Research Article

Explicit Scheme for Fixed Point Problem for Nonexpansive Semigroup and Split Equilibrium Problem in Hilbert Space

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We establish an iterative method for finding a common element of the set of fixed points of nonexpansive semigroup and the set of split equilibrium problems. Under suitable conditions, some strong convergence theorems are proved. Our works improve previous results for nonexpansive semigroup.

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ which is the set of real numbers. The equilibrium problem introduced by Blum and Oettli [1] for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1) (see [2–8]). The split equilibrium problem was introduced by Moudafi in [9]; he considers the following pair of equilibrium problems in different spaces: let H_1 and H_2 be two real Hilbert spaces, let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and let $A : H_1 \to H_2$ be a bounded linear operator, and consider the nonempty closed convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$; then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{2}$$

and such that

$$y^* \in Ax^* \in Q, \quad F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{3}$$

The solution set of SEP (2)-(3) is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

Recall that mapping T of C into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{4}$$

Let a family $S = (T(s))_{s \geq 0}$ be a nonexpansive semigroup on H, if it satisfies the following conditions:

(S1) $T(0)x = x \ \forall x \in H$;

(S2) $T(s + t) = T(s)T(t)$ for each $s, t \geq 0$;

(S3) $\|T(s)x - T(t)y\| \leq \|x - y\|$ for each $x, y \in H$ and $s \geq 0$;

(S4) $\forall x \in H, s \to T(s)x$ is continuous.

The set of all the common fixed points of family S is denoted by $F_{ix}(S)$; that is,

$$F_{ix}(S) = \{x \in C : T(s)x = x, 0 \leq s < \infty\} = \bigcap_{0 \leq s < \infty} F_{ix}(T(s)), \tag{5}$$

where $F_{ix}(T(s))$ is the set of fixed points of $T(s)$. It is well known that $F_{ix}(S)$ is closed and convex.

In 2010, Tian [10] introduced the following general iterative scheme for finding an element of set of solutions to the fixed point of nonexpansive mapping in a Hilbert space. Define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \mu \alpha_n B) T x_n, \tag{6}$$

where B is k-Lipschitzian and η-strongly monotone operator. Then he proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution $x^* \in F_{ix}(T)$ of the variational inequality

$$\langle (\gamma f - \mu B) x^*, x - x^* \rangle \le 0, \quad \forall x \in F_{ix}(T). \tag{7}$$

In 2011, Ceng et al. [11] added the metric project to the method of Tian (6) and studied the following explicit iterative scheme to find fixed points:

$$x_{n+1} = P_C \left[\alpha_n \gamma f(x_n) + (1 - \mu \alpha_n B) T x_n \right]. \tag{8}$$

They prove the strong convergence of x_n to a fixed point $x^* \in F_{ix}(T)$ of the same variational inequality (7).

In 2008, Plubtieng and Punpaeng [12] introduced the following implicit iterative algorithm to prove a strong convergence theorem for fixed point problem with nonexpansive semigroup:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s) x_n \, ds, \tag{9}$$

where x_n is a continuous net and s_n is a positive real divergent net.

In 2014, Kazmi and Rizvi [13] studied the following implicit iterative algorithm. Under some assumptions, they obtain some strong convergence theorem for EP (1) and the fixed point problem:

$$u_n = T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n \, ds, \tag{10}$$

where s_n and r_n are the continuous nets in $(0, 1)$.

Motivated and inspired by [10–13], we introduce an explicit iterative scheme for finding a common element of the set of solutions SEP and fixed point for a nonexpansive semigroup in real Hilbert spaces. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$u_n = T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$

$$x_{n+1} = P_C \left[\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n \, ds \right]. \tag{11}$$

Under suitable conditions, some strong convergence theorems for approximating to these common elements are proved.

2. Preliminaries

This section collects some results that will be used in the proofs.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively.

It is well known that, for all $x, y \in H$ and $\lambda \in [0, 1]$, the following holds:

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda) \|x - y\|^2. \tag{12}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z \rangle \ge 0, \quad \forall x \in H, \ z \in C. \tag{13}$$

It is easy to see that (13) is equivalent to

$$\|x - z\|^2 \ge \|x - P_C x\|^2 + \|P_C x - z\|^2, \quad \forall x \in H, \ z \in C. \tag{14}$$

Let $B : C \to H$ be a nonlinear mapping. Recall the following definitions.

Definition 1. B is said to be

(i) monotone if

$$\langle Bx - By, x - y \rangle \ge 0, \quad \forall x, y \in C, \tag{15}$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \ge \alpha \|x - y\|^2, \quad \forall x, y \in C, \tag{16}$$

for such a case, B is said to be α-strongly monotone,

(iii) α-inverse strongly monotone (α-ism) if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \ge \alpha \|Bx - By\|^2, \quad \forall x, y \in C, \tag{17}$$

(iv) k-Lipschitz continuous if there exists a constant $k \ge 0$ such that

$$\|Bx - By\| \le k \|x - y\|, \quad \forall x, y \in C. \tag{18}$$

Remark 2. Let $\mathcal{F} = \mu B - \gamma f$, where B is a k-Lipschitz and η-strongly monotone operator on H with $k > 0$ and f is a Lipschitz mapping on H with coefficient $L > 0, 0 < \gamma \le \mu \eta / L$. It is a simple matter to see that the operator $\mathcal{F}$ is $(\mu \eta - \gamma L)$-strongly monotone over H; that is,

$$\langle \mathcal{F} x - \mathcal{F} y, x - y \rangle \ge (\mu \eta - \gamma L) \|x - y\|^2, \tag{19}$$

$$\forall (x, y) \in H \times H.$$

Lemma 3 (see [14]). *Let T be a nonexpansive mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever the sequence of x_n is weakly convergent to x and $(I - T)x_n$ is strongly convergent to y, then $(I - T)x = y$.*

Lemma 4 (see [15]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $(T(s))_{s \geq 0}$ be a nonexpansive semigroup on C. Then, for each $h \geq 0$,*

$$\lim_{t \to +\infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s) \, x \, ds - T(h) \frac{1}{t} \int_0^t T(s) \, x \, ds \right\| = 0. \tag{20}$$

Definition 5 (see [9]). A mapping $T : H \to H$ is said to be averaged if it can be written as the average of the identity mapping and a nonexpansive mapping; that is,

$$T = (1 - \varepsilon) I + \varepsilon S, \tag{21}$$

where $\varepsilon \in (0, 1)$, $S : H \to H$ is nonexpansive, and I is the identity operator on H.

