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Research Article

On Hölder and Minkowski Type Inequalities

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We obtain inequalities of Hölder and Minkowski type with weights generalizing both the case of weights with alternating signs and the classical case of nonnegative weights.

1. Introduction

Recently Chunaev [1] obtained Hölder and Minkowski type inequalities with alternating signs. His results are a supplement to Jensen type inequalities with alternating signs obtained earlier by Szegö [2], Bellman [3, 4], Brunk [5], and others (see [6–11], [12, Section 5.38], and also Remark 7).

In this paper, we intend to give inequalities of Hölder and Minkowski type with more general weights, including both the case of weights with alternating signs and the classical case of nonnegative weights (see, e.g., [12, Section 4.2] and [1, 13]). Namely, weights p_k, $k = 1, \ldots, n$, satisfying the property

$$P_k \geqslant 0, \quad \text{where } P_k := \sum_{m=1}^{k} p_m, \ k = 1, \ldots, n, \tag{1}$$

are considered. We follow proofs in [1] with several changes in order to obtain our results.

In what follows, we denote nonnegative sequences of real numbers in bold print; for example, $\mathbf{a} = \{a_k\}_{k=1}^n$ or $\mathbf{b} = \{b_k\}_{k=1}^n$, where n is a positive integer or infinity. Expressions like $\mathbf{a} \equiv 1$ mean that all elements of $\mathbf{a}$ equal 1. In proofs we use several well-known inequalities for $\alpha, \beta \geqslant 0$ and $p \geqslant 1$:

$$(\alpha + \beta)^p \leqslant 2^{p-1} (\alpha^p + \beta^p) \quad (\text{Jensen's inequality}), \tag{2}$$

$$\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad (\text{Young's inequality}), \tag{3}$$

$$p\beta^{p-1} \leqslant \frac{\alpha^p - \beta^p}{\alpha - \beta} \leqslant p\alpha^{p-1}, \quad \alpha > \beta \quad (\text{see [14]}). \tag{4}$$

2. Hölder Type Inequalities

In this section, we show that there is not a direct analog of Hölder's inequality in the case of our weights, but one of reverse Hölder's inequalities exists. Note that reverse Hölder's inequalities for nonnegative weights are well studied (see [13]).

Theorem 1. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be nonincreasing such that*

$$0 < a \leqslant a_k \leqslant A < \infty, \quad 0 < b \leqslant b_k \leqslant B < \infty, \tag{5}$$

where $k = 1, \ldots, n$*. If, moreover,* $P_k \geqslant 0$*, and* $p, q > 1$*,* $1/p + 1/q = 1$*, then*

$$\begin{aligned} 0 &\leqslant \frac{\left(\sum_{k=1}^n p_k a_k^q\right)^{1/q} \left(\sum_{k=1}^n p_k b_k^p\right)^{1/p}}{\sum_{k=1}^n p_k a_k b_k} \\ &\leqslant (pA/a)^{1/p} (qB/b)^{1/q}. \end{aligned} \tag{6}$$

The left hand side of (6) *should be read as there exists no positive constant, depending on* $a, A, b, B, p,$ *or* q*, which bounds the fraction in* (6) *from below.*

Before the proof of Theorem 1, we establish the following fact.

Lemma 2. *Let* **a** *be nonincreasing and* **b** *nondecreasing such that* $b_k \leqslant B$ *for* $k = 1, \ldots, n$. *If, moreover,* $P_k \geqslant 0$ *for* $k = 1, \ldots, n$, *then*

$$\sum_{k=1}^{n} p_k a_k b_k \leqslant B \sum_{k=1}^{n} p_k a_k. \tag{7}$$

Proof. Applying the Abel transformation, we have

$$\begin{aligned} &B\sum_{k=1}^{n} p_k a_k - \sum_{k=1}^{n} p_k a_k b_k \\ &\quad = \sum_{k=1}^{n-1} P_k \left(a_k \left(B - b_k\right) - a_{k+1}\left(B - b_{k+1}\right)\right) \\ &\qquad + P_n a_n \left(B - b_n\right), \end{aligned} \tag{8}$$

where the latter expression is nonnegative since the sequences **a** and $\{B - b_k\}$ are nonincreasing, and $P_k \geqslant 0$. The equality holds, for example, if $\mathbf{b} \equiv B$. □

