-r)\eta} y(s) d_q s \right] \\ &\quad + \left(\alpha + \frac{\beta(T^2 - \eta^2)}{1+p} \right) \\ &\quad \left. \cdot \left(g(x) - \int_0^{\xi} (\xi - qs) y(s) d_q s \right) \right\}. \end{aligned} \tag{21}$$

Substituting into (14) the values of c_1 and c_2 we obtain (12). □

3. Existence Results: The Single-Valued Case

In view of Lemma 2, we define an operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \mathcal{Q}x(t) &= \int_0^t (t - qs) f(s, x(s)) d_q s \\ &\quad + \frac{t - \xi}{D} \left(\beta \int_{\eta}^T \int_0^s (s - q\tau) f(\tau, x(\tau)) d_q \tau d_p s \right. \\ &\qquad + \alpha \int_0^{r\eta} f(s, x(s)) d_q s \\ &\qquad \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f(s, x(s)) d_q s \right) \\ &\quad + \frac{1}{D} \left(\alpha + \frac{\beta(T^2 - \eta^2)}{1+p} - \beta(T - \eta) t \right) \\ &\quad \cdot \int_0^{\xi} (\xi - qs) f(s, x(s)) d_q s \\ &\quad - \frac{1}{D} \left(\alpha + \frac{\beta(T^2 - \eta^2)}{1+p} - \beta(T - \eta) t \right) g(x), \\ &\qquad t \in I_q^T. \end{aligned} \tag{22}$$

For convenience we set

$$\Lambda = \frac{T^2}{1+q} + \frac{\xi+T}{|D|} \cdot \left(\frac{|\beta|\left(T^3-\eta^3\right)}{(1+q)\left(1+p+p^2\right)} + |\alpha| r\eta + \frac{|\alpha|\eta(1-qr)}{(1+q)} \right) + k_0 \frac{\xi^2}{1+q}, \tag{23}$$

$$k_0 = \frac{1}{|D|}\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta| T (T-\eta) \right). \tag{24}$$

Theorem 3. *Let $f : I_q^T \times \mathbb{R} \to \mathbb{R}$ and $g : C(I_q^T, \mathbb{R}) \to \mathbb{R}$ be continuous functions. Assume that*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|$, $\forall t \in I_q^T$, $L > 0$, $x, y \in \mathbb{R}$;

(A_2) $|g(u) - g(v)| \le \ell \|u-v\|$, $\ell < k_0^{-1}$ *for all* $u, v \in C(I_q^T, \mathbb{R})$;

(A_3) $\gamma = L\Lambda + k_0\ell < 1$.

Then the boundary value problem (1) has a unique solution.

Proof. For $x, y \in \mathscr{C}$ and for each $t \in I_q^T$, from the definition of $\mathcal{Q}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
&|(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| \\
&\le \int_0^t (t-qs)\left|f(s,x(s)) - f(s,y(s))\right| d_q s + \frac{\xi+T}{|D|} \\
&\quad \cdot \left(|\beta| \int_\eta^T \int_0^s (s-q\tau)\left|f(\tau,x(\tau)) - f(\tau,y(\tau))\right| d_q\tau d_p s \right. \\
&\qquad + |\alpha| \int_0^{r\eta} \left|f(s,x(s)) - f(s,y(s))\right| d_q s \\
&\qquad \left. + \frac{|\alpha|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta-qs)\left|f(s,x(s)) - f(s,y(s))\right| d_q s \right) \\
&\quad + \frac{1}{|D|}\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta| T(T-\eta) \right) \\
&\quad \cdot \int_0^\xi (\xi-qs)\left|f(s,x(s)) - f(s,y(s))\right| d_q s \\
&\quad + \frac{1}{|D|}\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta| T(T-\eta) \right) \\
&\quad \cdot \left|g(x) - g(y)\right| \\
&\le L\|x-y\| \left\{ \frac{T^2}{1+q} + \frac{\xi+T}{|D|} \right. \\
&\qquad \cdot \left(\frac{|\beta|\left(T^3-\eta^3\right)}{(1+q)\left(1+p+p^2\right)} + |\alpha| r\eta \right. \\
&\qquad \left. + \frac{|\alpha|\eta(1-qr)}{(1+q)} \right) + \frac{1}{|D|} \\
&\qquad \cdot \left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta| T(T-\eta) \right) \\
&\qquad \left. \cdot \frac{\xi^2}{1+q} \right\} \\
&\quad + \frac{1}{|D|}\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta| T(T-\eta) \right) \ell \|x-y\|.
\end{aligned} \tag{25}$$

Hence

$$\|\mathcal{Q}x - \mathcal{Q}y\| \le \gamma \|x-y\|. \tag{26}$$

As $\gamma < 1$, by (A_3), F is a contraction map from the Banach space $\mathscr{C}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 4. Consider the following nonlocal boundary value problem of nonlinear q-difference equation

$$\begin{gathered}
D_{1/2}^2 x(t) = \frac{e^{-2t}}{2(t+5)^2} \cdot \frac{|x|}{1+|x|} + \frac{1}{2}, \quad t \in [0,2]_{1/2}, \\
x\left(\frac{1}{4}\right) = \frac{1}{30} x\left(\frac{1}{8}\right) + \frac{1}{3}, \\
\frac{2}{3} D_{3/4} x\left(\frac{1}{2}\right) - \frac{1}{3}\int_{1/2}^2 x(s)\, d_{1/4}s = 0.
\end{gathered} \tag{27}$$

Here, $q = 1/2$, $r = 3/4$, $p = 1/4$, $T = 2$, $\xi = 1/4$, $\eta = 1/2$, $\alpha = 2/3$, $\beta = -1/3$, $g(x) = (1/30)x + (1/3)$, and $f(t,x) = (e^{-2t}|x|)/(2(t+5)^2(1+|x|)) + (1/2)$. We find that $D = 5/24$, $k_0 = 192/15$, and $\Lambda = 981/45$.

