

## Research Article

# Limit Cycles Bifurcated from Some $Z_4$ -Equivariant Quintic Near-Hamiltonian Systems

Simin Qu, Cangxin Tang, Fengli Huang, and Xianbo Sun

Department of Applied Mathematics, Guangxi University of Finance and Economics, Nanning, Guangxi 530003, China

Correspondence should be addressed to Xianbo Sun; xianbo01@126.com

Received 3 December 2013; Accepted 14 January 2014; Published 3 March 2014

Academic Editor: Yonghua Xia

Copyright © 2014 Simin Qu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study the number and distribution of limit cycles of some planar  $Z_4$ -equivariant quintic near-Hamiltonian systems. By the theories of Hopf and heteroclinic bifurcation, it is proved that the perturbed system can have 24 limit cycles with some new distributions. The configurations of limit cycles obtained in this paper are new.

## 1. Introduction

In 1900, Hilbert proposed 23 open mathematical problems [1]; the second part of the 16th problem concerns the maximal number and relative position of limit cycles of the planar polynomial vector fields. Even though there have been many results of obtaining more limit cycles and various configuration patterns of their relative dispositions, it has not been solved completely. To reduce the difficulty one can study the systems with some symmetry. An important symmetry is the  $Z_q$ -equivariance which was first introduced in [2]. Here we mention some newer results; for more details, see summary work [3–5]. Li et al. [6] proved a cubic  $Z_2$ -equivariant system having 13 limit cycles; Zhao [7] proved that this system has 13 limit cycles with another distribution. Li and Liu [8] proved another cubic  $Z_2$ -equivariant system also having 13 limit cycles. Zhang et al. [9] found a quartic system having at least 15 limit cycles. Christopher [10] proved that a  $Z_2$ -equivariant system has 22 limit cycles. As to the case of quintic polynomial system, there are more results. Xu and Han [11] studied a cubic  $Z_4$ -equivariant system perturbed by quintic  $Z_4$ -equivariant polynomials having 13 limit cycles. Li et al. [12] studied a quintic system and obtained at least 23 limit cycles for  $Z_2$ -equivariant case and 17 limit cycles for  $Z_4$ -equivariant case. In [13], Wu et al. studied a  $Z_4$ -equivariant system and found 20 limit cycles. Li et al. [14] found that 24 limit cycles existing in a  $Z_6$ -equivariant quintic system. Yao and Yu [15] studied a  $Z_5$ -equivariant quintic planar vector

fields by normal form theory and proved that the maximal number of small limit cycles bifurcated from such vector fields is 25. Wu et al. [16] proved that a quintic  $Z_6$ -equivariant near-Hamiltonian system has 28 limit cycles. In [17], 24 limit cycles are found and two different configurations of them were shown in a  $Z_3$ -equivariant quintic planar polynomial system.

Our main result is that there can be 24 limit cycles with other distributions for the perturbed quintic  $Z_4$ -equivariant systems which are different from the known results. Using the methods of Hopf and heteroclinic bifurcation theories, the number and location of limit cycles of the following  $Z_4$ -equivariant quintic near-Hamiltonian system will be investigated:

$$\begin{aligned}\dot{x} &= H_y + \varepsilon P_5(x, y), \\ \dot{y} &= -H_x + \varepsilon Q_5(x, y),\end{aligned}\tag{1}$$

where  $\varepsilon$  is nonnegative and small and the Hamiltonian system is

$$H(x, y) = 2x^2 + 2y^2 - \frac{5}{4}x^4 - \frac{5}{4}y^4 + \frac{1}{6}x^6 + \frac{1}{6}y^6\tag{2}$$

with phase portrait of Figure 1.  $(P_5(x, y), Q_5(x, y))$  is the five-degree polynomial vector invariant under rotation of  $\pi/2$  with respect to the origin  $O$ . From [2] we know that

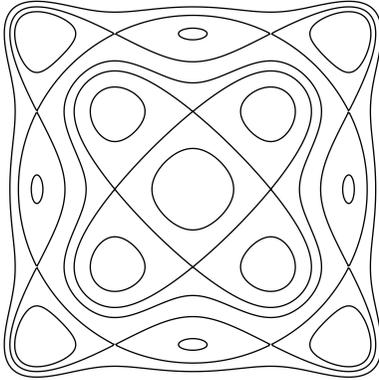


FIGURE 1: The phase portraits of  $(1)_{(\epsilon=0)}$ .

$(P_5(x, y), Q_5(x, y))$  is, respectively, the real and imaginary parts of the following complex function:

$$F_{5,4}(z, \bar{z}) = (A_0 + A_1|z|^2 + A_3|z|^4)z + (A_3 + A_4|z|^2)\bar{z}^3 + A_5z^5, \quad (3)$$

where  $A_k = a_k + ib_k$ ,  $k = 0, 1, 2, \dots, 5$ ,  $z = x + iy$ , and  $\bar{z} = x - iy$ . It is direct that

$$\begin{aligned} P_5 &= a_0x + a_1(x^3 + xy^2) + a_2(x^5 + 2x^3y^2 + xy^4) \\ &+ a_3(x^3 - 3xy^2) + a_4(x^5 - 2x^3y^2 - 3xy^4) \\ &+ a_5(x^5 - 10x^3y^2 + 5xy^4) + b_0(-y) \\ &+ b_1(-x^2y - y^3) + b_2(-x^4y - 2x^2y^3 - y^5) \\ &+ b_3(3x^2y - y^3) + b_4(3x^4y + 2x^2y^3 - y^5) \\ &+ b_5(10x^2y^3 - 5x^4y - y^5), \\ Q_5 &= a_0y + a_1(x^2y + y^3) + a_2(x^4y + 2x^2y^3 + y^5) \\ &+ a_3(y^3 - 3x^2y) + a_4(y^5 - 3x^4y - 2x^2y^3) \\ &+ a_5(5x^4y - 10x^2y^3 + y^5) + b_0x \\ &+ b_1(x^3 + xy^2) + b_2(x^5 + 2x^3y^2 + xy^4) \\ &+ b_3(x^3 - 3xy^2) + b_4(x^5 - 2x^3y^2 - 3xy^4) \\ &+ b_5(x^5 - 10x^3y^2 + 5xy^4). \end{aligned} \quad (4)$$

Our result is the following.