Proposition 6 (see [9]).

(i) *If $T = (1 - \varepsilon)S + \varepsilon V$, where $S : H \to H$ is averaged, $V : H \to H$ is nonexpansive, and $\varepsilon \in (0, 1)$, then T is averaged.*

(ii) *The composite of finitely many averaged mappings is averaged.*

(iii) *If T is ν-ism, then, for $\gamma > 0$, γT is (ν/γ)-ism.*

(iv) *T is averaged if, and only if, its complement $I - T$ is ν-ism for some $\nu > 1/2$.*

Assumption 7 (see [1]). For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{22}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 8 (see [2]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{23}$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \tag{24}$$

for all $x \in H$. Then the following hold:

(1) *T_r^F is single-valued;*

(2) *T_r^F is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\| T_r^F x - T_r^F y \right\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{25}$$

(3) *$F_{ix}(T_r^F) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 9 (see [16]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be a bifunction. Let $x \in C$ and $r_1, r_2 \in (0, \infty)$. Then*

$$\left\| T_{r_1}^F x - T_{r_2}^F x \right\| \leq \left| 1 - \frac{r_2}{r_1} \right| \left(\left\| T_{r_1}^F x \right\| + \| x \| \right). \tag{26}$$

Lemma 10 (see [17]). *Assume that $\lambda \in (0, 1)$ and $\mu > 0$. Let B be a $B : H \to H$ which is η-strongly monotone and θ-Lipschitzian on H with $\theta > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\theta^2$. Then the operator $I - \lambda \mu B$ is contraction; that is, $\|I - \lambda \mu B\| \leq (1 - \lambda \tau)$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\theta^2)} \in (0, 1]$.*

Lemma 11 (see [10]). *Let H be a Hilbert space, $f : H \to H$ is a contractive mapping with constant $\beta \in (0, 1)$. $B : H \to H$ is θ-Lipschitzian and η-strongly monotone operator with $\theta > 0$, $\eta > 0$. Then, for $0 < \sigma < \mu\eta/\beta$,*

$$\langle x - y, (\mu B - \sigma f) x - (\mu B - \sigma f) y \rangle \geq (\mu\eta - \sigma\beta) \| x - y \|^2,$$
$$x, y \in H. \tag{27}$$

That is, $\mu B - \sigma f$ is strongly monotone with coefficient $\mu\eta - \sigma\beta$.

Lemma 12. *Let H be a real Hilbert space. Then the following well-known results hold: $\forall x, y \in H$,*

$$\| x + y \|^2 \leq \| x \|^2 + 2 \langle y, x + y \rangle. \tag{28}$$

Lemma 13 (see [18]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \delta_n, \quad n \geq 0, \tag{29}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} (\delta_n/\gamma_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

3. The General Explicit Iterative Method

Let $f : H_1 \to H_1$ be a contractive mapping with constant $\beta \in (0, 1)$ and let $B : H_1 \to H_1$ be η-strongly monotone and θ-Lipschitzian with $\theta > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\theta^2$ and $0 < \gamma < \mu(\eta - \mu\theta^2/2)/\beta = \tau/\beta$. Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\Gamma = F_{ix}(S) \bigcap \Omega \neq \emptyset$. Assume $\{r_n\}$ and $\{s_n\}$ are the continuous nets of positive real numbers such that $\lim_{n \to 0} r_n = r > 0$ and $\lim_{n \to 0} s_n = +\infty$.

In this section, we introduce the following explicit iterative scheme that the nets $\{u_n\}$ and $\{x_n\}$ are generated by

$$u_n = T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$

$$x_{n+1} = P_C \left[\alpha_n \gamma f \left(x_n \right) + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \int_0^{s_n} T \left(s \right) u_n \, ds \right], \tag{30}$$

where $\delta \in (0, 1/L)$, L is the spectral radius of the operator $A^* A$, and A^* is the adjoint of A.

We prove the strong convergence of $\{u_n\}$ and $\{x_n\}$ to a fixed point x^* of S which solves the following variational inequality:

$$\langle (\mu F - \gamma f) x^*, x^* - \overline{x} \rangle \leq 0, \quad \forall \overline{x} \in \Gamma = F_{ix} (S) \bigcap \Omega. \tag{31}$$

In the sequel, we denote by y_n the sequence defined by

$$y_n = \frac{1}{s_n} \int_0^{s_n} T \left(s \right) u_n \, ds. \tag{32}$$

Next, we will prove first a lemma and then some corollaries to be used in the proofs for the main result of this section.

Theorem 14. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ are the bifunctions satisfying Assumption 7 and F_2 is upper semicontinuous in the first argument. Let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by (30), and suppose that the sequence $\{\alpha_n\}$ satisfies the following conditions:*

(a) *$\alpha_n \in (0, 1)$ and $\lim_{n \to \infty} \alpha_n = 0$;*

(b) *$\sum_{n=0}^{\infty} \alpha_n = 0$;*

(c) *either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \Rightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1$.*

Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $x^* \in \Gamma = F_{ix}(S) \bigcap \Omega$, where $x^* = P_\Gamma (I - \mu B + \gamma f) x^*$, which is the unique solution of the variational inequality (31).

Proof. We divide the proof into several steps.

(i) $\{x_n\}$ is well defined.