As $|f(t,x) - f(t,y)| \le (1/50)|x-y|$ and $|g(x) - g(y)| \le (1/30)|x-y|$, therefore, (A_1) and (A_2) are satisfied with $L = 1/50$ and $\ell = 1/30 < 15/192 = k_0^{-1}$, respectively. Hence $\gamma = L\Lambda + k_0\ell \approx 0.86267 < 1$. By the conclusion of Theorem 3, the boundary value problem (27) has a unique solution on $[0,2]_{1/2}$.

Next, we introduce the fixed point theorem which was established by O'Regan in [25]. This theorem will be adopted to prove the next main result.

Lemma 5. *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $\mathcal{F}(\overline{U})$ is bounded and that $\mathcal{F} : \overline{U} \to C$ is given by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, in* *which $\mathcal{F}_1 : \overline{U} \to E$ is continuous and completely continuous and $\mathcal{F}_2 : \overline{U} \to E$ is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\vartheta : [0,\infty) \to [0,\infty)$ satisfying $\vartheta(z) < z$ for $z > 0$, such that $\|\mathcal{F}_2(x) - \mathcal{F}_2(y)\| \leq \vartheta(\|x-y\|)$ for all $x, y \in \overline{U}$). Then, either*

(C1) *$\mathcal{F}$ has a fixed point $u \in \overline{U}$ or*

(C2) *there exist a point $u \in \partial U$ and $\kappa \in (0,1)$ with $u = \kappa\mathcal{F}(u)$, where $\overline{U}$ and ∂U, respectively, represent the closure and boundary of U on C.*

In the sequel, we will use Lemma 5 by taking C to be E. For more details of such fixed point theorems, we refer a paper [27] by Petryshyn.

Let

$$\Omega_r = \left\{x \in C\left(I_q^T, \mathbb{R}\right) : \|x\| < r\right\}. \tag{28}$$

Theorem 6. *Let $f : I_q^T \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_2) holds. In addition we assume that*

(A_4) *$g(0) = 0$;*

(A_5) *there exists a nonnegative function $p \in C(I_q^T, \mathbb{R})$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that*

$$\left|f(t,u)\right| \leq p(t)\,\psi(|u|) \quad \text{for any } (t,u) \in I_q^T \times \mathbb{R}; \tag{29}$$

(A_6) *$\sup_{r\in(0,\infty)}(r/\Lambda \| p \| \psi(r)) > 1/(1-k_0\ell)$, where Λ and k_0 are defined in (23) and (24), respectively.*

Then the boundary value problem (1) has at least one solution on I_q^T.

Proof. Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ as that defined in (30). We decompose $\mathcal{Q}$ into a sum of two operators

$$(\mathcal{Q}x)(t) = (\mathcal{Q}_1x)(t) + (\mathcal{Q}_2x)(t), \quad t \in I_q^T, \tag{30}$$

where

$$\begin{aligned}
&(\mathcal{Q}_1x)(t) \\
&\quad= \int_0^t (t-qs)\, f(s,x(s))\, d_qs \\
&\qquad+ \frac{t-\xi}{D}\left(\beta\int_\eta^T\int_0^s (s-q\tau)\, f(\tau,x(\tau))\, d_q\tau d_ps\right. \\
&\qquad\qquad+ \alpha\int_0^{r\eta} f(s,x(s))\, d_qs \\
&\qquad\qquad\left.+ \frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta} (\eta-qs)\, f(s,x(s))\, d_qs\right) \\
&\qquad+ \frac{1}{D}\left(\alpha + \frac{\beta\left(T^2-\eta^2\right)}{1+p} - \beta(T-\eta)t\right) \\
&\qquad\cdot\int_0^\xi (\xi-qs)\, f(s,x(s))\, d_qs, \quad t \in I_q^T, \\
&(\mathcal{Q}_2x)(t) = -\frac{1}{D}\left(\alpha + \frac{\beta\left(T^2-\eta^2\right)}{1+p} - \beta(T-\eta)t\right) g(x), \\
&\qquad t \in I_q^T.
\end{aligned} \tag{31}$$

From (A_6) there exists a number $r_0 > 0$ such that

$$\frac{r_0}{\Lambda\|p\|\psi(r_0)} > \frac{1}{1-k_0\ell}. \tag{32}$$

We will prove that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions in Lemma 5.