Proof. We denote the fraction in (6) by F_H. Applying the Abel transformation to the numerator and the denominator of F_H easily yields $F_H \geqslant 0$. But we prove even more, namely, that there exist no positive constants bounding F_H from below. Following [1], let $p_k = (-1)^{k+1}$, $k = 1, \ldots, n$, where n is even, and let $\mathbf{a} = \{a_1, a_1, a_3, a_3, \ldots, a_n, a_n, \ldots\}$ be positive and nondecreasing. The sequence **b** is arbitrary except such that $b_{2k-1} - b_{2k} = 0$ for all $k = 1, \ldots, n/2$. It follows that

$$F_H = \frac{0 \cdot \left(\sum_{k=1}^{n} (-1)^{k+1} b_k^p\right)^{1/p}}{\sum_{k=1}^{n/2} a_{2k-1}\left(b_{2k-1} - b_{2k}\right)} = 0. \tag{9}$$

Thus F_H cannot be bounded from below by a positive absolute constant or a constant depending on p,q, maximum or minimum elements of **a** and **b**.

Now we prove the right hand side of (6). Here N_H denotes the numerator of F_H. First we apply the Abel transformation:

$$\begin{aligned} N_H &= \left(\sum_{k=1}^{n-1} P_k \left(a_k^q - a_{k+1}^q\right) + P_n a_n^q\right)^{1/q} \\ &\quad \times \left(\sum_{k=1}^{n-1} P_k \left(b_k^p - b_{k+1}^p\right) + P_n b_n^p\right)^{1/p}. \end{aligned} \tag{10}$$

By the right hand side of (4) and the Abel transformation

$$\begin{aligned} N_H &\leqslant \frac{\left(qA^{q-1}\right)^{1/q}\left(pB^{p-1}\right)^{1/p}}{C^{1/q}D^{1/p}} \\ &\quad \times \left(\sum_{k=1}^{n} C p_k a_k\right)^{1/q} \left(\sum_{k=1}^{n} D p_k b_k\right)^{1/p}, \end{aligned} \tag{11}$$

where C and D are arbitrary positive constants. Therefore, (3) after several simplifications gives

$$N_H \leqslant \frac{(pA)^{1/p}(qB)^{1/q}}{C^{1/q}D^{1/p}} \left(\sum_{k=1}^{n} p_k \left(\frac{C}{qb_k} + \frac{D}{pa_k}\right) a_k b_k\right). \tag{12}$$

In the latter expression, $\{C/(qb_k) + D/(pa_k)\}$ is nondecreasing and $\{a_k b_k\}$ is nonincreasing, because **a** and **b** are nonincreasing. Hence by Lemma 2

$$\begin{aligned} N_H &\leqslant \frac{(pA)^{1/p}(qB)^{1/q}}{C^{1/q}D^{1/p}} \max_k \left\{\frac{C}{qb_k} + \frac{D}{pa_k}\right\} \sum_{k=1}^{n} p_k a_k b_k \\ &\leqslant (pA)^{1/p}(qB)^{1/q} \left(\frac{1}{qb}\left(\frac{C}{D}\right)^{1/p} + \frac{1}{pa}\left(\frac{D}{C}\right)^{1/q}\right) \\ &\quad \times \sum_{k=1}^{n} p_k a_k b_k. \end{aligned} \tag{13}$$

It is easily seen that, in order to get the smallest constant in the latter inequality, we must choose $C/D = b/a$. It gives the right hand side of (6). Note that the constant there belongs to $(1; \infty)$. □

Remark 3. From Theorem 1, it is seen that the constant in the right hand side of (6) tends to infinity as $a \to 0$ or $b \to 0$ (note that this constant is better than in [1]). Now we give an example of sequences confirming this [1]. In Theorem 1 we suppose that $p_k = (-1)^{k+1}$, $n = 2m + 1$, $\mathbf{a} \equiv 1$, and $b = b_{2m+1} \to 0$ in **b**. It gives

$$\begin{aligned} F_H &= \frac{\left(\sum_{k=1}^{2m+1} (-1)^{k+1} a_k^q\right)^{1/q} \left(\sum_{k=1}^{2m+1} (-1)^{k+1} b_k^p\right)^{1/p}}{\sum_{k=1}^{2m+1} (-1)^{k+1} a_k b_k} \\ &= \frac{\left(\sum_{k=1}^{2m} (-1)^{k+1} b_k^p\right)^{1/p}}{\sum_{k=1}^{2m} (-1)^{k+1} b_k}. \end{aligned} \tag{14}$$