For $\alpha_n \in (0, 1)$ and $\forall x \in H_1$, define a mapping $G : H_1 \rightarrow H_2$ by

$$Gx = P_C \left[\alpha_n \gamma f \left(x \right) + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \right.$$

$$\left. \times \int_0^{s_n} T \left(s \right) T_{r_n}^{F_1} \left(x + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x \right) ds \right]. \tag{33}$$

According to Lemma 8, we can easily know that $T_{r_1}^{F_1}$ and $T_{r_2}^{F_2}$ both are firmly nonexpansive mappings and are averaged

operators. From Proposition 6, we can obtain that the operator $(I + \delta A^* (T_{r_n}^{F_2} - I) A)$ is averaged and hence nonexpansive. Following Lemma 9 and $\forall x, y \in H_1$, we get

$$\| Gx - Gy \|$$

$$= \left\| P_C \left[\alpha_n \gamma f + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \right. \right.$$

$$\left. \times \int_0^{s_n} T \left(s \right) T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I \right) A \right) ds \right] x$$

$$- P_C \left[\alpha_n \gamma f + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \right.$$

$$\left. \left. \times \int_0^{s_n} T \left(s \right) T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I \right) A \right) ds \right] y \right\|$$

$$\leq \left\| \left[\alpha_n \gamma f + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \right. \right.$$

$$\left. \times \int_0^{s_n} T \left(s \right) T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I \right) A \right) ds \right] x$$

$$- \left[\alpha_n \gamma f + \left(I - \mu \alpha_n B \right) \frac{1}{s_n} \right.$$

$$\left. \left. \times \int_0^{s_n} T \left(s \right) T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I \right) A \right) ds \right] y \right\|$$

$$\leq \alpha_n \gamma \| f \left(x \right) - f \left(y \right) \| + \left(1 - \alpha_n \tau \right)$$

$$\times \left\| \frac{1}{s_n} \int_0^{s_n} T \left(s \right) \left[T_{r_n}^{F_1} \left(x + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x \right) ds \right. \right.$$

$$\left. \left. - T_{r_n}^{F_1} \left(y + \delta A^* \left(T_{r_n}^{F_2} - I \right) A y \right) ds \right] \right\|$$

$$\leq \alpha_n \gamma \beta \| x - y \| + \left(1 - \alpha_n \tau \right)$$

$$\times \left\| T_{r_n}^{F_1} \left(x + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x \right) ds \right.$$

$$\left. - T_{r_n}^{F_1} \left(y + \delta A^* \left(T_{r_n}^{F_2} - I \right) A y \right) ds \right\|$$

$$\leq \alpha_n \gamma \beta \| x - y \| + \left(1 - \alpha_n \tau \right) \| x - y \|$$

$$= \left(1 - \alpha_n \left(\tau - \alpha_n \beta \right) \right) \| x - y \|. \tag{34}$$

Since $0 < 1 - \alpha_n (\tau - \alpha_n \beta) < 1$, we can claim that the mapping G is a contraction mapping. Therefore, by Banach contraction principle, G has the unique fixed point x^*.

(ii) The sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ are bounded.

Letting $p \in \Gamma = F_{ix}(S) \bigcap \Omega$, we obtain that $p = T_{r_n}^{F_1} p$, $Ap = T_{r_n}^{F_2} Ap$, and $p = T(s)p$.

From (30), we obtain

$$\|u_n - p\|^2$$

$$= \left\|T_{r_n}^{F_1}(I + \delta A^*(T_{r_n}^{F_2} - I)A)x_n - p\right\|^2$$

$$= \left\|T_{r_n}^{F_1}(I + \delta A^*(T_{r_n}^{F_2} - I)A)x_n - T_{r_n}^{F_1}p\right\|^2$$

$$\leq \left\|x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p\right\|^2$$

$$\leq \|x_n - p\|^2 + \left\|\delta A^*(T_{r_n}^{F_2} - I)Ax_n\right\|^2$$

$$\quad + 2\delta \left\langle x_n - p, A^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle$$

$$\leq \|x_n - p\|^2 + \delta^2 \left\langle \left(T_{r_n}^{F_2} - I\right)Ax_n, AA^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle$$

$$\quad + 2\delta \left\langle A\left(x_n - p\right), \left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle$$

$$\leq \|x_n - p\|^2 + L\delta^2 \left\langle \left(T_{r_n}^{F_2} - I\right)Ax_n, \left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle$$

$$\quad + 2\delta \left\langle A\left(x_n - p\right) + \left(T_{r_n}^{F_2} - I\right)Ax_n\right.$$

$$\quad \left. - \left(T_{r_n}^{F_2} - I\right)Ax_n, A^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle$$

$$\leq \|x_n - p\|^2 + L\delta^2 \left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2$$

$$\quad + 2\delta \left\{ \left\langle T_{r_n}^{F_2}Ax_n - Ap, \left(T_{r_n}^{F_2} - I\right)Ax_n\right\rangle \right.$$

$$\quad \left. - \left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2 \right\}$$

$$\leq \|x_n - p\|^2 + L\delta^2 \left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2$$

$$\quad + 2\delta \left\{ \frac{1}{2}\left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2 - \left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2 \right\}$$

$$\leq \|x_n - p\|^2 + L\delta^2 \left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2 - \delta\left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2$$

$$= \|x_n - p\|^2 + \delta\left(L\delta - 1\right)\left\|\left(T_{r_n}^{F_2} - I\right)Ax_n\right\|^2. \tag{35}$$

