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Research Article

A Regularized Algorithm for the Proximal Split Feasibility Problem

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The proximal split feasibility problem has been studied. A regularized method has been presented for solving the proximal split feasibility problem. Strong convergence theorem is given.

1. Introduction

Throughout, we assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two real Hilbert spaces, $f : \mathscr{H}_1 \to \mathscr{R} \cup \{+\infty\}$ and $g : \mathscr{H}_2 \to \mathscr{R} \cup \{+\infty\}$ are two proper, lower semicontinuous convex functions, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator.

In the present paper, we are devoted to solving the following minimization problem:

$$\min_{x^\dagger \in \mathscr{H}_1} \left\{ f\left(x^\dagger\right) + g_\lambda\left(Ax^\dagger\right) \right\}, \tag{1}$$

where g_λ stands for the Moreau-Yosida approximation of the function g of parameter λ; that is,

$$g_\lambda(u) = \min_{v \in \mathscr{H}_2} \left\{ g(v) + \frac{1}{2\lambda}\|u - v\|^2 \right\}. \tag{2}$$

Problem (1) includes the split feasibility problem as a special case. In fact, we choose f and g as the indicator functions of two nonempty closed convex sets $C \subset \mathscr{H}_1$ and $Q \in \mathscr{H}_2$; that is,

$$f\left(x^\dagger\right) = \delta_C\left(x^\dagger\right) = \begin{cases} 0, & \text{if } x^\dagger \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$g\left(x^\dagger\right) = \delta_Q\left(x^\dagger\right) = \begin{cases} 0, & \text{if } x^\dagger \in Q, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

Then, problem (1) reduces to

$$\min_{x^\dagger \in \mathscr{H}_1} \left\{ \delta_C\left(x^\dagger\right) + \left(\delta_Q\right)_\lambda\left(Ax^\dagger\right) \right\}, \tag{4}$$

which equals

$$\min_{x^\dagger \in C} \left\{ \frac{1}{2\lambda} \left\| \left(I - \text{proj}_Q\right)\left(Ax^\dagger\right) \right\|^2 \right\}. \tag{5}$$

Now we know that solving (5) is exactly to solve the following split feasibility problem of finding $x^\ddagger$ such that

$$x^\ddagger \in C, \qquad Ax^\ddagger \in Q, \tag{6}$$

provided $C \cap A^{-1}(Q) \neq \emptyset$.

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Recently, the split feasibility problem (6) has been studied extensively by many authors; see, for instance, [2–8].

In order to solve (6), one of the key ideas is to use fixed point technique according to $x^\dagger$ which solves (6) if and only if

$$x^\dagger = \text{proj}_C\left(I - \gamma A^*\left(I - \text{proj}_Q\right)A\right)x^\dagger. \quad (7)$$

Next, we will use this idea to solve (1). First, by the differentiability of the Yosida approximation g_λ, we have

$$\begin{aligned}\partial\left(f\left(x^\dagger\right) + g_\lambda\left(Ax^\dagger\right)\right) &= \partial f\left(x^\dagger\right) + A^*\nabla g_\lambda\left(Ax^\dagger\right)\\ &= \partial f\left(x^\dagger\right) + A^*\left(\frac{I - \text{prox}_{\lambda g}}{\lambda}\right)\left(Ax^\dagger\right),\end{aligned} \quad (8)$$

where $\partial f(x^\dagger)$ denotes the subdifferential of f at $x^\dagger$ and $\text{prox}_{\lambda g}(x^\dagger)$ is the proximal mapping of g. That is,

$$\begin{aligned}\partial f\left(x^\dagger\right) &= \left\{x^* \in \mathscr{H}_1 : f\left(x^\ddagger\right) \geq f\left(x^\dagger\right) + \left\langle x^*, x^\ddagger - x^\dagger\right\rangle,\ \forall x^\ddagger \in \mathscr{H}_1\right\},\\ \text{prox}_{\lambda g}\left(x^\dagger\right) &= \arg\min_{x^\ddagger\in\mathscr{H}_2}\left\{g\left(x^\ddagger\right) + \frac{1}{2\lambda}\left\|x^\ddagger - x^\dagger\right\|^2\right\}.\end{aligned} \quad (9)$$