**Theorem 1.** *There exist some  $(a_0, a_1, a_2, a_3, a_4, a_5)$  such that system (1) can have 24 limit cycles with two different distributions, the distributions of these limit cycles are shown in Figure 2.*

The rest of this paper is organized as follows. Some useful preliminary theorems will be listed in Section 2. In Section 3,

some related coefficients of asymptotic expansions are firstly computed; then using this coefficients and preliminary lemmas we prove the main result.

## 2. Preliminary Lemmas

Let  $H(x, y)$ ,  $p(x, y, \delta)$ , and  $q(x, y, \delta)$  be analytic functions,  $\epsilon$  positive and small, and  $\delta \in D \subset \mathbb{R}^m$  with  $D$  compact; then the following system is a planar Hamiltonian system:

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \quad (5)$$

and the below system is usually called near-Hamiltonian system:

$$\begin{aligned} \dot{x} &= H_y(x, y) + \epsilon p(x, y, \delta), \\ \dot{y} &= -H_x(x, y) + \epsilon q(x, y, \delta). \end{aligned} \quad (6)$$

Let system (5) have at least one family of periodic orbits  $L_h$  defined by  $H(x, y) = h$  which form a periodic annulus  $\{L_h\}$ ; then the first-order approximation of the Poincaré map of system (6) is

$$M(h, \delta) = \oint_{L_h} (qdx - pdy) \quad (7)$$

which is called the Melnikov function or Abel integral. By the Poincaré-Pontryagin-Andronov theorem, an isolated zero of  $M(h, \delta)$  corresponds a limit cycle of system (6). A popular method to find limit cycles of (6) is to find zeros of  $M(h, \delta)$  and an efficient method to find zeros of  $M(h, \delta)$  is to investigate the asymptotic expansion of  $M(h, \delta)$  near the boundaries of  $\{L_h\}$ ; see [18].

Let the outer boundary of  $\{L_h\}$  be a homoclinic loop  $L_\beta$  defined by  $H(x, y) = \beta$  passing through a hyperbolic saddle at the origin; we have the following.

**Lemma 2** (see [19]). *(i) The function  $M(h, \delta)$  has the following expansion:*

$$\begin{aligned} M(h, \delta) &= c_0(\delta) + c_1(\delta)(h - \beta) \ln|h - \beta| \\ &+ c_2(\delta)(h - \beta) + c_3(\delta)(h - \beta)^2 \ln|h - \beta| \\ &+ O(|h - \beta|^2), \end{aligned} \quad (8)$$

for  $0 < \beta - h \ll 1$ ;  $c_i(\delta)$  depends on the parameters of  $H$ ,  $p$ , and  $q$ .

*(ii) Further suppose that, for  $(x, y)$  near  $(0, 0)$ ,*

$$\begin{aligned} H(x, y) &= \beta + \frac{\lambda}{2}(y^2 - x^2) + \sum_{i+j \geq 3} h_{ij}x^i y^j, \quad \lambda \neq 0, \\ p(x, y, \delta) &= \sum_{i+j \geq 0} a_{ij}x^i y^j, \\ q(x, y, \delta) &= \sum_{i+j \geq 0} b_{ij}x^i y^j. \end{aligned} \quad (9)$$

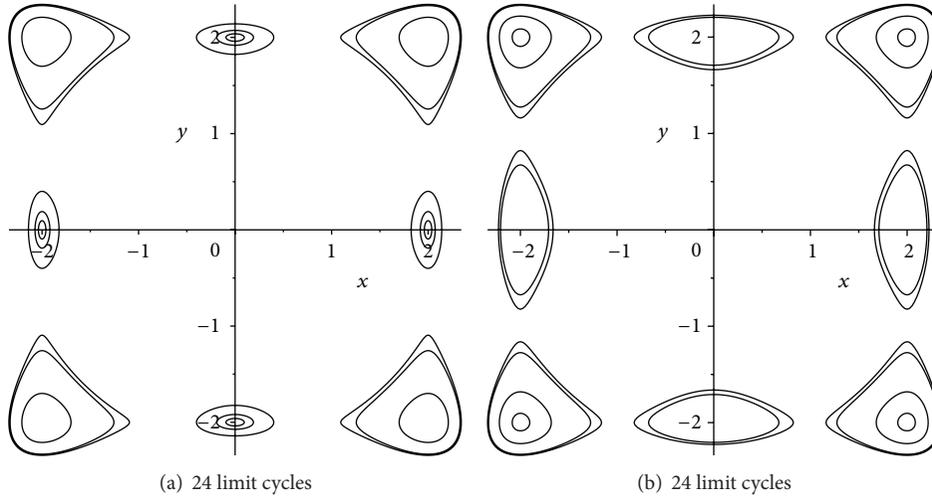


FIGURE 2: Two different distributions of 24 limit cycles of system (1).

Then,

$$\begin{aligned}
 c_0(\delta) &= \oint_{L_\beta} q dx - p dy, \\
 c_1(\delta) &= -\frac{1}{|\lambda|} (a_{10} + b_{01}), \\
 c_2(\delta) &= \oint_{L_\beta} (p_x + q_y) dt \quad \text{if } c_1(\delta) = 0, \\
 c_3(\delta) &= \frac{-1}{2|\lambda|\lambda} \{(-3a_{30} - b_{21} + a_{12} + 3b_{03}) \\
 &\quad - \frac{1}{\lambda} [(2b_{02} + a_{11})(3h_{03} - h_{21}) \\
 &\quad + (2a_{20} + b_{11})(3h_{30} - h_{12})]\}, \\
 &\quad \text{if } c_1(\delta) = 0.
 \end{aligned} \tag{10}$$

The values  $c_1(\delta)$  and  $c_3(\delta)$  are, respectively, called the first and second local Melnikov coefficients at the saddle  $O$ , denoted by  $c_1(O, \delta)$  and  $c_3(O, \delta)$ , respectively.