Since $\delta \in (0, 1/L)$, we have

$$\|u_n - p\| \leq \|x_n - p\|. \tag{36}$$

From (32), we have

$$\|y_n - p\| = \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|$$

$$\leq \frac{1}{s_n}\left\|\int_0^{s_n}\left(T(s)u_n - T(s)p\right)ds\right\| \tag{37}$$

$$\leq \|u_n - p\| \leq \|x_n - p\|.$$

Further, using (31) again, we obtain

$$\|x_n - p\|$$

$$= \left\|P_C\left[\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right] - p\right\|$$

$$\leq \left\|\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|$$

$$= \left\|\alpha_n\left(\gamma f(x_n) - \mu Bp\right) + (I - \mu\alpha_n B)\frac{1}{s_n}\right.$$

$$\quad \left. \times \int_0^{s_n} T(s)u_n ds - (I - \mu\alpha_n B)p\right\|$$

$$\leq \alpha_n\left\|\gamma f(x_n) - \mu Bp\right\| + (1 - \alpha_n\tau)$$

$$\quad \times \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|$$

$$\leq \alpha_n\left\|\gamma f(x_n) - \mu Bp\right\| + (1 - \alpha_n\tau)\frac{1}{s_n}$$

$$\quad \times \int_0^{s_n}\left\|T(s)u_n - T(s)p\right\|ds$$

$$\leq \alpha_n\gamma\left\|f(x_n) - f(p)\right\| + \alpha_n\left\|\gamma f(p) - \mu Bp\right\|$$

$$\quad + (1 - \alpha_n\tau)\left\|u_n - p\right\|$$

$$= \left[1 - \alpha_n\left(\tau - \gamma\beta\right)\right]\|u_n - p\| + \alpha_n\left\|\gamma f(p) - \mu Bp\right\|. \tag{38}$$

It follows from (40) and induction that

$$\|x_n - p\| \leq \frac{\|\gamma f(p) - \mu Bp\|}{\tau - \gamma\beta}. \tag{39}$$

Hence, the sequence $\{x_n\}$ is bounded and therefore $\{u_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded.

(iii) Consider that $\lim_{n\Rightarrow\infty}\|x_n - u_n\| = 0$.

According to (35) and Lemma 12, we obtain

$$\|x_n - p\|^2$$

$$\leq (1 - \alpha_n\tau)^2\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\|^2$$

$$\quad + 2\alpha_n\left\langle\gamma f(x_n) - \gamma f(p) + \gamma f(p) - \mu Bp, x_n - p\right\rangle$$

$$\leq \left(1 + \alpha_n^2\tau^2\right)\|u_n - p\|^2 + 2\alpha_n\gamma\beta\|x_n - p\|^2$$

$$\quad + 2\alpha_n\left\langle\gamma f(p) - \mu Bp, x_n - p\right\rangle$$

$$\leq \|u_n - p\|^2 + \alpha_n \tau^2 \|x_n - p\|^2 + 2\alpha_n \gamma \beta \|x_n - p\|^2$$

$$+ 2\alpha_n \|\gamma f(p) - \mu B p\| \|x_n - p\|$$

$$\leq \|x_n - p\|^2 + \delta (L\delta - 1) \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2$$

$$+ \alpha_n \tau^2 \|x_n - p\|^2 + 2\alpha_n \gamma \beta \|x_n - p\|^2$$

$$+ 2\alpha_n \|\gamma f(p) - \mu B p\| \|x_n - p\|. \tag{40}$$

From (40), we obtain

$$\delta (1 - L\delta) \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2$$

$$\leq \alpha_n \left(\left(\tau^2 + 2\gamma\beta\right) \|x_n - p\|^2 \right. \tag{41}$$

$$\left. + 2 \|\gamma f(p) - \mu B p\| \|x_n - p\| \right).$$

Since x_n is bounded, $\lim_{n\to\infty} \alpha_n = 0$, and $\delta(1 - L\delta) > 0$, we obtain that

$$\lim_{n\to\infty} \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2 = 0. \tag{42}$$

From (30), we have

$$\|u_n - p\|^2$$

$$= \left\| T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I\right) A\right) x_n - p \right\|^2$$

$$= \left\| T_{r_n}^{F_1} \left(I + \delta A^* \left(T_{r_n}^{F_2} - I\right) A\right) x_n - T_{r_n}^{F_1} p \right\|^2$$

$$\leq \left\langle u_n - p, x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n - p \right\rangle$$

$$= \frac{1}{2} \left\{ \|u_n - p\|^2 + \left\| x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n - p \right\|^2 \right.$$

$$\left. - \left\| u_n - p - \left[x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n - p\right] \right\|^2 \right\} \tag{43}$$

$$\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 \right.$$

$$\left. - \left\| u_n - x_n - \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2 \right\}$$

$$\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \right.$$

$$- \delta^2 \left\| A^* \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2$$

$$\left. + 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\| \right\}.$$

Hence, we obtain

$$\|u_n - p\|^2$$

$$\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 - \delta^2 \left\| A^* \left(T_{r_n}^{F_2} - I\right) A x_n \right\|^2$$

$$+ 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\|$$

$$\leq \|x_n - p\|^2 - \|u_n - x_n\|^2$$

$$+ 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\|. \tag{44}$$

It follows from (40) and (44) that

$$\|x_n - p\|^2$$

$$\leq \|u_n - p\|^2 + \alpha_n \tau^2 \|x_n - p\|^2 + 2\alpha_n \gamma \beta \|x_n - p\|^2$$

$$+ 2\alpha_n \|\gamma f(p) - \mu B p\| \|x_n - p\|$$

$$\leq \|x_n - p\|^2 - \|u_n - x_n\|^2$$

$$+ 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\| \tag{45}$$

$$+ \alpha_n \tau^2 \|x_n - p\|^2 + 2\alpha_n \gamma \beta \|x_n - p\|^2$$

$$+ 2\alpha_n \|\gamma f(p) - \mu B p\| \|x_n - p\|$$

$$= \|x_n - p\|^2 - \|u_n - x_n\|^2$$

$$+ 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\| + \alpha_n M_1,$$

where $M_1 = \tau^2 \|x_n - p\|^2 + 2\gamma\beta \|x_n - p\|^2 + 2\|\gamma f(p) - \mu B p\| \|x_n - p\|$. From (46), we obtain

$$\|u_n - x_n\|^2 \leq 2\delta \|A(u_n - x_n)\| \left\| \left(T_{r_n}^{F_2} - I\right) A x_n \right\| + \alpha_n M_1. \tag{46}$$

Since x_n is bounded, $\lim_{n\to\infty} \alpha_n = 0$, and $\delta > 0$ and considering (42), we can claim that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{47}$$

