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Research Article

A Generalized q-Grüss Inequality Involving the Riemann-Liouville Fractional q-Integrals

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The aim of this paper is to establish q-extension of the Grüss type integral inequality related to the integrable functions whose bounds are four integrable functions, involving Riemann-Liouville fractional q-integral operators. The results given earlier by Zhu et al. (2012) and Tariboon et al. (2014) follow the special cases of our findings.

1. Introduction

In [1], the well-known Grüss inequality is defined as follows (see also [2, p. 296]).

Let f and g be two continuous functions defined on $[a,b]$, such that $m \le f(t) \le M$ and $p \le g(t) \le P$, for each $t \in [a,b]$, where m, p, M, and P are given real constants; then

$$\left|\frac{1}{(b-a)}\int_a^b f(t)\,g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt\int_a^b g(t)\,dt\right| \le \frac{1}{4}(M-m)(P-p). \tag{1}$$

In the literature several generalizations of the Grüss type integral inequality are considered by many researchers (see [3–10]). Dahmani et al. [11] established a generalization of inequality (1), using Riemann-Liouville fractional integrals, as follows.

Let f and g be two integrable functions with constant bounds defined on $[0,\infty)$, such that

$$m \le f(t) \le M, \qquad p \le g(t) \le P; \qquad m,p,M,P \in \mathbb{R}, \quad t \in [0,\infty), \tag{2}$$

then for $\alpha > 0$

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha f(t)\,g(t) - I^\alpha f(t)\,I^\alpha g(t)\right| \le \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 (M-m)(P-p), \tag{3}$$

where $I^\alpha f(t)$ denote the Riemann-Liouville fractional integral operator of order α for the function $f(t)$. Recently, by replacing the constants appearing as bounds of the functions f and g by four integrable functions, Tariboon et al. [12] investigate more general forms of inequality (3).

The subject of q-calculus has gained noticeable importance due to applications in mathematics, statistics, and physics. Particularly, the q-analysis has found many applications in the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra, geometric functions theory, optimal control problems, q-difference, and q-integral equations [13–16]. This has led various workers in the field of q-theory for exploring the possible q-extensions to all the important results available in the classical theory. With this objective in mind, Gauchman [17] investigated q-analogues of some classical integral inequalities, including the well-known Grüss inequality (1).

Further, a number of authors have studied, in depth, the q-extension and applications of various classical integral inequalities (see [18–23]). Very recently, Zhu et al. [24] derived q-extension of inequality (3) and certain new fractional q-integral inequalities on the specific time scale.

It is fairly well-known that there are a number of different definitions of fractional integrals and their applications. Each definition has its own advantages and suitable for applications to different type of problems. All specialists of this field (Fractional Calculus) know the importance of different types of definitions of fractional calculus operators and their use in specified problems. It is to be noted that the problem considered here provides unifications to the results of [12, 24] and gives a generalized q-Grüss type integral inequality related to the integrable functions whose bounds are four integrable functions, involving Riemann-Liouville fractional q-integrals. Additionally, Riemann-Liouville fractional q-integral operator has the advantage that it generalizes the familiar Riemann-Liouville operator and also provides the results on time scales. Our main result provides q-extension of the result due to Tariboon et al. [12] and can be further applied to derive certain interesting consequent results and special cases.

We begin with the mathematical preliminaries of q-series and q-calculus. For more details of q-calculus and fractional q-calculus one can refer to [13, 16].

The q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha;q)_n = \begin{cases} 1; & n = 0 \\ (1-\alpha)(1-\alpha q)\cdots\left(1-\alpha q^{n-1}\right); & n \in \mathbb{N}, \end{cases} \tag{4}$$

and in terms of the basic analogue of the gamma function

$$\left(q^{\alpha};q\right)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n>0), \tag{5}$$

where the q-gamma function is defined by ([13, p. 16, eqn. (1.10.1)])

$$\Gamma_q(t) = \frac{(q;q)_\infty (1-q)^{1-t}}{\left(q^t;q\right)_\infty} \quad (0<q<1). \tag{6}$$

Further, we note that

$$\Gamma_q(1+t) = \frac{\left(1-q^t\right)\Gamma_q(t)}{1-q}, \tag{7}$$

and if $|q| < 1$, definition (4) remains meaningful for $n = \infty$, as a convergent infinite product given by