Now let the outer boundary of  $\{L_h\}$  be an 2-polycycle  $\Gamma^2$ :

$$\Gamma^2 = \bigcup_{i=1}^2 (L_i \cup S_i) \tag{11}$$

with 2 hyperbolic saddles,  $S_1$  and  $S_2$ , and 2 heteroclinic orbits,  $L_1$  and  $L_2$ , connecting them, defined by  $H(x, y) = \beta$ . The following lemma was proved in [19].

**Lemma 3** (see [19]). *Under the above assumptions,  $M(h, \delta)$  has the form, for  $0 < \beta - h \ll 1$ ,*

$$\begin{aligned}
 M(h, \delta) &= \sum_{j \geq 0} [c_{2j}(\delta) + c_{2j+1}(\delta)(h - \beta) \ln |h - \beta|] \\
 &\quad \times (h - \beta)^j,
 \end{aligned} \tag{12}$$

where

$$\begin{aligned}
 c_0(\delta) &= \sum_{i=1}^2 \int_{L_i} q dx - p dy, \\
 c_1(\delta) &= \sum_{i=1}^2 c_1(S_i, \delta), \\
 c_3(\delta) &= \sum_{i=1}^2 c_3(S_i, \delta),
 \end{aligned} \tag{13}$$

where  $c_1(S_i, \delta)$  and  $c_3(S_i, \delta)$  are, respectively, the first and the second local Melnikov coefficient at the saddle  $S_i$ ,  $i = 1, 2$ . In particular,

$$c_2(\delta) = \oint_{\Gamma^2} (p_x + q_y) dt = \sum_{i=1}^2 \int_{L_i} (p_x + q_y) dt, \tag{14}$$

if  $c_1(S_i, \delta) = 0$ ,  $i = 1, 2$ .

When the inner boundary of  $\{L_h\}$  is a elementary center  $(x_c, y_c)$  defined by  $H(x_c, y_c) = \alpha$ , the following lemma gives the asymptotic expansion of  $M(h, \delta)$ .

**Lemma 4** (see [20]).  *$M(h, \delta)$  has the form, for  $0 < h - \alpha \ll 1$ ,*

$$M(h, \delta) = \sum_{k \geq 0} B_k (h - \alpha)^{k+1}. \tag{15}$$

If for  $(x, y)$  near  $(x_c, y_c)$ ,

$$H(x, y) = \alpha + \frac{1}{2} ((x - x_c)^2 + (y - y_c)^2)$$

$$\begin{aligned}
 & + \sum_{i+j \geq 3} h_{ij}(x-x_c)^i(y-y_c)^j, \\
 p(x,y,\delta) &= \sum_{i+j \geq 1} a_{ij}(x-x_c)^i(y-y_c)^j, \\
 q(x,y,\delta) &= \sum_{i+j \geq 1} b_{ij}(x-x_c)^i(y-y_c)^j,
 \end{aligned} \tag{16}$$

the coefficients  $B_i$  can be obtained by the formulas in [20].

*Remark 5.* When the inner boundary is a nilpotent center, a new method of limit cycles bifurcated from the annulus near the center can be found in [21]. Lemma 3 has been developed in [22, 23].

In many cases the Hamiltonian function is not of the form presented in the above lemmas. Then to apply the lemmas we need first to introduce suitable linear change of variables which will cause a change in the first-order Melnikov function. The following lemma gives the relationship between the old and new Melnikov functions.

**Lemma 6** (see [24]). (i) Under the linear change of variables of the form:

$$\begin{aligned}
 u &= a(x-x_0) + b(y-y_0), \\
 v &= c(x-x_0) + d(y-y_0)
 \end{aligned} \tag{17}$$

and time rescaling  $\tau = kt$ , where  $D = ad - bc \neq 0$ , the system (6) becomes

$$\frac{du}{d\tau} = \tilde{H}_v + \varepsilon \tilde{p}, \quad \frac{dv}{d\tau} = -\tilde{H}_u + \varepsilon \tilde{q}, \tag{18}$$

where  $\tilde{H}(u,v) = (D/k)H(x,y)$ ,  $\tilde{p}(u,v,\delta) = (1/k)[ap(x,y,\delta) + bq(x,y,\delta)]$ , and  $\tilde{q}(u,v,\delta) = (1/k)[cp(x,y,\delta) + dq(x,y,\delta)]$ .

(ii) Let

$$\tilde{M}(h,\delta) = \oint_{L_h} \tilde{q} du - \tilde{p} dv \tag{19}$$

which is the Melnikov function of the system (18); then

$$M(h,\delta) = \frac{|k|}{D} \tilde{M}\left(\frac{D}{k}h,\delta\right). \tag{20}$$

When systems (5) and (6) are  $Z_4$ -equivariant, (5) has a compound cycle denoted by  $\Gamma^4$ , which consists of 8 hyperbolic saddles  $S_1, \dots, S_8$  and 16 heteroclinic orbits  $L_{12}, L_{21}, L_{23}, L_{32}, L_{34}, L_{43}, L_{45}, L_{54}, L_{56}, L_{65}, L_{67}, L_{76}, L_{78}, L_{87}, L_{81}$ , and  $L_{18}$  satisfying  $\alpha(L_{ij}) = S_i, \omega(L_{ij}) = S_j$ .  $\Gamma^4$  contains 8 two-polycycles  $L_i$  ( $i = 1, \dots, 8$ ), where  $L_1 = L_{12} \cup L_{21}$ ,  $L_2 = L_{23} \cup L_{32}$ ,  $L_3 = L_{34} \cup L_{43}$ ,  $L_4 = L_{45} \cup L_{54}$ ,  $L_5 = L_{56} \cup L_{65}$ ,  $L_6 = L_{67} \cup L_{76}$ ,  $L_7 = L_{78} \cup L_{87}$ , and  $L_8 = L_{81} \cup L_{18}$ . See Figure 3. We suppose that  $\Gamma^4$  is defined by  $H(x,y) = H(S_i) = \beta, i = 1, \dots, 8$ . There are 8 centers  $C_i(x_i, y_i)$  inside the 2-polycycle  $L_i$ , with  $H(C_1) = H(C_3) = H(C_5) = H(C_7) = \alpha_1$  and  $H(C_2) = H(C_4) = H(C_6) = H(C_8) = \alpha_2$ . There