Step 1. The operator $\mathcal{Q}_1$ is continuous and completely continuous. We first show that $\mathcal{Q}_1(\overline{\Omega}_{r_0})$ is bounded. For any $x \in \overline{\Omega}_{r_0}$ we have

$$\begin{aligned}
&\|\mathcal{Q}_1x\| \\
&\leq \int_0^t (t-qs)\left|f(s,x(s))\right| d_qs \\
&\quad+ \frac{|t-\xi|}{|D|}\left(|\beta|\int_\eta^T\int_0^s (s-q\tau)\left|f(\tau,x(\tau))\right| d_q\tau d_ps\right. \\
&\qquad+ |\alpha|\int_0^{r\eta}\left|f(s,x(s))\right| d_qs \\
&\qquad\left.+ \frac{|\alpha|}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)\left|f(s,x(s))\right| d_qs\right) \\
&\quad+ \frac{1}{D}\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta|\, T(T-\eta)\right) \\
&\quad\cdot\int_0^\xi (\xi-qs)\left|f(s,x(s))\right| d_qs \\
&\leq \|p\|\,\psi(r_0)\left\{\frac{T^2}{1+q} + \frac{\xi+T}{|D|}\right. \\
&\qquad\cdot\left(\frac{|\beta|\left(T^3-\eta^3\right)}{(1+q)\left(1+p+p^2\right)} + |\alpha|\, r\eta\right. \\
&\qquad\left.+ \frac{|\alpha|\,\eta(1-qr)}{(1+q)}\right) + \frac{1}{|D|} \\
&\qquad\cdot\left(|\alpha| + \frac{|\beta|\left(T^2-\eta^2\right)}{1+p} + |\beta|\, T(T-\eta)\right) \\
&\qquad\left.\cdot\frac{\xi^2}{1+q}\right\}.
\end{aligned} \tag{33}$$

This proves that $\mathcal{Q}_1(\overline{\Omega}_{r_0})$ is uniformly bounded.

In addition for any $t_1, t_2 \in I_q^T$, $t_1 < t_2$, we have

$$
\begin{aligned}
&\left|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)\right| \\
&\leq \left|\int_0^{t_2} (t_2 - qs) f(s, x(s)) d_q s - \int_0^{t_1} (t_1 - qs) f(s, x(s)) d_q s\right| \\
&\quad + \frac{|t_2 - t_1|}{|D|} \left(|\beta| \int_\eta^T \int_0^s (s - q\tau) |f(\tau, x(\tau))| d_q \tau d_p s\right. \\
&\qquad + |\alpha| \int_0^{r\eta} |f(s, x(s))| d_q s \\
&\qquad \left. + \frac{|\alpha|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) |f(s, x(s))| d_q s\right) \\
&\quad + \frac{|\beta| (T - \eta) |t_2 - t_1|}{|D|} \int_0^{\xi} (\xi - s) |f(s, x(s))| d_q s \\
&\leq \|p\| \psi(r_0) \left\{ \left|\int_0^{t_1} (t_2 - t_1) d_q s\right| + \left|\int_{t_1}^{t_2} (t_2 - qs) d_q s\right| \right. \\
&\quad + \frac{|t_2 - t_1|}{|D|} \left[\frac{|\beta| (T^3 - \eta^3)}{(1+q)(1+p+p^2)} \right. \\
&\qquad \left. + |\alpha| r\eta + \frac{|\alpha| \eta (1 - qr)}{(1+q)} \right] \\
&\quad \left. + \frac{|\beta| (T - \eta) |t_2 - t_1|}{|D|} \frac{\xi^2}{1+q} \right\},
\end{aligned} \tag{34}
$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{Q}_1(\overline{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \overline{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ uniformly valid on I_q^T. From the uniform continuity of $f(t, x)$ on the compact set $I_q^T \times [-r_0, r_0]$ it follows that $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ is uniformly valid on I_q^T. Hence $\|\mathcal{Q}_1 x_n - \mathcal{Q}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{Q}_1$. Hence Step 1 is completely proved.

Step 2. The operator $\mathcal{Q}_2 : \overline{\Omega}_{r_0} \to C(I_q^T, \mathbb{R})$ is contractive. This is a consequence of (A_2). Indeed, we have

$$
\begin{aligned}
&\left|(\mathcal{Q}_2 x)(t) - (\mathcal{Q}_2 y)(t)\right| \\
&\quad = \left|\frac{1}{D}\left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta) t\right)\right| |g(x) - g(y)| \\
&\quad \leq k_0 \ell \|x - y\|,
\end{aligned} \tag{35}
$$

or

$$
\|\mathcal{Q}_2 x - \mathcal{Q}_2 y\| \leq L_0 \|x - y\|, \quad L_0 = k_0 \ell < 1. \tag{36}
$$

Step 3. The set $F(\overline{\Omega}_{r_0})$ is bounded. By (A_2) and (A_4) we imply that

$$
\|\mathcal{Q}_2(x)\| \leq k_0 \ell r_0, \tag{37}
$$

for any $x \in \overline{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{Q}_1(\overline{\Omega}_{r_0})$, implies that the set $\mathcal{Q}(\overline{\Omega}_{r_0})$ is bounded.

Step 4. Finally, it is to show that the case $(C2)$ in Lemma 5 does not occur. To this end, we suppose that $(C2)$ holds. Then, we have that there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{Q} x$. So, we have $\|x\| = r_0$ and

$$
\begin{aligned}
&x(t) \\
&\quad = \lambda \left\{ \int_0^t (t - qs) f(s, x(s)) d_q s + \frac{t - \xi}{D} \right. \\
&\qquad \cdot \left(\beta \int_\eta^T \int_0^s (s - q\tau) f(\tau, x(\tau)) d_q \tau d_p s \right. \\
&\qquad + \alpha \int_0^{r\eta} f(s, x(s)) d_q s \\
&\qquad \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f(s, x(s)) d_q s\right) \\
&\qquad + \frac{1}{D}\left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta) t\right) \\
&\qquad \cdot \int_0^{\xi} (\xi - qs) f(s, x(s)) d_q s \\
&\qquad \left. - \frac{1}{D}\left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta) t\right) g(x) \right\}, \\
&\qquad t \in I_q^T.
\end{aligned} \tag{38}
$$