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}\left(1-\alpha\, q^j\right). \tag{8}$$

Also, the q-binomial expansion is given by

$$\begin{aligned}(x-y)_\nu &= x^\nu\left(-\frac{y}{x};q\right)_\nu \\ &= x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right].\end{aligned} \tag{9}$$

Jackson's q-derivative and q-integral of a function f defined on $\mathbb{T}$ are, respectively, given by (see [13, pp. 19, 22])

$$\begin{aligned} D_{q,t}f(t) &= \frac{f(t)-f(tq)}{t(1-q)} \quad (t\neq 0, q\neq 1), \\ \int_0^t f(\tau)\,d_q\tau &= t\,(1-q)\sum_{k=0}^{\infty} q^k f\left(tq^k\right). \end{aligned} \tag{10}$$

The Riemann-Liouville fractional q-integral operator of a function $f(t)$ of order α (due to Agarwal [25]) is given by

$$\begin{aligned} I_q^\alpha f(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t \left(\frac{q\tau}{t};q\right)_{\alpha-1} f(\tau)\,d_q\tau \\ (\alpha>0, 0<q<1), \end{aligned} \tag{11}$$

where

$$(a;q)_\alpha = \frac{(a;q)_\infty}{\left(aq^\alpha;q\right)_\infty} \quad (\alpha\in\mathbb{R}). \tag{12}$$

Following [25] (see also [26]), when $f(t) = t^\mu$, we obtain

$$I_q^\alpha t^\mu = \frac{\Gamma_q(1+\mu)}{\Gamma_q(1+\mu+\alpha)}t^{\mu+\alpha}, \quad (0<q<1, \mu>-1, t>0). \tag{13}$$

2. A Generalized q-Grüss Integral Inequality

Our results in this section are based on the following lemma, giving functional relation for Riemann-Liouville fractional q-integral operators, with the integrable functions.

Lemma 1. *Let f, φ_1, and φ_2 be integrable functions defined on $[0,\infty)$, such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \forall t\in[0,\infty). \tag{14}$$

Then, for $t>0$ and $\alpha>0$, we have

$$\begin{aligned} &\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f^2(t) - \left(I_q^\alpha f(t)\right)^2 \\ &\quad = \left(I_q^\alpha\varphi_2(t) - I_q^\alpha f(t)\right)\left(I_q^\alpha f(t) - I_q^\alpha\varphi_1(t)\right) \\ &\qquad - \frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right) \\ &\qquad + \frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_1(t)\,f(t) - I_q^\alpha\varphi_1(t)\,I_q^\alpha f(t) \\ &\qquad + \frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_2(t)\,f(t) - I_q^\alpha\varphi_2(t)\,I_q^\alpha f(t) \\ &\qquad + I_q^\alpha\varphi_1(t)\,I_q^\alpha\varphi_2(t) - \frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_1(t)\,\varphi_2(t). \end{aligned} \tag{15}$$

Proof. On using the hypothesis of inequality (14), for any $\tau, \rho > 0$, we can write

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) \\
&\quad + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \\
&\quad - (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) \\
&\quad - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \\
&= f^2(\tau) + f^2(\rho) - 2f(\tau) f(\rho) \\
&\quad + \varphi_2(\rho) f(\tau) + \varphi_1(\tau) f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\
&\quad + \varphi_2(\tau) f(\rho) + \varphi_1(\rho) f(\tau) - \varphi_1(\rho)\varphi_2(\tau) \\
&\quad - \varphi_2(\tau) f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\
&\quad - \varphi_1(\tau) f(\tau) - \varphi_2(\rho) f(\rho) \\
&\quad + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho) f(\rho).
\end{aligned} \tag{16}$$

Consider

$$F_q(t,\tau) = \frac{t^{\alpha-1}(q\tau/t;q)_{\alpha-1}}{\Gamma(\alpha)} \quad (\tau \in (0,t); t > 0), \tag{17}$$

for all $\tau \in (0,t)$ $(t > 0)$. Multiplying both sides of (16) by $F_q(t,\tau)$ and integrating the resulting identity with respect to τ from 0 to t, and using integral operator (11), we get