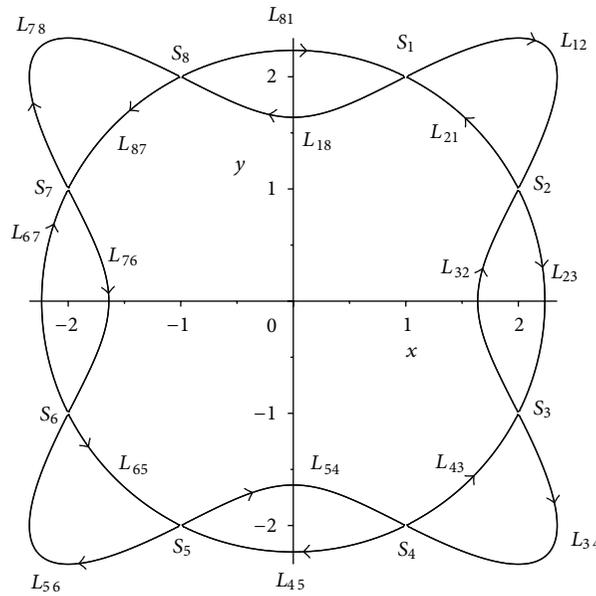


FIGURE 3

are 4 families of periodic orbits  $L_i^h$  inside the 2-polycycle  $L_i$ , defined by  $H(x,y) = h$  for  $h \in (\alpha_1, \beta), i = 1, 3, 5, 7$ , and 4 families of periodic orbits  $L_j^h$  inside the 2-polycycle  $L_j$ , defined by  $H(x,y) = h$  for  $h \in (\alpha_2, \beta), j = 2, 4, 6, 8$ . Then we have 8 Melnikov functions below:

$$\begin{aligned}
 M_i(h,\delta) &= \oint_{L_i^h} (q dx - p dy)|_{\varepsilon=0}, \\
 &\text{for } h \in (\alpha_1, \beta) \quad i = 1, 3, 5, 7. \\
 M_j(h,\delta) &= \oint_{L_j^h} (q dx - p dy)|_{\varepsilon=0}, \\
 &\text{for } h \in (\alpha_2, \beta) \quad j = 2, 4, 6, 8.
 \end{aligned} \tag{21}$$

By  $Z_4$ -equivariance,  $M_1(h,\delta) = M_3(h,\delta) = M_5(h,\delta) = M_7(h,\delta)$  and  $M_2(h,\delta) = M_4(h,\delta) = M_6(h,\delta) = M_8(h,\delta)$ , we can only study  $M_1(h,\delta)$  and  $M_8(h,\delta)$ . For convenience, the notations are introduced as follows:

$$\begin{aligned}
 c_{01}(\delta) &= \int_{L_{12}} q dx - p dy, \\
 c_{02}(\delta) &= \int_{L_{21}} q dx - p dy, \\
 e_{01}(\delta) &= \int_{L_{81}} q dx - p dy, \\
 e_{02}(\delta) &= \int_{L_{18}} q dx - p dy,
 \end{aligned} \tag{22}$$

and  $d_i(\delta) = (p_x + q_y)(S_i, \delta)$ ,  $1 \leq i \leq 8$ , where  $d_1 = d_3 = d_5 = d_7$  and  $d_2 = d_4 = d_6 = d_8$ . Letting  $d_1 = d_2 = 0$ , we introduce

$$\begin{aligned} c_{21}(\delta) &= \int_{L_{12}} (p_x + q_y) dt, \\ c_{22}(\delta) &= \int_{L_{21}} (p_x + q_y) dt, \\ e_{21}(\delta) &= \int_{L_{81}} (p_x + q_y) dt, \\ e_{22}(\delta) &= \int_{L_{18}} (p_x + q_y) dt. \end{aligned} \tag{23}$$

The following is directly from Lemma 3.

**Lemma 7.** *Under the above assumptions, we have the following expansions:*

$$\begin{aligned} M_i(h, \delta) &= c_0(\delta) + c_1(\delta)(h - \beta) \ln|h - \beta| \\ &+ c_2(\delta)(h - \beta) \\ &+ c_3(\delta)(h - \beta)^2 \ln|h - \beta| + \dots, \end{aligned} \tag{24}$$

for  $0 < h - \beta \ll 1$ ,  $i = 1, 3, 5, 7$ ,

$$\begin{aligned} M_j(h, \delta) &= e_0(\delta) + e_1(\delta)(h - \beta) \ln|h - \beta| \\ &+ e_2(\delta)(h - \beta) \\ &+ e_3(\delta)(h - \beta)^2 \ln|h - \beta| + \dots, \end{aligned} \tag{25}$$

for  $0 < h - \beta \ll 1$ ,  $j = 2, 4, 6, 8$ , where

$$\begin{aligned} c_0(\delta) &= c_{01}(\delta) + c_{02}(\delta), \\ e_0(\delta) &= e_{01}(\delta) + e_{02}(\delta), \\ c_1(\delta) &= c_1(S_1, \delta) + c_1(S_2, \delta) = -\frac{d_1(\delta)}{|\lambda_1|} - \frac{d_2(\delta)}{|\lambda_2|}, \\ e_1(\delta) &= c_1(S_1, \delta) + c_1(S_8, \delta) \\ &= c_1(S_1, \delta) + c_1(S_2, \delta) = c_1(\delta), \\ c_3(\delta) &= c_3(S_1, \delta) + c_3(S_2, \delta), \\ e_3(\delta) &= c_3(S_1, \delta) + c_3(S_8, \delta) \\ &= c_3(S_1, \delta) + c_3(S_2, \delta) = c_3(\delta), \\ c_2(\delta) &= c_{21}(\delta) + c_{22}(\delta), \\ e_2(\delta) &= e_{21}(\delta) + e_{22}(\delta) \end{aligned} \tag{26}$$