Note that the optimality condition of (8) is as follows:

$$0 \in \partial f\left(x^\dagger\right) + A^*\left(\frac{I - \text{prox}_{\lambda g}}{\lambda}\right)\left(Ax^\dagger\right), \quad (10)$$

which can be rewritten as

$$0 \in \mu\lambda\partial f\left(x^\dagger\right) + \mu A^*\left(I - \text{prox}_{\lambda g}\right)\left(Ax^\dagger\right), \quad (11)$$

which is equivalent to the fixed point equation

$$x^\dagger = \text{prox}_{\mu\lambda f}\left(x^\dagger - \mu A^*\left(I - \text{prox}_{\lambda g}\right)\right)\left(Ax^\dagger\right). \quad (12)$$

If $\arg\min f \cap A^{-1}(\arg\min g) \neq \emptyset$, then (1) is reduced to the following proximal split feasibility problem of finding $x^\dagger$ such that

$$x^\dagger \in \arg\min f, \qquad Ax^\dagger \in \arg\min g, \quad (13)$$

where

$$\begin{aligned}\arg\min f &= \left\{x^* \in \mathscr{H}_1 : f\left(x^*\right) \leq f\left(x^\dagger\right), \forall x^\dagger \in \mathscr{H}_1\right\},\\ \arg\min g &= \left\{x^\dagger \in \mathscr{H}_2 : g\left(x^\dagger\right) \leq g\left(x\right), \forall x \in \mathscr{H}_2\right\}.\end{aligned} \quad (14)$$

In the sequel, we will use Γ to denote the solution set of (13).

Recently, in order to solve (13), Moudafi and Thakur [9] presented the following split proximal algorithm with a way of selecting the stepsizes such that its implementation does not need any prior information about the operator norm.

Split Proximal Algorithm

Step 1 (initialization).

$$x_0 \in \mathscr{H}_1. \quad (15)$$

Step 2. Assume that x_n has been constructed and $\theta(x_n) \neq \emptyset$. Then compute x_{n+1} via the manner

$$x_{n+1} = \text{prox}_{\mu_n\lambda f}\left[x_n - \mu_n A^*\left(I - \text{prox}_{\lambda g}\right)Ax_n\right], \quad \forall n \geq 0, \quad (16)$$

where the stepsize $\mu_n = \rho_n((h(x_n) + l(x_n))/\theta^2(x_n))$ in which $0 < \rho_n < 4$, $h(x_n) = (1/2)\|(I - \text{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = (1/2)\|(I - \text{prox}_{\mu_n\lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (13) and the iterative process stops; otherwise, we set $n := n + 1$ and go to (16).

Consequently, they demonstrated the following weak convergence of the above split proximal algorithm.

Theorem 1. *Suppose that $\Gamma \neq \emptyset$. Assume that the parameters satisfy the condition:*

$$\epsilon \leq \rho_n \leq \frac{4h\left(x_n\right)}{h\left(x_n\right) + l\left(x_n\right)} - \epsilon \quad \text{for some } \epsilon > 0 \text{ small enough.} \quad (17)$$

Then the sequence x_n weakly converges to a solution of (13).

Note that the proximal mapping of g is firmly nonexpansive, namely,

$$\left\langle \text{prox}_{\lambda g}x - \text{prox}_{\lambda g}y, x - y\right\rangle \geq \left\|\text{prox}_{\lambda g}x - \text{prox}_{\lambda g}y\right\|^2, \quad \forall x, y \in \mathscr{H}_2, \quad (18)$$

and it is also the case for complement $I - \text{prox}_{\lambda g}$. Thus, $A^*(I - \text{prox}_{\lambda g})A$ is cocoercive with coefficient $1/\|A\|^2$ (recall that a mapping $B : \mathscr{H}_1 \to \mathscr{H}_1$ is said to be *cocoercive* if $\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2$ for all $x, y \in \mathscr{H}_1$ and some $\alpha > 0$). If $\mu \in (0, 1/\|A\|^2)$, then $I - \mu A^*(I - \text{prox}_{\lambda g})A$ is nonexpansive. Hence, we need to regularize (16) such that the strong convergence is obtained. This is the main purpose of this paper. In the next section, we will collect some useful lemmas and in the last section we will present our algorithm and prove its strong convergence.