From the left hand side of (4) we deduce

$$\begin{aligned} F_H &= \frac{\left(\sum_{k=1}^{m} \left(b_{2k-1}^p - b_{2k}^p\right)\right)^{1/p}}{\sum_{k=1}^{m} \left(b_{2k-1} - b_{2k}\right)} \\ &\geqslant p^{1/p} \left(\frac{b_{2m}}{\sum_{k=1}^{m} \left(b_{2k-1} - b_{2k}\right)}\right)^{1-1/p}, \end{aligned} \tag{15}$$

where $1 - 1/p > 0$. Therefore, for a fixed positive b_{2m} the sum in the denominator can be made sufficiently small by an appropriate choice of **b**. Consequently, F_H can be arbitrarily large. The same is for $a = a_{2m+1} \to 0$.

It is clear that if $p = q = 2$ then the constant in the right hand side of (6) equals $2\sqrt{AB(ab)^{-1}} \geqslant 2$. Now we give a more precise constant belonging to $[1; \infty)$ for the case when **a** and **b** satisfy several additional conditions.

Proposition 4. *Let* **a** *and* **b** *be nonincreasing such that the sequence* $\{a_k/b_k\}$ *is monotone and* $0 < m \leqslant a_k/b_k \leqslant M < \infty$. *If, moreover,* $P_k \geqslant 0$ *for* $k = 1, \ldots, n$, *then*

$$0 \leqslant \frac{\sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2}{\left(\sum_{k=1}^{n} p_k a_k b_k\right)^2} \leqslant \frac{1}{4}\left(\frac{m}{M} + \frac{M}{m}\right)^2. \tag{16}$$

The left hand side of (16) *should be read as there exists no positive constant, depending on m and M, which bounds the fraction in* (16) *from below.*

Proof. The left hand side inequality follows by the same method as in the proof of Theorem 1. To prove the right hand side we denote the numerator of the fraction in (16) by N_C. First we suppose $\{a_k/b_k\}$ to be nondecreasing, so $1 \leqslant a_k/(mb_k) \leqslant M/m$. Applying (3) with $p = q = 2$ yields

$$\begin{aligned} N_C &\leqslant \frac{1}{4m^2}\left(\sum_{k=1}^{n} p_k\left(a_k^2 + (mb_k)^2\right)\right)^2 \\ &= \frac{1}{4}\left(\sum_{k=1}^{n} p_k\left(\frac{a_k}{mb_k} + \frac{mb_k}{a_k}\right)a_k b_k\right)^2. \end{aligned} \tag{17}$$

In the latter expression, the sequence $\{c_k + 1/c_k\}$, where $c_k = a_k/(mb_k)$, is nondecreasing. Indeed, $\{c_k\}$ is nondecreasing and moreover $c_1 \geqslant 1$. Since $f(x) = x + 1/x$ is convex for $x \in (0;\infty)$ and has a minimum at $x = 1$, the sequence $\{f(c_k)\}$ is nondecreasing. From this by Lemma 2

$$N_C \leqslant \frac{1}{4}\left(\max_k\{f(c_k)\}\right)^2\left(\sum_{k=1}^{n}(-1)^{k+1}a_k b_k\right)^2, \tag{18}$$

where $\max_k\{f(c_k)\} = m/M + M/m$. Supposing $\{a_k/b_k\}$ to be nonincreasing and taking into account that $m/M \leqslant a_k/(Mb_k) \leqslant 1$, we obtain the right hand side of (16) by the same technique.

It is easily seen that equality in (16) holds, for example, if $\mathbf{a} \equiv \mathbf{b}$. The fact that the constant in the right hand side of (16) belongs to $[1;\infty)$ is obvious. □

From the well-known weighted inequality of arithmetic and geometric means (see, e.g., [14, Chapter 2]) supposing $a_m \geqslant 0$ and $v_m > 0$, we have

$$\prod_{m=1}^{M} a_m \leqslant \sum_{m=1}^{M} v_m a_m^{1/v_m}, \qquad \sum_{m=1}^{M} v_m = 1. \tag{19}$$

This is a multivariable version of Young's inequality (3). From this we obtain a multivariable version of Theorem 1 (but with less precise constant).