With hypotheses (A_4)–(A_6), and follows the computations of Step 1, we have

$$
\begin{aligned}
&|x(t)| \\
&\leq \int_0^t (t - qs) |f(s, x(s))| d_q s \\
&\quad + \frac{|t - \xi|}{|D|} \left(|\beta| \int_\eta^T \int_0^s (s - q\tau) |f(\tau, x(\tau))| d_q \tau d_p s\right. \\
&\qquad + |\alpha| \int_0^{r\eta} |f(s, x(s))| d_q s \\
&\qquad \left. + \frac{|\alpha|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) |f(s, x(s))| d_q s\right)
\end{aligned}
$$

$$+\frac{1}{|D|}\left(|\alpha|+\frac{|\beta|\left(T^2-\eta^2\right)}{1+p}+|\beta|T(T-\eta)\right)$$

$$\cdot\int_0^{\xi}(\xi-qs)|f(s,x(s))|d_qs$$

$$+\frac{1}{|D|}\left(|\alpha|+\frac{|\beta|\left(T^2-\eta^2\right)}{1+p}+|\beta|T(T-\eta)\right)|g(x)|$$

$$\leq\|p\|\psi(r_0)\left\{\frac{T^2}{1+q}+\frac{\xi+T}{|D|}\left(\frac{|\beta|\left(T^3-\eta^3\right)}{(1+q)(1+p+p^2)}+|\alpha|r\eta+\frac{|\alpha|\eta(1-qr)}{(1+q)}\right)+\frac{1}{|D|}\left(|\alpha|+\frac{|\beta|\left(T^2-\eta^2\right)}{1+p}+|\beta|T(T-\eta)\right)\cdot\frac{\xi^2}{1+q}\right\}$$

$$+\frac{1}{|D|}\left(|\alpha|+\frac{|\beta|\left(T^2-\eta^2\right)}{1+p}+|\beta|T(T-\eta)\right)|g(x)|, \tag{39}$$

which implies

$$r_0\leq\Lambda\|p\|\psi(r_0)+k_0\ell r_0. \tag{40}$$

Thus,

$$\frac{r_0}{\Lambda\|p\|\psi(r_0)}\leq\frac{1}{1-k_0\ell}, \tag{41}$$

which contradicts to (32). Consequently, we have proved that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions in Lemma 5. Hence, the operator $\mathcal{Q}$ has at least one fixed point $x\in\overline{\Omega}_{r_0}$, which is the solution of the boundary value problem (1). The proof is completed. □

Example 7. Consider the following nonlocal boundary value problem of nonlinear q-difference equation

$$D^2_{1/2}x(t)=\frac{t}{25}\left(|x|+\frac{|x|+1}{|x|+2}\right),\quad t\in[0,1]_{1/2},$$
$$x\left(\frac{1}{8}\right)=\frac{1}{33}\sin\left(x\left(\frac{1}{4}\right)\right), \tag{42}$$
$$-\frac{3}{4}D_{2/3}x\left(\frac{1}{2}\right)+\frac{4}{5}\int_{1/2}^{1}x(s)d_{1/5}s=0.$$

Here, $q=1/2$, $r=2/3$, $p=1/5$, $T=1$, $\xi=1/8$, $\eta=1/2$, $\alpha=-3/4$, $\beta=4/5$, $g(x)=(1/33)\sin x$, and $f(t,x)=(t/25)(|x|+((|x|+1)/(|x|+2)))$. We find that $D=3/10$, $k_0=11/2$, and $\Lambda=22009/5952$.

As $|g(x)-g(y)|\leq(1/33)|x-y|$ with $\ell=(1/33)<(2/11)=k_0^{-1}$ and $g(0)=0$; therefore, (A_2) and (A_4) are satisfied, respectively. Since $|f(t,x)|=|(t/25)(|x|+((|x|+1)/(|x|+2)))|\leq(t/25)(x^2+3|x|+1)$, we choose $p(t)=t/25$ and $\psi(|x|)=x^2+3|x|+1$. We can show that

$$\sup_{r\in(0,\infty)}\frac{r}{\Lambda\|p\|\psi(r)}\approx1.35240>\frac{6}{5}=\frac{1}{1-k_0\ell}. \tag{43}$$

Therefore, by Theorem 6, the boundary value problem (42) has at least one solution on $[0,1]_{1/2}$.

4. Existence Results: The Multivalued Case

Let us recall some basic definitions on multivalued maps [28, 29].

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{cl}(X)=\{Y\in\mathcal{P}(X):Y\text{ is closed}\}$, $\mathcal{P}_b(X)=\{Y\in\mathcal{P}(X):Y\text{ is bounded}\}$, $P_{cp}(X)=\{Y\in\mathcal{P}(X):Y\text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X)=\{Y\in\mathcal{P}(X):Y\text{ is compact and convex}\}$. A multivalued map $G:X\to\mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. The map G is bounded on bounded sets if $G(\mathbb{B})=\cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B}\in\mathcal{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y|:y\in G(x)\}\}<\infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0)\subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph; that is, $x_n\to x_*$, $y_n\to y_*$, $y_n\in G(x_n)$ imply $y_*\in G(x_*)$. G has a fixed point if there is $x\in X$ such that $x\in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G:[0;1]\to\mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if, for every $y\in\mathbb{R}$, the function,

$$t\longmapsto d(y,G(t))=\inf\{|y-z|:z\in G(t)\}, \tag{44}$$

is measurable.