(iv) Consider that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From (30) and Lemma 13, we have

$$\|u_n - u_{n-1}\|$$

$$= \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n\right) \right.$$

$$\left. - T_{r_{n-1}}^{F_1} \left(x_{n-1} + \delta A^* \left(T_{r_{n-1}}^{F_2} - I\right) A x_{n-1}\right) \right\|$$

$$\leq \left\| \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n\right) \right.$$

$$\left. - \left(x_{n-1} + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_{n-1}\right) \right\|$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n\right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\|$$

$$\leq \left\| x_n - x_{n-1} - \delta A^* A \left(x_n - x_{n-1} \right) \right\|$$

$$+ \delta \left\| A \right\| \left\| T_{r_n}^{F_2} A x_n - T_{r_{n-1}}^{F_2} A x_{n-1} \right\|$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|$$

$$\leq \left(\left\| x_n - x_{n-1} \right\|^2 - 2\delta \left\| A \left(x_n - x_{n-1} \right) \right\|^2 \right.$$

$$\left. + \delta^2 \left\| A \right\|^4 \left\| x_n - x_{n-1} \right\|^2 \right)^{1/2}$$

$$+ \delta \left\| A \right\| \left(\left\| A \left(x_n - x_{n-1} \right) \right\| \right.$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_2} A x_n - A x_{n-1} \right\| \right)$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|$$

$$\leq \left(1 - 2\delta \left\| A \right\|^2 + \delta^2 \left\| A \right\|^4 \right)^{1/2} \left\| x_n - x_{n-1} \right\|$$

$$+ \delta \left\| A \right\|^2 \left(\left\| x_n - x_{n-1} \right\| \right.$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_2} A x_n - A x_{n-1} \right\| \right)$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|$$

$$\leq \left(1 - \delta \left\| A \right\|^2 \right) \left\| x_n - x_{n-1} \right\|$$

$$+ \delta \left\| A \right\|^2 \left(\left\| x_n - x_{n-1} \right\| + \left| 1 - \delta \left\| A \right\| \frac{r_{n-1}}{r_n} \right| \right.$$

$$\times \left\| T_{r_n}^{F_2} A x_n - A x_{n-1} \right\| \right)$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|$$

$$= \left\| x_n - x_{n-1} \right\| + \delta \left\| A \right\| \left| 1 - \frac{r_{n-1}}{r_n} \right|$$

$$\times \left\| T_{r_n}^{F_2} A x_n - A x_{n-1} \right\|$$

$$+ \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right.$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|$$

$$= \left\| x_n - x_{n-1} \right\| + \left| 1 - \frac{r_{n-1}}{r_n} \right| \left(\delta \left\| A \right\| \varepsilon_n + \xi_n \right),$$

$$(48)$$

where

$$\varepsilon_n = \left\| T_{r_n}^{F_2} A x_n - A x_{n-1} \right\|,$$

$$\xi_n = \left\| T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right. \quad (49)$$

$$\left. - \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right) \right\|.$$

From (32), we obtain

$$\left\| y_n - y_{n-1} \right\|$$

$$= \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s) u_{n-1} ds \right\|$$

$$\leq \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - \frac{1}{s_n} \int_0^{s_n} T(s) u_{n-1} ds \right\|$$

$$+ \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_{n-1} ds - \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s) u_{n-1} ds \right\|$$

$$\leq \frac{1}{s_n} \int_0^{s_n} \left\| T(s) \left(u_n - u_{n-1} \right) \right\| ds$$

$$+ \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_{n-1} ds - \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s) u_{n-1} ds \right\|$$

$$\leq \left\| u_n - u_{n-1} \right\| + \left| \frac{1}{s_n} - \frac{1}{s_{n-1}} \right| \left\| \int_0^{s_{n-1}} T(s) u_{n-1} ds \right\|$$

$$+ \frac{1}{s_n} \left\| \int_{s_{n-1}}^{s_n} T(s) u_{n-1} ds \right\|.$$

$$(50)$$

Following (48) and (50), we obtain

$$\left\| y_n - y_{n-1} \right\|$$

$$\leq \left(\left\| x_n - x_{n-1} \right\| + \left| 1 - \frac{r_{n-1}}{r_n} \right| \left(\delta \left\| A \right\| \varepsilon_n + \xi_n \right) \right)$$

$$+ \left| \frac{1}{s_n} - \frac{1}{s_{n-1}} \right| \left\| \int_0^{s_{n-1}} T(s) u_{n-1} ds \right\| + \frac{1}{s_n} \left\| \int_{s_{n-1}}^{s_n} T(s) u_{n-1} ds \right\|.$$

$$(51)$$

Using (30) again, we obtain

$$\left\| x_{n+1} - x_n \right\|$$

$$= \left\| P_C \left[\alpha_n \gamma f \left(x_n \right) + \left(I - \mu \alpha_n B \right) y_n \right] \right.$$

$$\left. - P_C \left[\alpha_{n-1} \gamma f \left(x_{n-1} \right) + \left(I - \mu \alpha_{n-1} B \right) y_{n-1} \right] \right\|$$

$$\leq \left\| \alpha_n \gamma f \left(x_n \right) + \left(I - \mu \alpha_n B \right) y_n - \alpha_{n-1} \gamma f \left(x_{n-1} \right) \right.$$

$$\left. - \left(I - \mu \alpha_{n-1} B \right) y_{n-1} \right\|$$

$$= \left\| \alpha_n \gamma \left(f \left(x_n \right) - f \left(x_{n-1} \right) \right) + \gamma \left(\alpha_n - \alpha_{n-1} \right) f \left(x_{n-1} \right) \right.$$

$$\left. + \left(I - \mu \alpha_n B \right) \left(y_n - y_{n-1} \right) + \mu \left(\alpha_n - \alpha_{n-1} \right) y_{n-1} \right\|$$

$$\leq \alpha_n \gamma \beta \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\|$$

$$+ (1 - \alpha_n \tau) \|y_n - y_{n-1}\| + \mu |\alpha_n - \alpha_{n-1}| \|y_{n-1}\|$$