Proposition 5. *Let* $\mathbf{x}_m := \{x_{m,k}\}_{k=1}^n$ *be nonincreasing sequences such that* $0 < a_m \leqslant x_{m,k} \leqslant A_m < \infty$, *where* $m = 1,\ldots,M$. *If, moreover,* $P_k \geqslant 0$, $k = 1,\ldots,n$, *and* $w_k > 0$, $k = 1,\ldots,n$, *are such that* $\sum_{m=1}^{M} w_m = 1$, *then*

$$\begin{aligned} 0 &\leqslant \frac{\prod_{m=1}^{M}\left(\sum_{k=1}^{n} p_k x_{m,k}^{1/w_m}\right)^{w_m}}{\sum_{k=1}^{n} p_k \prod_{m=1}^{M} x_{m,k}} \\ &\leqslant \sum_{m=1}^{M} A_m^{1/w_m - 1}\prod_{j=1, j\neq m}^{M}\frac{A_j^{1/w_j - 1}}{w_j a_j}. \end{aligned} \tag{20}$$

The left hand side of (20) *should be read as there exists no positive constant, depending on* a_m, A_m, *and* w_m, *which bounds the fraction in* (20) *from below.*

Proof. Set F_H is the fraction in (20). Nonexistence of a positive constant bounding F_H from below follows from Theorem 1. To prove the right hand side we denote the numerator of F_H by N_H. By the Abel transformation

$$N_H = \prod_{m=1}^{M}\left(\sum_{k=1}^{n-1} P_k\left(x_{m,k}^{1/w_m} - x_{m,k+1}^{1/w_m}\right) + P_n x_{m,n}^{1/w_m}\right)^{w_m}. \tag{21}$$

The right hand side of (4) and the Abel transformation yields

$$N_H \leqslant \prod_{m=1}^{M}\frac{A_m^{1/w_m - 1}}{w_m}\prod_{m=1}^{M}\left(\sum_{k=1}^{n} p_k x_{m,k}\right)^{w_m}. \tag{22}$$

Supposing $v_m = w_m$ in (19), we obtain

$$\begin{aligned} N_H &\leqslant \prod_{m=1}^{M}\frac{A_m^{1/w_m - 1}}{w_m} \\ &\quad \times \left(\sum_{k=1}^{n} p_k\left(\sum_{m=1}^{M} w_m \prod_{m=1, m\neq k}^{M} x_{m,k}^{-1}\right)\prod_{m=1}^{M} x_{m,k}\right), \end{aligned} \tag{23}$$

where it is obvious that $\{\sum_{m=1}^{M} w_m \prod_{m=1,m\neq k}^{M} x_{m,k}^{-1}\}_{k=1}^n$ is nondecreasing and that $\{\prod_{m=1}^{M} x_{m,k}\}_{k=1}^n$ is nonincreasing. Thus by Lemma 2

$$\begin{aligned} N_H &\leqslant \prod_{m=1}^{M}\frac{A_m^{1/w_m - 1}}{w_m}\max_k\left\{\sum_{m=1}^{M} w_m \prod_{m=1, j\neq m}^{M} x_{j,k}^{-1}\right\} \\ &\quad \times \sum_{k=1}^{n} p_k \prod_{m=1}^{M} x_{m,k}. \end{aligned} \tag{24}$$

Several simplifications give the right hand side of (20). □

3. Minkowski Type Inequalities

In this section we prove precise Minkowski type inequalities with our weights. As we have already mentioned, these generalize both the case of weight with alternating signs and the case of nonnegative weights (see [1]).

Theorem 6. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be nonnegative nonincreasing sequences, and* $P_k \geqslant 0$ *for* $k = 1,\ldots,n$. *Then for* $p \geqslant 1$

$$0 \leqslant \frac{\left(\sum_{k=1}^{n} p_k a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} p_k b_k^p\right)^{1/p}}{\left(\sum_{k=1}^{n} p_k (a_k + b_k)^p\right)^{1/p}} \leqslant 2^{1-1/p}, \tag{25}$$

The constant $2^{1-1/p}$ *is best possible. The left hand side of* (25) *should be read as there exists no positive constant, depending on only p, which bounds the fraction in* (25) *from below.*

Proof. Throughout the proof, F_M denotes the fraction in (25). Applying the Abel transformation for the numerator and the denominator of F_M easily yields $F_M \geqslant 0$. Moreover, there exists no positive constant depending on p only that bounds F_M from below. Indeed [1], for each $p > 1$ there exists a sequence such that F_M tends to zero. Supposing that $p_k = (-1)^{k+1}, n \geqslant 2$, $\mathbf{a} = \{1, 1, 0, \ldots, 0, \ldots\}$ and $\mathbf{b} = \{b, 0, \ldots, 0, \ldots\}$ with some $b > 0$, from the left hand side of (4) we deduce

$$F_M = \frac{b}{((1+b)^p - 1)^{1/p}} \leqslant \frac{b}{(pb)^{1/p}} < b^{1-1/p}. \quad (26)$$

In this way $F_M \to 0$ as $b \to 0$ since $1 - 1/p > 0$ for all $p > 1$.