Let $L^1(I_q^T,\mathbb{R})$ denote the space of all functions f defined on I_q^T such that $\|x\|_{L^1}=\int_0^T|x(t)|dt<\infty$.

Definition 8. A function $x\in\mathcal{C}$ is a solution of the problem (2) if $x(\xi)=g(x)$, $\alpha D_rx(\eta)+\beta\int_\eta^T x(s)d_qs=0$, and there exists a function $f\in L^1(I_q^T,\mathbb{R})$ such that it is continuous at $t=0,T$ and $f(t)\in F(t,x(t))$ on I_q^T and

$$x(t)=\int_0^t(t-qs)f(s)d_qs+\frac{t-\xi}{D}\left(\beta\int_\eta^T\int_0^s(s-q\tau)f(\tau)d_q\tau d_ps+\alpha\int_0^{r\eta}f(s)d_qs+\frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)f(s)d_qs\right)$$

$$+\frac{1}{D}\left(\alpha+\frac{\beta\left(T^2-\eta^2\right)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)f(s)\,d_qs$$
$$-\frac{1}{D}\left(\alpha+\frac{\beta\left(T^2-\eta^2\right)}{1+p}-\beta(T-\eta)t\right)g(x),\quad t\in I_q^T. \tag{45}$$

Definition 9. A multivalued map $F : I_q^T\times\mathbb{R}\rightarrow\mathscr{P}(\mathbb{R})$ is said to be Carathéodory (in the sense of q-calculus) if $x\mapsto F(t,x)$ is upper semicontinuous on I_q^T. Further a Carathéodory function F is called L^1—Carathéodory if there exists $\varphi_\alpha\in L^1(I_q^T,\mathbb{R}^+)$ such that $\|F(t,x)\|=\sup\{|v| : v\in F(t,x)\}\le\varphi_\alpha(t)$ for all $\|x\|\le\alpha$ on I_q^T for each $\alpha>0$.

For each $y\in\mathscr{C}$, define the set of selections of F by

$$S_{F,y}:=\left\{v\in\mathscr{C} : v(t)\in F\left(t,y(t)\right)\ \text{ on }\ I_q^T\right\}. \tag{46}$$

The following lemma will be used in the sequel.

Lemma 10 (see [30]). *Let X be a Banach space. Let $F : J\times\mathbb{R}\rightarrow\mathscr{P}_{\mathrm{cp},c}(X)$ be an L^1—Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\begin{gathered}\Theta\circ S_F : C(J,X)\longrightarrow\mathscr{P}_{\mathrm{cp},c}(C(J,X)),\\ x\longmapsto\left(\Theta\circ S_F\right)(x)=\Theta\left(S_{F,x}\right),\end{gathered} \tag{47}$$

is a closed graph operator in $C(J,X)\times C(J,X)$.

To prove our main result in this section we will use the following form of the nonlinear alternative for contractive maps [31, Corollary 3.8].

Theorem 11. *Let X be a Banach space and D a bounded neighborhood of $0\in X$. Let $Z_1 : X\rightarrow\mathscr{P}_{\mathrm{cp},c}(X)$ and $Z_2 : \overline{D}\rightarrow\mathscr{P}_{\mathrm{cp},c}(X)$ two multivalued operators satisfying the following:*

(a) *Z_1 is contraction,*

(b) *Z_2 is u.s.c and compact.*

Then, if $G=Z_1+Z_2$, either

(i) *G has a fixed point in $\overline{D}$ or*

(ii) *there is a point $u\in\partial D$ and $\lambda\in(0,1)$ with $u\in\lambda G(u)$.*

Theorem 12. *Assume that (A_2) holds. In addition we suppose that*

(H_1) *$F : I_q^T\times\mathbb{R}\rightarrow\mathscr{P}_{\mathrm{cp},c}(\mathbb{R})$ is such that $x\rightarrow F(t,x)$ is u.s.c. on I_q^T;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty)\rightarrow(0,\infty)$ and a function $p\in L^1(I_q^T,\mathbb{R}^+)$ such that*

$$\begin{gathered}\|F(t,x)\|_{\mathscr{P}}:=\sup\{|y| : y\in F(t,x)\}\le p(t)\psi(\|x\|)\\ \text{for each }(t,x)\in I_q^T\times\mathbb{R};\end{gathered} \tag{48}$$

(H_3) *there exists a number $M>0$ such that*

$$\frac{\left(1-k_0\ell\right)M}{\Lambda\|p\|\psi(M)}>1, \tag{49}$$

where Λ and k_0 are defined in (23) and (24), respectively.

Then the boundary value problem (2) has at least one solution on I_q.

Proof. Transform the problem (2) into a fixed point problem. Consider the operator $\mathscr{N} : C(I_q^T,\mathbb{R})\rightarrow\mathscr{P}(C(I_q^T,\mathbb{R}))$ defined by

$$\mathscr{N}(x)$$
$$=\Bigg\{h\in C\left(I_q^T,\mathbb{R}\right) :$$
$$h(t)=\int_0^t(t-qs)f(s)\,d_qs+\frac{t-\xi}{D}$$
$$\cdot\left(\beta\int_\eta^T\int_0^s(s-q\tau)f(\tau)\,d_q\tau\, d_ps\right.$$
$$+\alpha\int_0^{r\eta}f(s)\,d_qs$$
$$\left.+\frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)f(s)\,d_qs\right)$$
$$+\frac{1}{D}\left(\alpha+\frac{\beta\left(T^2-\eta^2\right)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)f(s)\,d_qs$$
$$\left.-\frac{1}{D}\left(\alpha+\frac{\beta\left(T^2-\eta^2\right)}{1+p}-\beta(T-\eta)t\right)g(x)\right\}, \tag{50}$$

for $f\in S_{F,x}$.