2. Lemmas

Lemma 2 (see [10]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq \left(1 - \alpha_n\right)a_n + \alpha_n\sigma_n + \delta_n, \quad n \geq 0, \quad (19)$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\limsup_{n\to\infty}\sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty}\delta_n < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 3 (see [11]). *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_i}\}_{i\in\mathbb{N}}$ of $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_{n_i} < \gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\gamma_{m_k} \le \gamma_{m_k+1}, \qquad \gamma_k \le \gamma_{m_k+1}. \tag{20}$$

In fact, m_k is the largest number n in the set $\{1,\ldots,k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main results

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let $f : \mathscr{H}_1 \to \mathscr{R} \cup \{+\infty\}$ and $g : \mathscr{H}_2 \to \mathscr{R} \cup \{+\infty\}$ be two proper, lower semicontinuous convex functions and $A : \mathscr{H}_1 \to \mathscr{H}_2$ a bounded linear operator.

Now, we firstly introduce our algorithm.

Algorithm 4

Step 1 (initialization).

$$x_0 \in \mathscr{H}_1. \tag{21}$$

Step 2. Assume that x_n has been constructed. Set $h(x_n) = (1/2)\|(I - \operatorname{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = (1/2)\|(I - \operatorname{prox}_{\mu_n\lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ for all $n \in \mathbb{N}$.

If $\theta(x_n) \neq \emptyset$, then compute x_{n+1} via the manner

$$x_{n+1} = \operatorname{prox}_{\mu_n\lambda f}\left[\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n\right], \quad \forall n \ge 0, \tag{22}$$

where $u \in \mathscr{H}_1$ is a fixed point and $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ is a real number sequence and μ_n is the stepsize satisfying $\mu_n = \rho_n((h(x_n) + l(x_n))/\theta^2(x_n))$ with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (13) and the iterative process stops; otherwise, we set $n := n+1$ and go to (22).

Theorem 5. *Suppose that $\Gamma \neq \emptyset$. Assume that the parameters $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $\epsilon \le \rho_n \le (4h(x_n)/(h(x_n) + l(x_n))) - \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Then the sequence x_n converges strongly to $proj_\Gamma(u)$.

Proof. Let $x^* \in \Gamma$. Since minimizers of any function are exactly fixed points of its proximal mappings, we have $x^* = \operatorname{prox}_{\mu_n\lambda f}x^*$ and $Ax^* = \operatorname{prox}_{\lambda g}Ax^*$. By (22) and the nonexpansivity of $\operatorname{prox}_{\mu_n\lambda f}$, we derive

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \left\|\operatorname{prox}_{\mu_n\lambda f}\left[\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n\right] - \operatorname{prox}_{\mu_n\lambda f}x^*\right\|^2 \\
&\le \left\|\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n - x^*\right\|^2 \\
&= \left\|\alpha_n(u - x^*) + (1-\alpha_n) \times \left[x_n - \frac{\mu_n}{1-\alpha_n}A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n - x^*\right]\right\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + (1-\alpha_n) \times \left\|x_n - \frac{\mu_n}{1-\alpha_n}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - x^*\right\|^2.
\end{aligned} \tag{23}$$

Since $\operatorname{prox}_{\lambda g}$ is firmly nonexpansive, we deduce that $I - \operatorname{prox}_{\lambda g}$ is also firmly nonexpansive. Hence, we have

$$\begin{aligned}
&\left\langle A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n, x_n - x^*\right\rangle \\
&= \left\langle\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n, Ax_n - Ax^*\right\rangle \\
&= \left\langle\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n - \left(I - \operatorname{prox}_{\lambda g}\right)Ax^*, Ax_n - Ax^*\right\rangle \\
&\ge \left\|\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n\right\|^2 = 2h(x_n).
\end{aligned} \tag{24}$$