Now we prove the right hand side of (25). From (2) we have

$$\begin{aligned} &\left(\left(\sum_{k=1}^{n} p_k a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} p_k b_k^p\right)^{1/p}\right)^p \\ &\quad \leqslant 2^{p-1}\left(\sum_{k=1}^{n} p_k \left(a_k^p + b_k^p\right)\right). \end{aligned} \quad (27)$$

Now, before extraction of the pth root, it is enough to show that

$$\sum_{k=1}^{n} p_k \left(a_k^p + b_k^p\right) \leqslant \sum_{k=1}^{n} p_k (a_k + b_k)^p, \quad p \geqslant 1. \quad (28)$$

Inequality (28) by the Abel transformation is equivalent to

$$\sum_{k=1}^{n} p_k c_k = \sum_{k=1}^{n-1} P_k \left(c_k - c_{k+1}\right) + P_n c_n \geqslant 0, \quad (29)$$

where $c_k := (a_k + b_k)^p - (a_k^p + b_k^p)$. The latter inequality holds since $P_k \geqslant 0$ for all k and $c_k \geqslant c_{k+1}$ for $k = 1, \ldots, n-1$. Indeed, for the function $f(x, y) = (x + y)^p - (x^p + y^p)$, where $x \geqslant 0$, $y \geqslant 0$, and $p \geqslant 1$, we have $f'_x \geqslant 0$ and $f'_y \geqslant 0$. Therefore,

$$\begin{aligned} &f\left(a_k, y\right) \geqslant f\left(a_{k+1}, y\right), \\ &f\left(x, b_k\right) \geqslant f\left(x, b_{k+1}\right) \Longrightarrow f\left(a_k, b_k\right) \geqslant f\left(a_{k+1}, b_{k+1}\right). \end{aligned} \quad (30)$$

This completes the proof of (28).

The precision of the constant $2^{1-1/p}$ comes out from the following observation from [1]. If $p_k = 1$ for all k, $\mathbf{a} = \{1, \ldots, 1, 0, \ldots, 0\}$ (first n elements are units) and $\mathbf{b} = \{n^{1/p}, 0, \ldots, 0\}$, then after several simplifications we get

$$\begin{aligned} &\frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}}{\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p}} \\ &\quad = 2\left(1 - \frac{1}{n} + \left(1 + \frac{1}{n^{1/p}}\right)^p\right)^{-1/p} \\ &\quad = 2^{1-1/p} - \varepsilon_n, \end{aligned} \quad (31)$$

where positive $\varepsilon_n \to 0$ as $n \to \infty$. □

Remark 7. The following Jensen-Steffensen type statement was proved in [15] (see also [12, Section 2.2]).

Let $\mathbf{a}$ be a nonincreasing positive sequence and φ a function convex on $[a_n; a_1]$ such that $\varphi(0) = 0$. Then the necessary and sufficient condition on weights p_k in order that

$$\varphi\left(\sum_{k=1}^{n} p_k a_k\right) \leqslant \sum_{k=1}^{n} p_k \varphi\left(a_k\right), \quad P_k = \sum_{m=1}^{k} p_m, \quad (32)$$

is $0 \leqslant P_k \leqslant 1$, $k = 1, \ldots, n$.

From this point of view, the sufficient condition $P_k \geqslant 0$, $k = 1, \ldots, n$, in Theorems 1 and 6 seems to be quite close to the necessary one.

4. Further Generalizations

Now we give integral versions of Lemma 2 and Theorems 1 and 6. In what follows, we use the notation

$$P(x) := \int_{\alpha}^{x} p(t)\, dt, \quad x \in [\alpha; \beta], \quad (33)$$

and suppose that all functions of x are integrable and differentiable on $[\alpha; \beta]$.