$$\leq \alpha_n \gamma \beta \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\|$$

$$+ (1 - \alpha_n \tau) \left(\|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right| (\delta \|A\| \varepsilon_n + \xi_n) \right)$$

$$+ \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s) u_{n-1} ds\right\|$$

$$+ \frac{1}{s_n} \left\|\int_{s_{n-1}}^{s_n} T(s) u_{n-1} ds\right\| + \mu |\alpha_n - \alpha_{n-1}| \|y_{n-1}\|$$

$$= (1 - \alpha_n (\tau - \gamma \beta))$$

$$\times \left(\|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \right.$$

$$+ \left|1 - \frac{r_{n-1}}{r_n}\right| (\delta \|A\| \varepsilon_n + \xi_n) \Big)$$

$$+ \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s) u_{n-1} ds\right\|$$

$$+ \frac{1}{s_n} \left\|\int_{s_{n-1}}^{s_n} T(s) u_{n-1} ds\right\| + \mu |\alpha_n - \alpha_{n-1}| \|y_{n-1}\|$$

$$\leq (1 - \alpha_n (\tau - \gamma \beta)) \|x_n - x_{n-1}\|$$

$$+ M_2 \left(\gamma |\alpha_n - \alpha_{n-1}| + \left|1 - \frac{r_{n-1}}{r_n}\right| + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \right.$$

$$+ \frac{1}{s_{n-1}} + \mu |\alpha_n - \alpha_{n-1}| \Big),$$

$$(52)$$

where

$$M_2 = \max \left\{ \sup_{n \geq 1} (\delta \|A\| \varepsilon_n + \xi_n), \right.$$

$$\sup_{n \geq 1} \left(\left\|\int_{s_n}^{s_{n-1}} T(s) u_{n-1} ds\right\| \right), \sup_{n \geq 1} \|y_{n-1}\| \right\}.$$

$$(53)$$

Since $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ are bounded, we can claim that $\{Ax_n\}$ and $\{T(s)u_{n-1}\}$ are bounded. We can deduce that $\sup_{n \geq 1}(\delta \|A\| \varepsilon_n + \xi_n) < \infty$, $\sup_{n \geq 1}(\| \int_{s_n}^{s_{n-1}} T(s)u_{n-1} ds\|) < \infty$ and $\sup_{n \geq 1} \|y_{n-1}\| < \infty$, and $M_2 < \infty$.

Following conditions (a)–(c), $\lim_{n \to 0} r_n = r > 0$, $\lim_{n \to 0} s_n = +\infty$, and Lemma 13, we obtain that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \qquad (54)$$

(v) Consider that $\lim_{n \to \infty} \|T(s)x_n - x_n\| = 0$.

From (30) and (32), we obtain

$$\|x_{n+1} - y_n\|$$

$$\leq \left\| P_C \left[\alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right] \right.$$

$$\left. - P_C y_n \right\| \qquad (55)$$

$$\leq \left\| \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - y_n \right\|$$

$$\leq \alpha_n \left\| \gamma f(x_n) - \mu B \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|.$$

Since $\lim_{n \to \infty} \alpha_n = 0$ and x_n and u_n are bounded, we obtain

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \qquad (56)$$

By (54) and (56), we get

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|. \qquad (57)$$

Next, we have

$$\lim_{n \to \infty} \|x_n - y_n\| = 0. \qquad (58)$$

On the other hand, by (30), we have

$$\|T(s) x_n - x_n\|$$

$$\leq \left\| T(s) x_n - T(s) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|$$

$$+ \left\| T(s) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|$$

$$+ \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - x_n \right\|$$

$$\leq \left\| x_n - \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\| \qquad (59)$$

$$+ \left\| T(s) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|$$

$$+ \left\| \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - x_n \right\|$$

$$\leq 2 \|x_n - y_n\| + \left\| T(s) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right.$$

$$\left. - \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|.$$

So without loss of generality, we may assume that $S = \{T(s) : 0 \leq s < +\infty\}$ is nonexpansive semigroup on K, and, by Lemma 4, we obtain

$$\lim_{n \to \infty} \left\| T(s) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds \right\|. \qquad (60)$$

From (58), (59), and (60), we have

$$\lim_{n \to \infty} \|T(s) x_n - x_n\| = 0. \tag{61}$$

(vi) Consider that $\omega \in \mathrm{EP}(F_1) \bigcap \mathrm{EP}(F_2)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to ω. From (47), we obtain $\{u_{n_i}\}$ which converges weakly to ω. From (58), $y_{n_i} \rightharpoonup \omega$ follows. We show $\omega \in \mathrm{EP}(F_1)$. According to (30) and (A2),

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n) \tag{62}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F_1(y, u_{n_i}). \tag{63}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup \omega$, from (A4), it follows that $0 \geq F_1(y, \omega)$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)\omega$; then we get $0 \geq F_1(y_t, \omega)$. So, from (A1) and (A4) we have

$$0 = F_1(y_t, y_t) \leq t F_1(y_t, y) + (1-t) F(y_t, \omega) \leq t F_1(y_t, y) \tag{64}$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F_1(\omega, y)$ for all $y \in H$. Therefore, $\omega \in \mathrm{EP}(F_1)$.

Since $x_{n_i} \rightharpoonup \omega$ and A is a bounded linear operator, we obtain $Ax_{n_i} \rightharpoonup A\omega$. Let $v_{n_j} = Ax_{n_j} - T_{r_{n_j}}^{F_2} x_{n_j}$. Following (42), we obtain $\lim_{n \to \infty} v_{n_j} = 0$ and $Ax_{n_j} - v_{n_j} = T_{r_{n_j}}^{F_2} x_{n_j}$. Then from Lemma 8, we get

$$F_2\left(Ax_{n_j} - v_{n_j}, y\right) + \frac{1}{r_{n_j}} \left\langle y - \left(Ax_{n_j} - v_{n_j}\right), \left(Ax_{n_j} - v_{n_j}\right) \right.$$
$$\left. - Ax_{n_j} \right\rangle \geq 0, \quad \forall y \in Q. \tag{65}$$

Since F_2 is upper semicontinuous in the first argument, taking lim sup to above inequality as $j \to \infty$ and using $\limsup_{n \to \infty} r_n = r > 0$, we obtain that

$$F_2(A\omega, y) \geq 0, \quad \forall y \in Q, \tag{66}$$

which means that $A\omega \in \mathrm{EP}(F_2)$ and hence $\omega \in \Omega$.