Now, we define two operators as follows. $\mathscr{A} : C(I_q^T,\mathbb{R})\rightarrow C(I_q^T,\mathbb{R})$ by

$$\mathscr{A}x(t)=-\frac{1}{D}\left(\alpha+\frac{\beta\left(T^2-\eta^2\right)}{1+p}-\beta(T-\eta)t\right)g(x), \tag{51}$$

and the multivalued operator $\mathscr{B} : C(I_q^T, \mathbb{R}) \to \mathscr{P}(C(I_q^T, \mathbb{R}))$ by

$$\begin{aligned}
&\mathscr{B}(x) \\
&= \Bigg\{ h \in C(I_q^T, \mathbb{R}) : \\
&\quad h(t) = \int_0^t (t - qs) f(s) \, d_q s + \frac{t-\xi}{D} \\
&\quad \cdot \left(\beta \int_\eta^T \int_0^s (s - q\tau) f(\tau) \, d_q \tau d_p s \right. \\
&\quad + \alpha \int_0^{r\eta} f(s) \, d_q s \\
&\quad \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f(s) \, d_q s \right) \\
&\quad + \frac{1}{D} \left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta) t \right) \\
&\quad \cdot \int_0^\xi (\xi - qs) f(s) \, d_q s \Bigg\}.
\end{aligned} \tag{52}$$

Then $\mathscr{N} = \mathscr{A} + \mathscr{B}$. We will show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 11 on I_q^T. For better readability, we break the proof into a sequence of steps and claims.

Step 1. We show that $\mathscr{A}$ *is a contraction on* $C(I_q^T, \mathbb{R})$. The proof is similar to the one for the operator $\mathscr{Q}_2$ in Step 2 of Theorem 6.

Step 2. We will show that the operator $\mathscr{B}$ *is compact and convex valued and it is completely continuous*. This will be given in several claims.

Claim I. $\mathscr{B}$ *maps bounded sets into bounded sets in* $C(I_q^T, \mathbb{R})$. To see this, let $B_r = \{x \in C(I_q^T, \mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C(I_q^T, \mathbb{R})$. Then, for each $h \in \mathscr{B}(x)$, $x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
h(t) &= \int_0^t (t - qs) f(s) \, d_q s + \frac{t-\xi}{D} \\
&\quad \cdot \left(\beta \int_\eta^T \int_0^s (s - q\tau) f(\tau) \, d_q \tau d_p s + \alpha \int_0^{r\eta} f(s) \, d_q s \right. \\
&\quad \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f(s) \, d_q s \right) \\
&\quad + \frac{1}{D} \left(\alpha + \frac{\beta (T^2 - \eta^2)}{1+p} - \beta (T - \eta) t \right) \\
&\quad \cdot \int_0^\xi (\xi - qs) f(s) \, d_q s.
\end{aligned} \tag{53}$$

Then for $t \in I_q^T$ we have

$$\begin{aligned}
&|h(t)| \\
&\le \int_0^t (t - qs) |f(s, x(s))| \, d_q s \\
&\quad + \frac{|t-\xi|}{|D|} \left(|\beta| \int_\eta^T \int_0^s (s - q\tau) |f(\tau)| \, d_q \tau d_p s \right. \\
&\quad + |\alpha| \int_0^{r\eta} |f(s)| \, d_q s \\
&\quad \left. + \frac{|\alpha|}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) |f(s)| \, d_q s \right) \\
&\quad + \frac{1}{|D|} \left(|\alpha| + \frac{|\beta| (T^2 - \eta^2)}{1+p} + |\beta| T (T - \eta) \right) \\
&\quad \cdot \int_0^\xi (\xi - qs) |f(s)| \, d_q s \\
&\le \|p\| \psi(\|x\|) \\
&\quad \cdot \left\{ \frac{T^2}{1+q} + \frac{\xi + T}{|D|} \left(\frac{|\beta| (T^3 - \eta^3)}{(1+q)(1+p+p^2)} + |\alpha| r\eta \right. \right. \\
&\quad \left. + \frac{|\alpha| \eta (1 - qr)}{(1+q)} \right) + \frac{1}{|D|} \\
&\quad \left. \cdot \left(|\alpha| + \frac{|\beta| (T^2 - \eta^2)}{1+p} + |\beta| T (T - \eta) \right) \frac{\xi^2}{1+q} \right\}.
\end{aligned} \tag{54}$$

Thus,

$$\begin{aligned}
\|h\| &\le \psi(\rho) \|p\| \left\{ \frac{T^2}{1+q} + \frac{\xi + T}{|D|} \right. \\
&\quad \cdot \left(\frac{|\beta| (T^3 - \eta^3)}{(1+q)(1+p+p^2)} + |\alpha| r\eta \right. \\
&\quad \left. + \frac{|\alpha| \eta (1 - qr)}{(1+q)} \right) + \frac{1}{|D|} \\
&\quad \cdot \left(|\alpha| + \frac{|\beta| (T^2 - \eta^2)}{1+p} + |\beta| T (T - \eta) \right) \\
&\quad \left. \cdot \frac{\xi^2}{1+q} \right\}.
\end{aligned} \tag{55}$$