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))\left(I_q^\alpha f(t) - I_q^\alpha \varphi_1(t)\right) \\
&\quad + \left(I_q^\alpha \varphi_2(t) - I_q^\alpha f(t)\right)(f(\rho) - \varphi_1(\rho)) \\
&\quad - I_q^\alpha(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \\
&\quad - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) I_q^\alpha\{1\} \\
&= I_q^\alpha f^2(t) + f^2(\rho) I_q^\alpha\{1\} \\
&\quad - 2f(\rho) I_q^\alpha f(t) + \varphi_2(\rho) I_q^\alpha f(t) \\
&\quad + f(\rho) I_q^\alpha \varphi_1(t) - \varphi_2(\rho) I_q^\alpha \varphi_1(t) \\
&\quad + f(\rho) I_q^\alpha \varphi_2(t) + \varphi_1(\rho) I_q^\alpha f(t) \\
&\quad - \varphi_1(\rho) I_q^\alpha \varphi_2(t) - I_q^\alpha \varphi_2(t) f(t) \\
&\quad + I_q^\alpha \varphi_1(t)\varphi_2(t) - I_q^\alpha \varphi_1(t) f(t) \\
&\quad - \varphi_2(\rho) f(\rho) I_q^\alpha\{1\} + \varphi_1(\rho)\varphi_2(\rho) I_q^\alpha\{1\} \\
&\quad - \varphi_1(\rho) f(\rho) I_q^\alpha\{1\}.
\end{aligned} \tag{18}$$

Next, on multiplying both sides of (18) by $F_q(t,\rho)$ $(\rho \in (0,t), t > 0)$, where $F_q(t,\rho)$ is given by (17), and integrating with respect to ρ from 0 to t, we obtain

$$\begin{aligned}
&2\left(I_q^\alpha \varphi_2(t) - I_q^\alpha f(t)\right)\left(I_q^\alpha f(t) - I_q^\alpha \varphi_1(t)\right) \\
&\quad - 2I_q^\alpha(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) I_q^\alpha\{1\} \\
&\quad = 2I_q^\alpha\{1\} I_q^\alpha f^2(t) - 2\left(I_q^\alpha f(t)\right)^2 \\
&\quad\quad + 2I_q^\alpha \varphi_1(t) I_q^\alpha f(t) - 2I_q^\alpha\{1\} I_q^\alpha \varphi_1(t) f(t) \\
&\quad\quad + 2I_q^\alpha \varphi_2(t) I_q^\alpha f(t) - 2I_q^\alpha\{1\} I_q^\alpha \varphi_2(t) f(t) \\
&\quad\quad - 2I_q^\alpha \varphi_1(t) I_q^\alpha \varphi_2(t) + 2I_q^\alpha\{1\} I_q^\alpha \varphi_1(t)\varphi_2(t),
\end{aligned} \tag{19}$$

which upon using formula (13) (for $\mu = 0$), we easily arrive at the identity (15). □

Now, we obtain a generalized q-Grüss integral inequality, which gives an estimation for the fractional q-integral of a product in terms of the product of the individual function fractional q-integrals, involving Riemann-Liouville fractional hypergeometric operators. Our inequality is related to the integrable functions f and g, whose bounds are integrable functions and satisfying the Cauchy-Schwarz inequality.

Theorem 2. *Let f and g be two integrable functions on $[0,\infty)$ and φ_1, φ_2, ψ_1, and ψ_2 are four integrable functions on $[0,\infty)$, such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \psi_1(t) \le g(t) \le \psi_2(t), \quad \forall t \in [0,\infty). \tag{20}$$

Then, for $t > 0$ and $\alpha > 0$, one has

$$\begin{aligned}
&\left|\frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha f(t) g(t) - I_q^\alpha f(t) I_q^\alpha g(t)\right| \\
&\quad \le \sqrt{\mathcal{T}_q(f,\varphi_1,\varphi_2)\,\mathcal{T}_q(g,\psi_1,\psi_2)},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\mathcal{T}_q(u,v,w) \\
&\quad = \left(I_q^\alpha w(t) - I_q^\alpha u(t)\right)\left(I_q^\alpha u(t) - I_q^\alpha v(t)\right) \\
&\quad\quad + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha v(t) u(t) - I_q^\alpha v(t) I_q^\alpha u(t) \\
&\quad\quad + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha w(t) u(t) - I_q^\alpha w(t) I_q^\alpha u(t) \\
&\quad\quad + I_q^\alpha v(t) I_q^\alpha w(t) - \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha v(t) w(t).
\end{aligned} \tag{22}$$