(vii) Consider that $\langle (\mu F - \gamma f) x^*, x^* - \tilde{x} \rangle \leq 0$, $\forall \tilde{x} \in \Gamma = F_{ix}(S) \bigcap \Omega$.

According to (30), letting

$$z_n = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n \, ds, \tag{67}$$

we can observe that

$$x_{n+1} = P_C z_n = P_C z_n - z_n + \alpha_n \gamma f(x_n)$$
$$+ (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n \, ds. \tag{68}$$

Observe that

$$(\mu B - \gamma f) x_n = \frac{1}{\alpha_n} (P_C z_n - z_n) + \frac{1}{\alpha_n} (x_n - x_{n+1})$$
$$+ (I - \mu \alpha_n B)(y_n - x_n). \tag{69}$$

Hence, for each $p \in \Gamma = F_{ix}(S) \bigcap \Omega$, we can obtain

$$\langle (\mu B - \gamma f) x_n, x_n - p \rangle$$
$$= \frac{1}{\alpha_n} \langle P_C z_n - z_n, x_n - p \rangle, + \frac{1}{\alpha_n} \langle x_n - x_{n+1}, x_n - p \rangle$$
$$+ \frac{1}{\alpha_n} \langle (I - \mu \alpha_n B)(y_n - x_n), x_n - p \rangle$$
$$= \frac{1}{\alpha_n} \langle P_C z_n - z_n, x_n - p \rangle, + \frac{1}{\alpha_n} \langle x_n - x_{n+1}, x_n - p \rangle$$
$$+ \frac{1}{\alpha_n} \langle y_n - x_n, x_n - p \rangle - \mu \langle B y_n - B x_n, x_n - p \rangle. \tag{70}$$

Taking limit $n \to \infty$ in (70) and noticing that $B y_n - B x_n \to B x^* - B x^* = 0$, $y_n - x_n \to \infty$, and $P_C z_n - z_n \to P_C x^* - x^* = 0$, we obtain that

$$\langle (\mu B - \gamma f) \omega, \omega - p \rangle \leq 0, \tag{71}$$

which implies $\omega = P_\Gamma (I - \mu B + \gamma f)$.

(viii) Consider that $\omega \in \Gamma = F_{ix}(S) \bigcap \Omega$.

From (30), we have $x_{n+1} = P_C z_n$, and, for any given $x^* \in \Gamma$,

$$x_{n+1} - x^* = P_C z_n - z_n + z_n - x^*$$
$$= P_C z_n - z_n + \alpha_n (\gamma f(x_n) - \mu B x^*)$$
$$+ (I - \mu \alpha_n B) y_n - (I - \mu \alpha_n B) x^*. \tag{72}$$

Since P_C is the metric projection from H_1 onto C, we obtain

$$\langle P_C z_n - z_n, P_C z_n - x^* \rangle \leq 0. \tag{73}$$

It follows from (72) and (37) that

$$\|x_{n+1} - x\|^2$$
$$= \langle P_C z_n - z_n, x_{n+1} - x^* \rangle + \alpha_n \langle \gamma f(x_n) - \mu B x^*, x_{n+1} - x^* \rangle$$
$$+ \langle (I - \mu \alpha_n B)(y_n - x^*), x_{n+1} - x^* \rangle$$

$$\leq \alpha_n \left\langle \gamma f(x_n) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$+ \left\langle (I - \mu \alpha_n B)(y_n - x^*), x_{n+1} - x^* \right\rangle$$
$$\leq \alpha_n \gamma \left\langle f(x_n) - f(x^*), x_{n+1} - x^* \right\rangle$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$+ \left\langle (I - \mu \alpha_n B)(y_n - x^*), x_{n+1} - x^* \right\rangle$$
$$\leq \alpha_n \gamma \beta \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$+ (1 - \alpha_n \tau) \left\| y_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$\leq \alpha_n \gamma \beta \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$+ (1 - \alpha_n \tau) \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$\leq (1 - \alpha_n (\tau - \gamma \beta)) \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$\leq \alpha_n \gamma \beta \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$+ (1 - \alpha_n \tau) \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|$$
$$\leq \frac{(1 - \alpha_n (\tau - \gamma \beta))}{2} \left(\left\| x_n - x^* \right\|^2 + \left\| x_{n+1} - x^* \right\|^2 \right)$$
$$+ \alpha_n \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle, \tag{74}$$

which hence implies that

$$\left\| x_{n+1} - x \right\|^2$$
$$\leq \frac{(1 - \alpha_n (\tau - \gamma \beta))}{(1 + \alpha_n (\tau - \gamma \beta))} \left\| x_n - x^* \right\|^2$$
$$+ \frac{2 \alpha_n}{1 + \alpha_n (\tau - \gamma \beta)} \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle \tag{75}$$
$$\leq (1 - \alpha_n (\tau - \gamma \beta)) \left\| x_n - x^* \right\|^2$$
$$+ \frac{2 \alpha_n}{1 + \alpha_n (\tau - \gamma \beta)} \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle$$
$$\leq (1 - a_n) \left\| x_n - x^* \right\|^2 + \alpha_n b_n,$$

where

$$a_n = \alpha_n (\tau - \gamma \beta),$$
$$b_n = \frac{2}{1 + \alpha_n (\tau - \gamma \beta)} \left\langle \gamma f(x^*) - \mu B x^*, x_{n+1} - x^* \right\rangle. \tag{76}$$

It is easily seen that $\sum_{n=0}^{\infty} a_n = \infty$ and $\limsup_{n \to \infty} b_n \leq 0$ (due to (a), (b), (c), and (72)). According to Lemma 13, we can conclude that $x_n \to x^*$. This completes the proof. □

Corollary 15. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 7 and F_2 is upper semicontinuous in the first argument. Let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$u_n = T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds. \tag{77}$$

Suppose that the sequence $\{\alpha_n\}$ satisfies the following conditions: (a), (b), and (c) (the same in Theorem 14). Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $x^* \in \Gamma = F_{ix}(S) \bigcap \Omega$, where $x^* = P_\Gamma (I - \mu B + \gamma f) x^*$, which is the unique solution of the variational inequality (31).