Claim II. Next we show that $\mathscr{B}$ *maps bounded sets into equicontinuous sets*. Let $t_1, t_2 \in I_q$ with $t_1 < t_2$ and $x \in B_\rho$. For each $h \in \mathscr{B}(x)$, we obtain

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\le \left| \int_0^{t_2} (t_2 - qs) f(s) \, d_q s - \int_0^{t_1} (t_1 - qs) |f(s)| \, d_q s \right|
\end{aligned}$$

$$+\frac{|t_2-t_1|}{|D|}\left(|\beta|\int_0^T\int_0^s(s-q\tau)|f(\tau)|d_q\tau d_ps\right.$$
$$+|\alpha|\int_0^{r\eta}|f(s)|d_qs$$
$$\left.+\frac{|\alpha|}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)|f(s)|d_qs\right)$$
$$+\frac{|\beta|(T-\eta)|t_2-t_1|}{|D|}\int_0^{\xi}(\xi-s)|f(s)|d_qs$$
$$\leq\|p\|\psi(\rho)\left\{\left|\int_0^{t_1}(t_2-t_1)d_qs\right|+\left|\int_{t_1}^{t_2}(t_2-qs)d_qs\right|\right.$$
$$+\frac{|t_2-t_1|}{|D|}\left[\frac{|\beta|(T^3-\eta^3)}{(1+q)(1+p+p^2)}\right.$$
$$\left.+|\alpha|r\eta+\frac{|\alpha|\eta(1-qr)}{(1+q)}\right]$$
$$\left.+\frac{|\beta|(T-\eta)|t_2-t_1|}{|D|}\frac{\xi^2}{1+q}\right\}. \tag{56}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. As $\mathscr{B}$ satisfies the above three assumptions; therefore, it follows by the Ascoli-Arzelá theorem that $\mathscr{B}:C(I_q^T,\mathbb{R})\to\mathscr{P}(C(I_q^T,\mathbb{R}))$ is completely continuous.

Claim III. Next we prove that $\mathscr{B}$ *has a closed graph.* Let $x_n\to x_*, h_n\in\mathscr{B}(x_n)$, and $h_n\to h_*$. Then we need to show that $h_*\in\mathscr{B}(x_*)$. Associated with $h_n\in\mathscr{B}(x_n)$, there exists $f_n\in S_{F,x_n}$ such that, for each $t\in I_q^T$,

$$h_n(t)$$
$$=\int_0^t(t-qs)f_n(s)d_qs$$
$$+\frac{t-\xi}{D}\left(\beta\int_\eta^T\int_0^s(s-q\tau)f_n(\tau)d_q\tau d_ps+\alpha\int_0^{r\eta}f_n(s)d_qs\right.$$
$$\left.+\frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)f_n(s)d_qs\right)$$
$$+\frac{1}{D}\left(\alpha+\frac{\beta(T^2-\eta^2)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)f_n(s)d_qs. \tag{57}$$

Thus it suffices to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in I_q^T$,

$$h_*(t)=\int_0^t(t-qs)f_*(s)d_qs+\frac{t-\xi}{D}$$
$$\cdot\left(\beta\int_\eta^T\int_0^s(s-q\tau)f_*(\tau)d_q\tau d_ps+\alpha\int_0^{r\eta}f_*(s)d_qs\right.$$
$$\left.+\frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)f_*(s)d_qs\right)$$
$$+\frac{1}{D}\left(\alpha+\frac{\beta(T^2-\eta^2)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)f_*(s)d_qs. \tag{58}$$

Let us consider the linear operator $\Theta:L^1(I_q^T,\mathbb{R})\to C(I_q^T,\mathbb{R})$ given by

$$f\longmapsto\Theta(f)(t)$$
$$=\int_0^t(t-qs)f(s)d_qs+\frac{t-\xi}{D}$$
$$\cdot\left(\beta\int_\eta^T\int_0^s(s-q\tau)f(\tau)d_q\tau d_ps+\alpha\int_0^{r\eta}f(s)d_qs\right.$$
$$\left.+\frac{\alpha}{(1-r)\eta}\int_{r\eta}^{\eta}(\eta-qs)f(s)d_qs\right)$$
$$+\frac{1}{D}\left(\alpha+\frac{\beta(T^2-\eta^2)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)f(s)d_qs. \tag{59}$$

Observe that

$$\|h_n(t)-h_*(t)\|$$
$$=\int_0^t(t-qs)(f_n(s)-f_*(s))d_qs$$
$$+\frac{t-\xi}{D}\left(\beta\int_\eta^T\int_0^s(s-q\tau)(f_n(\tau)-f_*(\tau))d_q\tau d_ps\right.$$
$$+\alpha\int_0^{r\eta}(f_n(s)-f_*(s))d_qs+\frac{\alpha}{(1-r)\eta}$$
$$\left.\cdot\int_{r\eta}^{\eta}(\eta-qs)(f_n(s)-f_*(s))d_qs\right)$$
$$+\frac{1}{D}\left(\alpha+\frac{\beta(T^2-\eta^2)}{1+p}-\beta(T-\eta)t\right)$$
$$\cdot\int_0^{\xi}(\xi-qs)(f_n(s)-f_*(s))d_qs. \tag{60}$$

Thus, it follows by Lemma 10 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$
\begin{aligned}
h_*(t) = {} & \int_0^t (t-qs) f_*(s)\, d_q s + \frac{t-\xi}{D} \\
& \cdot \left(\beta \int_\eta^T \int_0^s (s-q\tau) f(\tau)\, d_q\tau d_p s + \alpha \int_0^{r\eta} f_*(s)\, d_q s \right. \\
& \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f_*(s)\, d_q s \right) \\
& + \frac{1}{D} \left(\alpha + \frac{\beta(T^2-\eta^2)}{1+p} - \beta(T-\eta)t \right) \\
& \cdot \int_0^\xi (\xi - qs) f_*(s)\, d_q s,
\end{aligned}
\tag{61}
$$

for some $f_* \in S_{F,x_*}$. Hence $\mathscr{B}$ has a closed graph (and, therefore, has closed values). As a result $\mathscr{B}$ is compact valued.