if  $d_1(\delta) = d_2(\delta) = 0$ .  $\lambda_i$  denotes an eigenvalue of  $S_i$  for (6) and  $c_1(S_i, \delta)$  and  $c_3(S_i, \delta)$  are the first and the second Melnikov coefficients at the saddle  $S_i$  ( $i = 1, 2$ ) as defined after Lemma 2.

By Lemma 4, for the expansions of  $M_i(h, \delta)$  near the center, we have

$$\begin{aligned} M_1(h, \delta) &= \sum_{k \geq 0} B_k(\delta)(h - \alpha_1)^{k+1} \\ &\text{for } 0 < h - \alpha_1 \ll 1, \\ M_8(h, \delta) &= \sum_{k \geq 0} d_k(\delta)(h - \alpha_2)^{k+1} \\ &\text{for } 0 < h - \alpha_2 \ll 1. \end{aligned} \tag{27}$$

To obtain more limit cycles, we have the following.

**Theorem 8.** *Let (24), (25), and (27) hold.*

(1) *Suppose that there exists  $\delta_0 \in D$  such that*

$$\begin{aligned} c_0(\delta_0) &= c_1(\delta_0) = d_0(\delta_0) = d_1(\delta_0) = 0, \\ c_2(\delta_0) &\neq 0, \quad d_2(\delta_0) \neq 0, \\ \text{rank} \frac{\partial(c_0, c_1, d_0, d_1)}{\partial(\delta_1, \delta_2, \dots, \delta_m)} &= 4. \end{aligned} \tag{28}$$

*Then there exist some  $(\varepsilon, \delta)$  near  $(0, \delta_0)$  such that (5) has*

$$\begin{aligned} 4 + \frac{1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0))}{2} \\ + \frac{1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0))}{2} \end{aligned} \tag{29}$$

*limit cycles in the 2-polycycles  $L_1$  and  $L_8$ , where  $h_1 = \beta - \varepsilon_0$ ,  $h_2 = \alpha_1 + \varepsilon_0$ ,  $h_3 = \beta - \varepsilon_0$ , and  $h_4 = \alpha_2 + \varepsilon_0$  with  $\varepsilon_0$  being positive and very small, and the location of these limit cycles is as follows: 2 limit cycles near the 2-polycycle  $L_1$ , 2 limit cycles near the center  $C_8$ ,  $(1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2$  limit cycles between the center  $C_1$  and the polycycle  $L_1$ , and  $(1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2$  limit cycles between the center  $C_8$  and the polycycle  $L_8$ .*

(2) *Suppose that there exists  $\delta_0 \in D$  such that*

$$\begin{aligned} c_0(\delta_0) &= c_1(\delta_0) = B_0(\delta_0) = e_0(\delta_0) = 0, \\ c_2(\delta_0) &\neq 0, \quad e_2(\delta_0) \neq 0, \quad B_1(\delta_0) \neq 0, \\ \text{rank} \frac{\partial(c_0, c_1, B_0, e_0)}{\partial(\delta_1, \delta_2, \dots, \delta_m)} \Big|_{\delta=\delta_0} &= 4. \end{aligned} \tag{30}$$

*Then there exist some  $(\varepsilon, \delta)$  near  $(0, \delta_0)$  such that (5) has*

$$\begin{aligned} 4 + \frac{1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0))}{2} \\ + \frac{1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0))}{2} \\ + \frac{1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0))}{2} \end{aligned} \tag{31}$$

*limit cycles in the 2-polycycles  $L_1$  and  $L_8$ , where  $h_1 = \beta - \varepsilon_0$ ,  $h_2 = \alpha_1 + \varepsilon_0$ ,  $h_3 = \beta - \varepsilon_0$ , and  $h_4 = \alpha_2 + \varepsilon_0$  with  $\varepsilon_0$  being positive*

and very small, and the location of these limit cycles is the following: 2 limit cycles near the 2-polycycle  $L_1$ , 1 limit cycles near the center  $C_1$ ,  $1 + ((1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0)))/2)$  limit cycles near the 2-polycycle  $L_8$ ,  $(1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2$  limit cycles between the center  $C_1$ , and the polycycle  $L_1$  and  $(1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2$  limit cycles between the center  $C_8$  and the polycycle  $L_8$ .

*Proof.* Because of the similarity, we only prove the last conclusion. For  $\delta = \delta_0$ , by continuity, there exist  $(1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2$  zeros of  $M_1(h, \delta_0)$  between  $h_1$  and  $h_2$  and  $(1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2$  zeros of  $M_8(h, \delta_0)$  between  $h_3$  and  $h_4$ . Thus, for all  $\delta$  near  $\delta_0$  or  $\delta \in U_0 = \{\delta \mid |\delta| < \varepsilon^*, \varepsilon^* \text{ is very small}\}$  there exist  $(1 - \text{sgn}(M_1(h_1, \delta)M_1(h_2, \delta)))/2$  zeros of  $M_1(h, \delta)$  between  $h_1$  and  $h_2$  and  $(1 - \text{sgn}(M_8(h_3, \delta)M_8(h_4, \delta)))/2$  zeros of  $M_8(h, \delta)$  between  $h_3$  and  $h_4$ .