Proof. Let f and g be two integrable functions on $[0,\infty)$ and satisfying inequality (20); then for any $\tau, \rho > 0$, we define a function

$$\mathcal{H}_q(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau,\rho \in (0,t), \quad t > 0. \tag{23}$$

On multiplying both sides of (23) by $F_q(t,\tau)F_q(t,\rho)$, where $F_q(t,\tau)$ and $F_q(t,\rho)$ are given by (17), and integrating with respect to τ and ρ, respectively, from 0 to t, we obtain

$$\frac{t^{2\alpha-2}}{2\Gamma_q^2(\alpha)}\iint_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\times\left(\frac{q\rho}{t};q\right)_{\alpha-1}\mathscr{H}_q(\tau,\rho)\,d_q\tau\,d_q\rho = \frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f(t)\,g(t)-I_q^\alpha f(t)\,I_q^\alpha g(t). \tag{24}$$

Now, upon using the Cauchy-Schwarz inequality for q-integrals (for details, see [24]), we get

$$\left(\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f(t)g(t)-I_q^\alpha f(t)I_q^\alpha g(t)\right)^2 \le\left(\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f^2(t)-\left(I_q^\alpha f(t)\right)^2\right)\times\left(\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha g^2(t)-\left(I_q^\alpha g(t)\right)^2\right). \tag{25}$$

On the other hand, we observe that each term of the series in (17) is positive, and hence, the function $F_q(t,\tau)$ remains positive, for all $\tau\in(0,t)$ $(t>0)$. Therefore, under the hypothesis of Lemma 1, it is obvious to see that either if a function f is integrable and nonnegative on $[0,\infty)$, then $I_q^\alpha f(t)\ge 0$ or if a function f is integrable and nonpositive on $[0,\infty)$, then $I_q^\alpha f(t)\le 0$.

Now, by noting the relation that, for all $t\in[0,\infty)$,

$$\begin{aligned}&\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\ge 0,\\&\left(\psi_2(t)-g(t)\right)\left(g(t)-\psi_1(t)\right)\ge 0,\end{aligned} \tag{26}$$

we have

$$\begin{aligned}&\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\ge 0,\\&\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\left(\psi_2(t)-g(t)\right)\left(g(t)-\psi_1(t)\right)\ge 0.\end{aligned} \tag{27}$$

Thus, upon using Lemma 1, we get

$$\begin{aligned}&\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f^2(t)-\left(I_q^\alpha f(t)\right)^2\\&\le\left(I_q^\alpha\varphi_2(t)-I_q^\alpha f(t)\right)\left(I_q^\alpha f(t)-I_q^\alpha\varphi_1(t)\right)\\&\quad+\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_1(t)\,f(t)-I_q^\alpha\varphi_1(t)\,I_q^\alpha f(t)\\&\quad+\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_2(t)\,f(t)-I_q^\alpha\varphi_2(t)\,I_q^\alpha f(t)\\&\quad+I_q^\alpha\varphi_1(t)\,I_q^\alpha\varphi_2(t)-\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha\varphi_1(t)\,\varphi_2(t)\\&=\mathscr{T}_q(f,\varphi_1,\varphi_2).\end{aligned} \tag{28}$$

Similarly, we can write

$$\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha g^2(t)-\left(I_q^\alpha g(t)\right)^2\le\mathscr{T}_q(g,\psi_1,\psi_2). \tag{29}$$

On making use of inequalities (25), (28), and (29), we easily arrive at the main result (21). □

Now, we briefly consider some special cases of the result derived in the preceding section. If we let $q\to 1^-$, and we make use of the limit formulas:

$$\begin{aligned}&\lim_{q\to1^-}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n,\\&\lim_{q\to1^-}\Gamma_q(\alpha)=\Gamma(\alpha),\end{aligned} \tag{30}$$

we observe that inequality (21) of Theorem 2 provides the q-extension of the known result due to Tariboon et al. [12, p. 5, Theorem 9].