Therefore, the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 11 and hence an application of it yields that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathscr{A}(x) + \lambda\mathscr{B}(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$
\begin{aligned}
x(t) = {} & \lambda \int_0^t (t-qs) f(s)\, d_q s + \lambda \frac{t-\xi}{D} \\
& \cdot \left(\beta \int_\eta^T \int_0^s (s-q\tau) f(\tau)\, d_q\tau d_p s + \alpha \int_0^{r\eta} f(s)\, d_q s \right. \\
& \left. + \frac{\alpha}{(1-r)\eta} \int_{r\eta}^{\eta} (\eta - qs) f(s)\, d_q s \right) \\
& + \lambda \frac{1}{D} \left(\alpha + \frac{\beta(T^2-\eta^2)}{1+p} - \beta(T-\eta)t \right) \\
& \cdot \int_0^\xi (\xi - qs) f(s)\, d_q s \\
& - \lambda \frac{1}{D} \left(\alpha + \frac{\beta(T^2-\eta^2)}{1+p} - \beta(T-\eta)t \right) g(x), \\
& \qquad t \in I_q^T.
\end{aligned}
\tag{62}
$$

Consequently, we have

$$
\begin{aligned}
\|x\| \le {} & \psi(\|x\|) \|p\| \\
& \cdot \left\{ \frac{T^2}{1+q} + \frac{\xi+T}{|D|} \left(\frac{|\beta|(T^3-\eta^3)}{(1+q)(1+p+p^2)} \right.\right. \\
& \left. + |\alpha| r\eta + \frac{|\alpha|\eta(1-qr)}{(1+q)} \right) \\
& + \frac{1}{|D|} \left(|\alpha| + \frac{|\beta|(T^2-\eta^2)}{1+p} + |\beta| T(T-\eta) \right) \\
& \left. \cdot \frac{\xi^2}{1+q} \right\} + k_0 \ell \|x\|
\end{aligned}
\tag{63}
$$

or

$$\|x\| \le \Lambda \|p\| \psi(\|x\|) + k_0 \ell \|x\|. \tag{64}$$

If condition (ii) of Theorem 11 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_M$ with $x = \lambda\mathscr{N}(x)$. Then, x is a solution of (30) with $\|x\| = M$. Now, the previous inequality implies

$$\frac{(1-k_0\ell)M}{\Lambda \|p\| \psi(M)} \le 1, \tag{65}$$

which contradicts to (49). Hence, $\mathscr{N}$ has a fixed point in I_q^T by Theorem 11, and consequently the boundary value problem (2) has a solution. This completes the proof. □

Example 13. Consider the following nonlocal boundary value problem of nonlinear q-difference inclusion

$$
\begin{gathered}
D_{1/2}^2 x(t) \in F(t,x), \quad t \in \left[0, \frac{1}{2}\right]_{1/2}, \\
x\left(\frac{1}{8}\right) = \frac{|x(1/16)|}{4(1+|x(1/16)|)}, \\
\frac{1}{2} D_{1/3} x\left(\frac{1}{4}\right) + \frac{3}{5} \int_{1/4}^{1/2} x(s)\, d_{1/4} s = 0,
\end{gathered}
\tag{66}
$$

where $F : [0,1/2] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$x \longrightarrow F(t,x) = \left[\frac{t|x|(1+\cos^2 x)}{1+|x|}, \frac{(t+1)(|x|+1)e^{-x^2}}{1+\sin^2 x} \right]. \tag{67}$$

Here, $q = 1/2$, $r = 1/3$, $p = 1/4$, $T = 1/2$, $\xi = 1/8$, $\eta = 1/4$, $\alpha = 1/2$, $\beta = 3/5$, and $g(x) = (1/4)(|x|/(1+|x|))$. We find that $D = -457/800$, $k_0 = 532/457$, and $\Lambda = 11083/32904$.

As $|g(x) - g(y)| \le (1/4)|x-y|$, therefore, (A_2) is satisfied with $\ell = (1/4) < (457/532) = k_0^{-1}$. For $f \in F$, we have

$$
\begin{aligned}
|f| & \le \max\left(\frac{t|x|(1+\cos^2 x)}{1+|x|}, \frac{(t+1)(|x|+1)e^{-x^2}}{1+\sin^2 x} \right) \\
& \le (t+1)(|x|+1), \quad x \in \mathbb{R}.
\end{aligned}
\tag{68}
$$

Thus

$$
\begin{aligned}
\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|), \\
x \in \mathbb{R},
\end{aligned}
\tag{69}
$$

with $p(t) = t+1$ and $\psi(\|x\|) = \|x\| + 1$. From the given data, it is found that $M > 2.47997$. Clearly, all the conditions of Theorem 12 are satisfied. Hence, the nonlocal boundary value problem (66) has at least one solution on $[0,1/2]_{1/2}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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