According to the condition, we can take  $c_0, c_1, e_0$ , and  $B_0$  as free parameters. Hence, we first take  $c_1$  satisfying

$$|c_1| \ll |c_2|, \quad |c_1| \ll |e_2|, \quad c_1 c_2 > 0, \quad (32)$$

so that there is a zero of  $M_1(h, \delta)$  denoted by  $\tilde{h}_1$  near  $\beta$  satisfying  $h_1 < \tilde{h}_1 < \beta$ . On the other hand, if  $c_2 e_2 > 0$ , considering  $c_1 = e_1$ , we have  $|e_1| \ll |e_2|$  and  $e_1 e_2 > 0$  which implies that there is a zero of  $M_8(h, \delta)$  denoted by  $\tilde{h}_1$  near  $\beta$  satisfying  $h_3 < \tilde{h}_1 < \beta$ . If  $c_2 e_2 < 0$ , we are not sure if  $M_8(h, \delta)$  has a zero. Thus, so far we obtain 1 zero of  $M_1(h, \delta)$  and  $(1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$  zeros of  $M_8(h, \delta)$  for  $\delta \in U_1 = \{\delta \mid |c_1| \ll |c_2|, |c_1| \ll |e_2|, c_1 c_2 > 0\}$ .

Next, we take  $c_0, e_0$ , and  $B_0$  satisfying

$$\begin{aligned} |c_0| \ll |c_1|, \quad c_0 c_1 < 0, \quad |e_0| \ll |e_1|, \\ e_0 e_1 < 0, \quad |B_0| \ll |B_1|, \quad B_0 B_1 < 0. \end{aligned} \quad (33)$$

Then  $M_1(h, \delta)$  has two new zeros  $\tilde{h}_2$  near  $\beta$  and  $\tilde{h}_3$  near  $\alpha_2$  satisfying  $h_1 < \tilde{h}_1 < \tilde{h}_2 < \beta$  and  $\alpha_2 < \tilde{h}_3 < h_4$ , and  $M_8(h, \delta)$  has a new zero  $\tilde{h}_2$  near  $\beta$  satisfying  $h_3 < \tilde{h}_2 < \beta$ . In this step, we get 2 more zeros of  $M_1(h, \delta)$  and 1 more zero of  $M_1(h, \delta)$  for  $\delta \in U_2 = \{\delta \mid |c_0| \ll |c_1|, c_0 c_1 < 0, |e_0| \ll |e_1|, e_0 e_1 < 0, |B_0| \ll |B_1|, \text{ and } B_0 B_1 < 0\}$ . Then totally we have  $4 + (1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2 + (1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2 + (1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$  zeros for  $\delta \in U_0 \cap U_1 \cap U_2$ . Therefore, there are  $4 + (1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2 + (1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2 + (1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$  limit cycles for some  $\delta$  near  $\delta_0$ . This completes the proof.  $\square$

*Remark 9.* The signs of  $M_1(h_1, \delta_0)$ ,  $M_1(h_2, \delta_0)$ ,  $M_8(h_3, \delta_0)$ , and  $M_8(h_4, \delta_0)$  can be determined by using the first nonzero coefficients in their expansions.

### 3. Main Result

In this section, we investigate the distributions of limit cycles of system (1). For  $\varepsilon = 0$ , (1) has two level sets,  $\tilde{\Gamma}_1$  and  $\tilde{\Gamma}_2$ , defined by  $H(x, y) = -5/12$  and  $H(x, y) = -5/12$

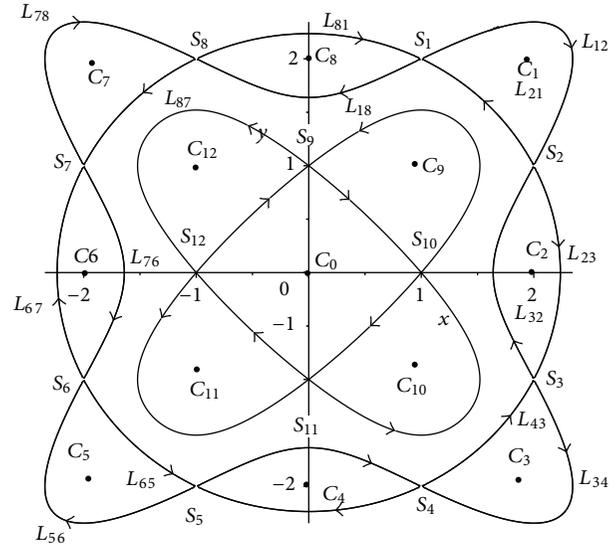


FIGURE 4

respectively.  $\tilde{\Gamma}_1$  consists of 8 saddles,  $S_1 = (1, 2)$ ,  $S_2 = (2, 1)$ ,  $S_3 = (2, -1)$ ,  $S_4 = (1, -2)$ ,  $S_5 = (-1, -2)$ ,  $S_6 = (-2, -1)$ ,  $S_7 = (-2, 1)$ , and  $S_8 = (-1, 2)$ , as shown in Figure 4. Let  $L_{ij}, L_i, L_j^h, c_{01}, c_{02}, e_{01}, e_{02}, c_{21}, c_{22}, e_{21}, e_{22}$ , and so forth are the same as those for Lemma 7. We can write the compound cycles  $\tilde{\Gamma}_1$  as

$$\begin{aligned} \tilde{\Gamma}_1 = \bigcup_{i=1}^8 \{S_i\} \\ \cup \{L_{12}, L_{21}, L_{23}, L_{32}, \dots, L_{78}, L_{87}, L_{81}, L_{18}\}. \end{aligned} \quad (34)$$

The equation  $H(x, y) = -8/3$  defines four centers,  $C_1 = (2, 2)$ ,  $C_3 = (2, -2)$ ,  $C_5 = (-2, -2)$ , and  $C_7 = (-2, 2)$ , where the equation  $H(x, y) = -4/3$  defines four centers of  $C_2 = (2, 0)$ ,  $C_4 = (0, -2)$ ,  $C_6 = (-2, 0)$ , and  $C_8 = (0, 2)$ . The center  $C_i$  is inside the 2-polycycle  $L_i$  for  $i = 1, \dots, 8$ .