Note that $\nabla h(x_n) = A^*(I - \operatorname{prox}_{\lambda g})Ax_n$ and $\nabla l(x_n) = (I - \operatorname{prox}_{\mu_n\lambda f})x_n$. From (24), we obtain

$$\begin{aligned}
&\left\|x_n - \frac{\mu_n}{1-\alpha_n}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - x^*\right\|^2 \\
&= \|x_n - x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\left\|A^*(I - \operatorname{prox}_{\lambda g})Ax_n\right\|^2 - \frac{2\mu_n}{1-\alpha_n}\left\langle A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n, x_n - x^*\right\rangle \\
&= \|x_n - x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - \frac{2\mu_n}{1-\alpha_n}\langle\nabla h(x_n), x_n - x^*\rangle \\
&\le \|x_n - x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - \frac{4\mu_n h(x_n)}{1-\alpha_n}
\end{aligned}$$

$$\begin{aligned}
&= \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2 \theta^4(x_n)} \|\nabla h(x_n)\|^2 \\
&\quad - 4\rho_n \frac{h(x_n) + l(x_n)}{(1-\alpha_n)\theta^2(x_n)} h(x_n) \\
&\leq \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2 \theta^2(x_n)} \\
&\quad - 4\rho_n \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)} \frac{h(x_n)}{h(x_n) + l(x_n)} \\
&= \|x_n - x^*\|^2 - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \\
&\quad \times \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}.
\end{aligned} \tag{25}$$

By condition (C3), without loss of generality, we can assume that $(4h(x_n)/(h(x_n)+l(x_n))) - (\rho_n/(1-\alpha_n)) \geq 0$ for all $n \geq 0$. Thus, from (23) and (25), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \\
&\quad \times \Bigg[\|x_n - x^*\|^2 \\
&\qquad - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)} \Bigg] \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\leq \max\left\{ \|u - x^*\|^2, \|x_n - x^*\|^2 \right\}.
\end{aligned} \tag{26}$$

Hence, $\{x_n\}$ is bounded.

Let $z = P_\Gamma u$. From (26), we deduce

$$\begin{aligned}
0 &\leq \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\leq \alpha_n \|u - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\end{aligned} \tag{27}$$

We consider the following two cases.

Case 1. One has $\|x_{n+1} - z\| \leq \|x_n - z\|$ for every $n \geq n_0$ large enough.

In this case, $\lim_{n\to\infty} \| x_n - z \|$ exists as finite and hence

$$\lim_{n\to\infty} (\|x_{n+1} - z\| - \|x_n - z\|) = 0. \tag{28}$$

This together with (27) implies that

$$\rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \longrightarrow 0. \tag{29}$$

Since $\liminf_{n\to\infty} \rho_n((4h(x_n)/(h(x_n)+l(x_n))) - (\rho_n/(1-\alpha_n))) \geq 2\epsilon$ (by condition (C3)), we get

$$\frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \longrightarrow 0. \tag{30}$$

Noting that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded, we deduce immediately that

$$\lim_{n\to\infty} (h(x_n) + l(x_n)) = 0. \tag{31}$$

Therefore,

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0. \tag{32}$$

Next, we prove

$$\limsup_{n\to\infty} \langle u - z, x_n - z \rangle \leq 0. \tag{33}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ satisfying $x_{n_i} \rightharpoonup z^\dagger$ and

$$\limsup_{n\to\infty} \langle u - z, x_n - z \rangle = \lim_{i\to\infty} \langle u - z, x_{n_i} - z \rangle. \tag{34}$$

By the lower semicontinuity of h, we get

$$0 \leq h(z^\dagger) \leq \liminf_{i\to\infty} h(x_{n_i}) = \lim_{n\to\infty} h(x_n) = 0. \tag{35}$$

So,

$$h(z^\dagger) = \frac{1}{2} \left\| \left(I - \operatorname{prox}_{\lambda g} \right) A z^\dagger \right\| = 0. \tag{36}$$

That is, $Az^\dagger$ is a fixed point of the proximal mapping of g or equivalently $0 \in \partial g(Az^\dagger)$. In other words, $Az^\dagger$ is a minimizer of g.