Further, if we set $\varphi_1(t)=m$, $\varphi_2(t)=M$, $\psi_1(t)=p$, and $\psi_2(t)=P$, where m, M, p, and P are real constants, then Theorem 2 yields the following q-Grüss integral inequality, which may be regarded as q-extensions of inequality (3).

Corollary 3. *Let f and g be two integrable functions on $[0,\infty)$, such that*

$$\begin{aligned}&m\le f(t)\le M,\qquad p\le g(t)\le P,\\&m,p,M,P\in\mathbb{R}\quad\forall t\in[0,\infty).\end{aligned} \tag{31}$$

Then, for $t>0$ and $\alpha>0$, one has

$$\left|\frac{t^\alpha}{\Gamma_q(1+\alpha)}I_q^\alpha f(t)\,g(t)-I_q^\alpha f(t)\,I_q^\alpha g(t)\right|\le\left(\frac{t^\alpha}{2\Gamma_q(1+\alpha)}\right)^2(M-m)(P-p). \tag{32}$$

A similar type of fractional q-integral inequality (32) has been derived by Zhu et al. [24] on the specific time scale.

Again, if we set $\varphi_1(t)=t$, $\varphi_2(t)=t+1$, $\psi_1(t)=t-1$, and $\psi_2(t)=t$, then Theorem 2 leads to the following q-integral inequality.

Corollary 4. *Let f and g be two integrable functions on $[0,\infty)$, such that*

$$t \le f(t) \le t+1, \quad t-1 \le g(t) \le t, \quad \forall t \in [0,\infty). \tag{33}$$

Then, for $t > 0$ and $\alpha > 0$, one has

$$\left| \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha f(t)\, g(t) - I_q^\alpha f(t)\, I_q^\alpha g(t) \right| \le \sqrt{\mathcal{T}_q(f,t,t+1)\,\mathcal{T}_q(g,t-1,t)}, \tag{34}$$

where

$$\begin{aligned}
\mathcal{T}_q(f,t,t+1) &= \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} + \frac{t^\alpha}{\Gamma_q(1+\alpha)} - I_q^\alpha f(t) \right) \\
&\quad \times \left(I_q^\alpha f(t) - \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha t f(t) \\
&\quad - \frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} I_q^\alpha f(t) + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha (t+1) f(t) \\
&\quad - \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} + \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) I_q^\alpha f(t) \\
&\quad + \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} \right) \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} + \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) \\
&\quad - \frac{t^\alpha}{\Gamma_q(1+\alpha)} \left(\frac{\Gamma_q(3)\,t^{\alpha+3}}{\Gamma_q(3+\alpha)} + \frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} \right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_q(f,t-1,t) &= \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} - I_q^\alpha g(t) \right) \\
&\quad \times \left(I_q^\alpha g(t) - \frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} + \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) \\
&\quad + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha (t-1) f(t) \\
&\quad - \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} - \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) I_q^\alpha f(t) \\
&\quad + \frac{t^\alpha}{\Gamma_q(1+\alpha)} I_q^\alpha t f(t) - \frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} I_q^\alpha f(t) \\
&\quad + \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} - \frac{t^\alpha}{\Gamma_q(1+\alpha)} \right) \left(\frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} \right) \\
&\quad - \frac{t^\alpha}{\Gamma_q(1+\alpha)} \left(\frac{\Gamma_q(3)\,t^{\alpha+3}}{\Gamma_q(3+\alpha)} - \frac{\Gamma_q(2)\,t^{\alpha+1}}{\Gamma_q(2+\alpha)} \right).
\end{aligned} \tag{35}$$

We conclude this paper by remarking that we have introduced a new general extension of q-Grüss type integral inequality, which gives an estimation for the fractional q-integral of a product in terms of the product of the individual function fractional q-integrals involving Riemann-Liouville fractional integral operators. Our main result is related to the integrable functions f and g, whose bounds are integrable functions. Therefore, by suitably specializing the arbitrary functions $\varphi_1(t)$, $\varphi_2(t)$, $\psi_1(t)$, and $\psi_2(t)$, one can easily investigate additional integral inequalities from our main result Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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