We first investigate the 2-polycycles  $L_1$  and  $L_8$ . Here,  $L_1^h$  denotes the periodic orbit defined by  $H(x, y) = h$  surrounding the unique center  $C_1$  and  $L_8^h$  denotes the periodic orbit defined by  $H(x, y) = h$  surrounding the unique center  $C_8$ . Then

$$\begin{aligned} M_1(h, \delta) &= \oint_{L_1^h} Q_5 dx - P_5 dy \\ &= \oint_{L_1^h} \left( Q_5 - P_5 \frac{dy}{dx} \right) dx, \quad h \in \left( -\frac{8}{3}, -\frac{5}{12} \right), \end{aligned} \quad (35)$$

$$\begin{aligned} M_8(h, \delta) &= \oint_{L_8^h} Q_5 dx - P_5 dy \\ &= \oint_{L_8^h} \left( Q_5 - P_5 \frac{dy}{dx} \right) dx, \quad h \in \left( -\frac{4}{3}, -\frac{5}{12} \right), \end{aligned}$$

where  $\delta = (a_0, a_1, \dots, a_5) \in R^6$ . By Lemma 7, we have

$$\begin{aligned}
 M_1(h, \delta) &= c_0(\delta) + c_1(\delta) \left( h + \frac{5}{12} \right) \ln \left| h + \frac{5}{12} \right| \\
 &\quad + c_2(\delta) \left( h + \frac{5}{12} \right) \\
 &\quad + c_3(\delta) \left( h + \frac{5}{12} \right)^2 \ln \left| h + \frac{5}{12} \right| + \dots,
 \end{aligned} \tag{36}$$

for  $0 < -(h + 5/12) \ll 1$ ,

$$\begin{aligned}
 M_8(h, \delta) &= e_0(\delta) + e_1(\delta) \left( h + \frac{5}{12} \right) \ln \left| h + \frac{5}{12} \right| \\
 &\quad + e_2(\delta) \left( h + \frac{5}{12} \right) \\
 &\quad + e_3(\delta) \left( h + \frac{5}{12} \right)^2 \ln \left| h + \frac{5}{12} \right| + \dots,
 \end{aligned} \tag{37}$$

for  $0 < -(h + 5/12) \ll 1$ .

By Lemma 4, for  $0 < h + 8/3 \ll 1$ ,  $i = 1, 3, 5, 7$ ,

$$M_i(h, \delta) = \sum_{k \geq 0} B_k(\delta) \left( h + \frac{8}{3} \right)^{k+1}, \tag{38}$$

and, for  $0 < h + 4/3 \ll 1$ ,  $j = 2, 4, 6, 8$ ,

$$M_j(h, \delta) = \sum_{k \geq 0} d_k(\delta) \left( h + \frac{4}{3} \right)^{k+1}. \tag{39}$$

To find the zeros of  $M_1(h, \delta)$  and  $M_8(h, \delta)$ , the coefficients in these asymptotic expansions need to be calculated. In order to calculate the coefficients  $c_{01}(\delta)$ ,  $c_{02}(\delta)$ ,  $e_{01}(\delta)$ ,  $e_{02}(\delta)$ , and so forth, the expressions of heteroclinic orbits are found as follows:

$$\begin{aligned}
 L_{12}: y_1(x) &= \frac{1}{2} \sqrt{2x^2 + 5 + \sqrt{-12x^4 + 60x^2 + 33}}, \\
 &\quad 1 \leq x \leq \frac{1}{2} \sqrt{22}, \\
 y_2(x) &= \frac{1}{2} \sqrt{2x^2 + 5 - \sqrt{-12x^4 + 60x^2 + 33}}, \\
 &\quad \frac{1}{2} \sqrt{22} \leq x \leq 2,
 \end{aligned} \tag{40}$$

$$L_{21}: y_3(x) = \sqrt{5 - x^2}, \quad 2 \geq x \geq 1,$$

$$L_{81}: y_4(x) = \sqrt{5 - x^2}, \quad -1 \leq x \leq 1,$$

$$\begin{aligned}
 L_{18}: y_5(x) &= \frac{1}{2} \sqrt{2x^2 + 5 + \sqrt{-12x^4 + 60x^2 + 33}}, \\
 &\quad 1 \geq x \geq -1.
 \end{aligned}$$

Figure 5 may be helpful to understand the step to calculate the coefficients in the following. By (4),  $P_5(x, y)$  and  $Q_5(x, y)$  are written as follows:

$$\begin{aligned}
 P_5(x, y) &= \sum_{i=0}^5 p_{1i} a_i + p_{2i} b_i, \\
 Q_5(x, y) &= \sum_{i=0}^5 q_{1i} a_i + q_{2i} b_i,
 \end{aligned} \tag{41}$$

and introduce the following notations:

$$\begin{aligned}
 p_{1i}^j &= p_{1i}|_{y=y_j(x)}, & p_{2i}^j &= p_{2i}|_{y=y_j(x)}, \\
 q_{1i}^j &= q_{1i}|_{y=y_j(x)}, & q_{2i}^j &= q_{2i}|_{y=y_j(x)},
 \end{aligned} \tag{42}$$