If B is a strongly positive bounded linear self-adjoint operator on H_1 with constant $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}/\alpha < \gamma + 1/\alpha$, we can easily obtain the following corollary.

Corollary 16. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 7 and F_2 is upper semicontinuous in the first argument. Let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$u_n = T_{r_n}^{F_1} \left(x_n + \delta A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds. \tag{78}$$

Suppose that the sequence $\{\alpha_n\}$ satisfies the following conditions: (a), (b), and (c) (the same in Theorem 14). Then the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to $x^* \in \Gamma = F_{ix}(S) \bigcap \Omega$, where $x^* = P_\Gamma (I - \mu B + \gamma f) x^*$, which is the unique solution of the variational inequality

$$\left\langle (\gamma f - B) x^*, \overline{x} - x^* \right\rangle \leq 0, \quad \forall \overline{x} \in \Gamma. \tag{79}$$

4. Application

In this section, we introduce an example of numerical test to illustrate the algorithm given in Corollary 16.

Example 17. Let $C, Q = [0, +\infty) \in H_1 = H_2 = R$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy$, $\forall x, y \in R$, and induced usual norm $\| \cdot \|$. Assume that $F_1 : C \times C \to R$ and $F_2 : Q \times Q \to R$ are defined by $F_1(x, y) = (x - 4)(y - x)$, $\forall x, y \in C$, and $F_2(u, v) = (u + 2)(v - u)$, $\forall u, v \in Q$. It is easy to claim that the bifunctions F_1 and F_2 satisfy Assumption 7 and the bifunction F_2 is upper semicontinuous. Next, we can find the formula of $T_{r_n}^{F_1} x$. From
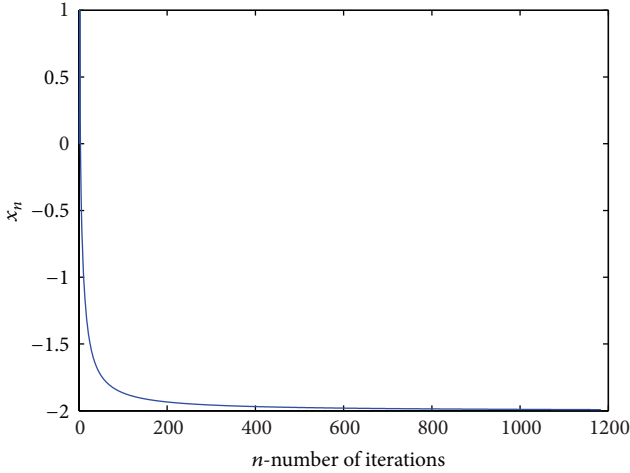
FIGURE 1: Convergence of iterative sequence $\{x_n\}$.

Lemma 8, we can claim that $T_{r_n}^{F_1} x$ is single-valued; for any $y \in C$, $r > 0$,

$$F_1(x, y) + \frac{1}{r} \langle x - z, y - x \rangle \geq 0$$

$$\Longleftrightarrow -(r + 1)x^2 + (rx + x - 4r - z)y + (4r + z)x \geq 0. \tag{80}$$

Letting $H(y) = -(r+1)x^2 + (rx + x - 4r - z)y + (4r + z)x$, we obtain that

$$z = (r + 1)x - 4r, \tag{81}$$

and so

$$T_{r_n}^{F_1} x = (r_n + 1)x - 4r_n. \tag{82}$$

Similarly, we can also obtain that

$$T_{r_n}^{F_2} u = (r_n + 1)u + 2r_n. \tag{83}$$

For all $x \in R$, we can define mapping $f(x) = (1/4)x$, $A(x) = -(1/2)x$, $B(x) = 2x$, and the nonexpansive mapping T is satisfied as $Tx = x \; \forall x \in C$. It is claimed that the mapping f is contraction with constant $\beta = 1/2$, A is a bounded linear operator on R with adjoint operator A^* and $\|A\| = \|A^*\| = 1/2$, and B is a strongly positive bounded linear self-adjoint operator with constant $\overline{\gamma} = 1$ on R. On the other hand, we can take $\gamma = 2$ which satisfies $0 < \gamma < \overline{\gamma}/\alpha < \gamma + 1/\alpha$. Hence, it is easy to observe that $F_{ix}(T) = (0, \infty)$, $\mathrm{EP}(F_1) = \{4\}$, and $\mathrm{EP}(F_2) = \{-2\}$. Furthermore, we can obtain that $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\} = \{-2\}$. Consequently, $\Gamma = F_{ix}(S) \bigcap \Omega = \{-2\}$. Therefore, all the assumptions in Corollary 16 are satisfied. We can obtain the following numerical algorithm:

$$u_n = T_{r_n}^{F_1} \left(x_n + \frac{1}{8} A^* \left(T_{r_n}^{F_2} - I \right) A x_n \right),$$
$$x_{n+1} = \frac{2}{n+1} \left(\frac{1}{4} x_n \right) + \left(1 - \frac{2}{n+1} \right) u_n, \tag{84}$$

where $\alpha_n = 1/(n+1)$ and $r_n = 1$. Then, by Corollary 16, the sequence $\{x_n\}$ converges to a solution of Example 17. For a number $\varepsilon = 10^{-5}$, using the MATLAB, we generated a sequence $\{x_n\}$ as in Figure 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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