Similarly, from the lower semicontinuity of l, we get

$$0 \leq l(z^\dagger) \leq \liminf_{i\to\infty} l(x_{n_i}) = \lim_{n\to\infty} l(x_n) = 0. \tag{37}$$

Therefore,

$$l(z^\dagger) = \frac{1}{2} \left\| \left(I - \operatorname{prox}_{\mu_n \lambda f} \right) z^\dagger \right\| = 0. \tag{38}$$

That is, $z^\dagger$ is a fixed point of the proximal mapping of f or equivalently $0 \in \partial f(z^\dagger)$. In other words, $z^\dagger$ is a minimizer of f. Hence, $z^\dagger \in \Gamma$. Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - z, x_n - z \rangle &= \lim_{i\to\infty} \langle u - z, x_{n_i} - z \rangle \\
&= \langle u - z, z^\dagger - z \rangle \leq 0.
\end{aligned} \tag{39}$$

From (22), we have

$$\begin{aligned}
&\left\|x_{n+1}-z\right\|^2 \\
&\leq \Big\| \alpha_n(u-z)+(1-\alpha_n) \\
&\qquad \times \left[x_n-\frac{\mu_n}{1-\alpha_n}A^*\left(I-\operatorname{prox}_{\lambda g}\right)Ax_n-z\right]\Big\|^2 \\
&=(1-\alpha_n)^2\left\|x_n-\frac{\mu_n}{1-\alpha_n}A^*(I-\operatorname{prox}_{\lambda g})Ax_n-z\right\|^2 \\
&\quad +\alpha_n^2\|u-z\|^2+2\alpha_n(1-\alpha_n) \\
&\quad \times\left\langle x_n-\frac{\mu_n}{1-\alpha_n}A^*\left(I-\operatorname{prox}_{\lambda g}\right)Ax_n-z,u-z\right\rangle \\
&\leq(1-\alpha_n)^2\left\|x_n-z\right\|^2+\alpha_n^2\|u-z\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle x_n-z,u-z\rangle \\
&\quad -2\alpha_n\mu_n\langle\nabla h(x_n),u-z\rangle \\
&\leq(1-\alpha_n)\left\|x_n-z\right\|^2 \\
&\quad +\alpha_n\Big(\alpha_n\|u-z\|^2+2(1-\alpha_n)\langle x_n-z,u-z\rangle \\
&\qquad +2\mu_n\left\|\nabla h(x_n)\right\|\|u-z\|\Big).
\end{aligned} \tag{40}$$

Since ∇h is Lipschitz continuous with Lipschitzian constant $\|A\|^2$ and ∇l is nonexpansive, $\nabla h(u_n)$,$\nabla l(u_n)$, and $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ are bounded. Note that $\mu_n\|\nabla h(x_n)\| = \rho_n((h(x_n)+l(x_n))/\theta^2(x_n))\|\nabla h(x_n)\|$. Thus, $\mu_n\|\nabla h(x_n)\| \to 0$ by (32). From Lemma 2, (39), and (40) we deduce that $x_n \to z$.

Case 2. There exists a subsequence $\{\|x_{n_j}-z\|\}$ of $\{\|x_n-z\|\}$ such that

$$\left\|x_{n_j}-z\right\|<\left\|x_{n_j+1}-z\right\|, \tag{41}$$

for all $j \geq 1$. By Lemma 3, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty}m_k = +\infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\left\|x_{m_k}-z\right\|\leq\left\|x_{m_{k+1}}-z\right\|,\qquad \left\|x_k-z\right\|\leq\left\|x_{m_{k+1}}-z\right\|. \tag{42}$$

Consequently,

$$\begin{aligned}
0&\leq\lim_{k\to\infty}\left(\left\|x_{m_{k+1}}-z\right\|-\left\|x_{m_k}-z\right\|\right) \\
&\leq\limsup_{n\to\infty}\left(\left\|x_{n+1}-z\right\|-\left\|x_n-z\right\|\right) \\
&\leq\limsup_{n\to\infty}\left(\alpha_n\|u-z\|+(1-\alpha_n)\left\|x_n-z\right\|-\left\|x_n-z\right\|\right) \\
&=\limsup_{n\to\infty}\alpha_n\left(\|u-z\|-\left\|x_n-z\right\|\right)=0.
\end{aligned} \tag{43}$$

Hence,

$$\lim_{k\to\infty}\left(\left\|x_{m_{k+1}}-z\right\|-\left\|x_{m_k}-z\right\|\right)=0. \tag{44}$$