Lemma 8. *For $x \in [\alpha; \beta]$, let $f(x)$ be nonnegative and nonincreasing, and let $g(x)$ be nondecreasing such that $0 \leqslant g(x) \leqslant B$, and $P(x) \geqslant 0$. Then*

$$\int_{\alpha}^{\beta} f(x)\, g(x)\, dP(x) \leqslant B \int_{\alpha}^{\beta} f(x)\, dP(x). \quad (34)$$

Proof. Applying integration by parts gives

$$\begin{aligned} &B \int_{\alpha}^{\beta} f(x)\, dP(x) - \int_{\alpha}^{\beta} f(x)\, g(x)\, dP(x) \\ &\quad = P(x) f(x) (B - g(x))\Big|_{\alpha}^{\beta} \\ &\qquad - \int_{\alpha}^{\beta} P(x)\, d\left(f(x)(B - g(x))\right) \\ &\quad = P(\beta) f(\beta) (B - g(\beta)) \\ &\qquad + \int_{\alpha}^{\beta} P(x) \left(f(x) g'(x) - f'(x)(B - g(x))\right) dx \geqslant 0. \end{aligned} \quad (35)$$

Here we took into account that $P(\alpha) = 0$; $P(x)$, $f(x)$, $g'(x)$, $B - g(x)$ are nonnegative and $f'(x)$ is nonpositive for $x \in [\alpha; \beta]$. It is easily seen that equality holds, for example, if $g(x) \equiv B$. □

Using Lemma 8 and integration by parts instead of the Abel transformation, we obtain the following results by essential repeating proofs of Theorems 1 and 6. We emphasize that $dP(x)$ may be negative here in contrast to the classical case.

Theorem 9. *For $x \in [\alpha; \beta]$, let $f(x)$ and $g(x)$ be nonincreasing and let*

$$0 < a \leqslant f(x) \leqslant A < \infty, \qquad 0 < b \leqslant g(x) \leqslant B < \infty. \quad (36)$$

If, moreover, $P(x) \geqslant 0$, $x \in [\alpha;\beta]$, *and* $p, q > 1$, $1/p + 1/q = 1$, *then*

$$0 \leqslant \frac{\left(\int_\alpha^\beta f^q(x)\,dP(x)\right)^{1/q}\left(\int_\alpha^\beta g^p(x)\,dP(x)\right)^{1/p}}{\int_\alpha^\beta f(x)\,g(x)\,dP(x)} \leqslant (pA/a)^{1/p}(qB/b)^{1/q}. \tag{37}$$

The left hand side of (37) *should be read as there exists no positive constant, depending on* a, A, b, B, p*, and* q*, which bounds the fraction in* (37) *from below.*

Theorem 10. *For* $x \in [\alpha;\beta]$, *let* $f(x)$ *and* $g(x)$ *be nonnegative and nonincreasing, and* $P(x) \geqslant 0$. *Then*

$$0 \leqslant \frac{\left(\int_\alpha^\beta f^p(x)\,dP(x)\right)^{1/p} + \left(\int_\alpha^\beta g^p(x)\,dP(x)\right)^{1/p}}{\left(\int_\alpha^\beta (f(x)+g(x))^p\,dP(x)\right)^{1/p}} \leqslant 2^{1-1/p}, \quad p \geqslant 1. \tag{38}$$

The constant $2^{1-1/p}$ *is best possible. The left hand side of* (38) *should be read as there exists no positive constant, depending only on* p*, which bounds the fraction in* (38) *from below.*

In conclusion we give several examples concerning Theorems 9 and 10. Let $p(t) = \sin t$ and $x \in [0;\infty)$ in (33); then $P(x) = 1 - \cos x \geqslant 0$, and thus

$$\begin{aligned} 0 &\leqslant \frac{\left(\int_0^\infty f^q(x)\sin x\,dx\right)^{1/q}\left(\int_0^\infty g^p(x)\sin x\,dx\right)^{1/p}}{\int_0^\infty f(x)\,g(x)\sin x\,dx} \leqslant (pA/a)^{1/p}(qB/b)^{1/q}, \\ 0 &\leqslant \frac{\left(\int_0^\infty f^p(x)\sin x\,dx\right)^{1/p} + \left(\int_0^\infty g^p(x)\sin x\,dx\right)^{1/p}}{\left(\int_0^\infty (f(x)+g(x))^p\sin x\,dx\right)^{1/p}} \leqslant 2^{1-1/p}, \quad p \geqslant 1. \end{aligned} \tag{39}$$

Appropriate discretization yields inequalities with alternating signs obtained earlier in [1] (the case $p_k = (-1)^{k+1}$ in Theorems 1 and 6).

If $P(x)$ is nondecreasing for $x \in [\alpha;\beta]$ (i.e., $dP(x)$ is nonnegative), Theorems 9 and 10 give the classical case of nonnegative weights, for which we can put 1 instead of 0 in the left hand sides of (37) and (38) due to Hölder's and Minkowski's inequalities.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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