By a similar argument as that of Case 1, we can prove that

$$\begin{aligned}
&\limsup_{k\to\infty}\left\langle u-z,x_{m_k}-z\right\rangle\leq 0, \\
&\left\|x_{m_k+1}-z\right\|^2\leq\left(1-\alpha_{m_k}\right)\left\|x_{m_k}-z\right\|^2+\alpha_{m_k}\sigma_{m_k},
\end{aligned} \tag{45}$$

where $\sigma_{m_k} = \alpha_{m_k}\|u-z\|^2 + 2(1-\alpha_{m_k})\langle x_{m_k}-z,u-z\rangle + 2\mu_{m_k}\|\nabla h(x_{m_k})\|\|u-z\|$.

In particular, we get

$$\begin{aligned}
&\alpha_{m_k}\left\|x_{m_k}-z\right\|^2 \\
&\leq\left\|x_{m_k}-z\right\|^2-\left\|x_{m_k+1}-z\right\|+\alpha_{m_k}\sigma_{m_k} \\
&\leq\alpha_{m_k}\sigma_{m_k}.
\end{aligned} \tag{46}$$

Then,

$$\limsup_{k\to\infty}\left\|x_{m_k}-z\right\|^2\leq\limsup_{k\to\infty}\sigma_{m_k}\leq 0. \tag{47}$$

Thus, from (42) and (44), we conclude that

$$\limsup_{k\to\infty}\left\|x_k-z\right\|\leq\limsup_{k\to\infty}\left\|x_{m_k+1}-z\right\|=0. \tag{48}$$

Therefore, $x_n \to z$. This completes the proof. □

Remark 6. Note that problem (13) was considered, for example, in [12, 13]; however, the iterative methods proposed to solve it need to know a priori the norm of the bounded linear operator A.

Remark 7. We would like also to emphasize that by taking $f = \delta_C$, $g = \delta_Q$ the indicator functions of two nonempty closed convex sets C, Q of H_1 and H_2 respectively, our algorithm (22) reduces to

$$\begin{aligned}
&x_{n+1} \\
&\quad=\operatorname{proj}_C\left[\alpha_n u+(1-\alpha_n)x_n-\mu_n A^*\left(I-\operatorname{proj}_Q\right)Ax_n\right], \\
&\qquad\forall n\geq 0.
\end{aligned} \tag{49}$$

We observe that (49) is simpler than the one in [14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2–4, pp. 221–239, 1994.

[2] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[3] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces," *Inverse Problems*, vol. 26, no. 10, Article ID 105018, 17 pages, 2010.

[4] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[5] J. Zhao and Q. Yang, "Several solution methods for the split feasibility problem," *Inverse Problems*, vol. 21, no. 5, pp. 1791–1799, 2005.

[6] Y. Dang and Y. Gao, "The strong convergence of a KM-CQ-like algorithm for a split feasibility problem," *Inverse Problems*, vol. 27, no. 1, Article ID 015007, 9 pages, 2011.

[7] F. Wang and H. Xu, "Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem," *Journal of Inequalities and Applications*, vol. 2010, Article ID 102085, 13 pages, 2010.

[8] Y. Yao, W. Jigang, and Y. Liou, "Regularized methods for the split feasibility problem," *Abstract and Applied Analysis*, vol. 2012, Article ID 140679, 13 pages, 2012.

[9] A. Moudafi and B. S. Thakur, "Solving proximal split feasibility problems without prior knowledge of operator norms," *Optimization Letters*, 2013.

[10] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[11] P. E. Maingé, "Strong convergence of projected subgradient methods for non-smooth and non-strictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.

[12] A. Moudafi, "Split monotone variational inclusions," *Journal of Optimization Theory and Applications*, vol. 150, no. 2, pp. 275–283, 2011.

[13] C. Byrne, Y. Censor, A. Gibaliand, and S. Reich, "Weak and strong convergence of algorithms for the split common null point problem," *Journal of Nonlinear and Convex Analysis*, vol. 13, pp. 759–775, 2012.

[14] Y. Yu, "An explicit method for the split feasibility problem with self-adaptive step sizes," *Abstract and Applied Analysis*, vol. 2012, Article ID 432501, 9 pages, 2012.