$i, j = 1, 2, 3, 4, 5,$

and  $\tilde{f}_j = \left. \frac{dy}{dx} \right|_{y=y_j(x)} = \left. \frac{(-2x + 3x^3 - x^5)}{(2y - 3y^3 + y^5)} \right|_{y=y_j(x)}$ . By Lemma 7, we have

$$\begin{aligned}
 c_{01}(\delta) &= \int_{L_{12}} Q_5 dx - P_5 dy \\
 &= \int_1^{\sqrt{22}/2} \left( Q_5 - P_5 \frac{dy}{dx} \right) \Big|_{y=y_1(x)} dx \\
 &\quad + \int_{\sqrt{22}/2}^2 \left( Q_5 - P_5 \frac{dy}{dx} \right) \Big|_{y=y_2(x)} dx \\
 &= \sum_{i=0}^5 a_i I_{1i}^1 + b_i I_{2i}^1 + a_i I_{1i}^2 + b_i I_{2i}^2, \\
 c_{02}(\delta) &= \int_{L_{21}} Q_5 dx - P_5 dy \\
 &= \int_2^1 \left( Q_5 - P_5 \frac{dy}{dx} \right) \Big|_{y=y_3(x)} dx \\
 &= \sum_{i=0}^5 a_i I_{1i}^3 + b_i I_{2i}^3, \\
 e_{01}(\delta) &= \int_{L_{81}} Q_5 dx - P_5 dy \\
 &= \int_{-1}^1 \left( Q_5 - P_5 \frac{dy}{dx} \right) \Big|_{y=y_4(x)} dx \\
 &= \sum_{i=0}^5 a_i I_{1i}^4 + b_i I_{2i}^4, \\
 e_{02}(\delta) &= \int_{L_{18}} Q_5 dx - P_5 dy \\
 &= \int_1^{-1} \left( Q_5 - P_5 \frac{dy}{dx} \right) \Big|_{y=y_5(x)} dx \\
 &= \sum_{i=0}^5 a_i I_{1i}^4 + b_i I_{2i}^4,
 \end{aligned} \tag{43}$$

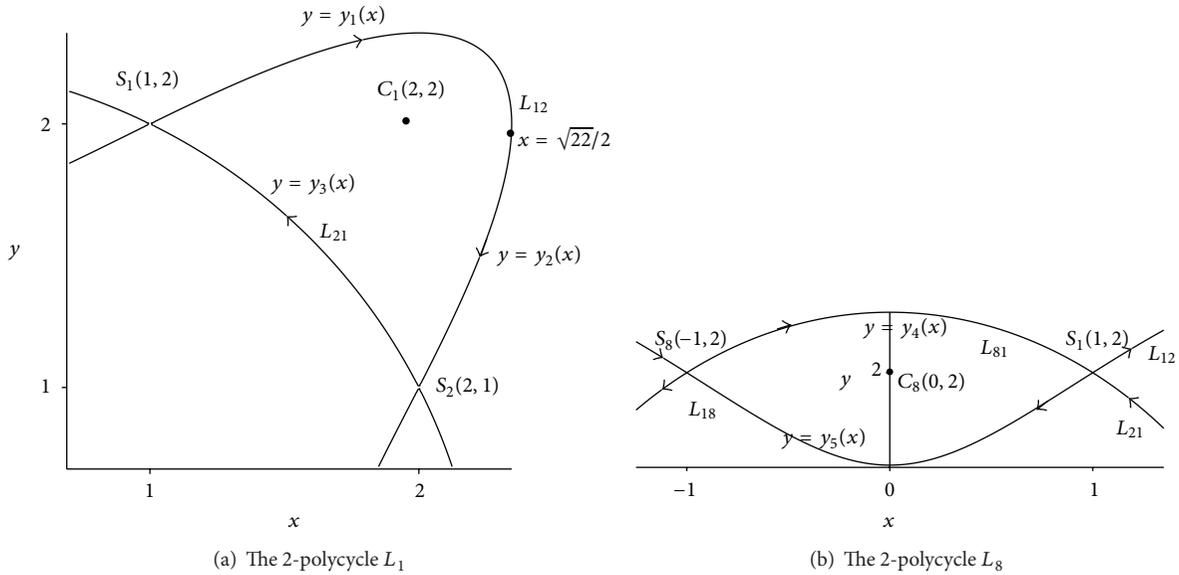


FIGURE 5

where, for  $i = 0, 1, 2, 3, 4, 5$ ,

$$\begin{aligned}
 I_{1i}^1 &= \int_1^{\sqrt{22}/2} (p_{1i}^1 + q_{1i}^1 \tilde{f}_1) dx, \\
 I_{2i}^1 &= \int_1^{\sqrt{22}/2} (p_{2i}^1 + q_{2i}^1 \tilde{f}_1) dx, \\
 I_{1i}^2 &= \int_{\sqrt{22}/2}^2 (p_{1i}^2 + q_{1i}^2 \tilde{f}_2) dx, \\
 I_{2i}^2 &= \int_{\sqrt{22}/2}^2 (p_{2i}^2 + q_{2i}^2 \tilde{f}_2) dx, \\
 I_{1i}^3 &= \int_2^1 (p_{1i}^3 + q_{1i}^3 \tilde{f}_3) dx, \\
 I_{2i}^3 &= \int_2^1 (p_{2i}^3 + q_{2i}^3 \tilde{f}_3) dx, \\
 I_{1i}^4 &= \int_{-1}^1 (p_{1i}^4 + q_{1i}^4 \tilde{f}_4) dx, \\
 I_{2i}^4 &= \int_{-1}^1 (p_{2i}^4 + q_{2i}^4 \tilde{f}_4) dx, \\
 I_{1i}^5 &= \int_1^{-1} (p_{1i}^5 + q_{1i}^5 \tilde{f}_5) dx, \\
 I_{2i}^5 &= \int_1^{-1} (p_{2i}^5 + q_{2i}^5 \tilde{f}_5) dx.
 \end{aligned}
 \tag{44}$$

Thus,

$$\begin{aligned}
 c_0(\delta) &= c_{01}(\delta) + c_{02}(\delta) \\
 &= \sum_{i=0}^5 [(I_{1i}^1 + I_{1i}^2 + I_{1i}^3) a_i + (I_{2i}^1 + I_{2i}^2 + I_{2i}^3) b_i]
 \end{aligned}$$

$$\equiv l_0 a_0 + l_1 a_1 + l_2 a_2 + l_3 a_3 + l_4 a_4 + l_5 a_5,$$

$$\begin{aligned}
 e_0(\delta) &= e_{01}(\delta) + e_{02}(\delta) \\
 &= \sum_{i=0}^5 [(I_{1i}^4 + I_{1i}^5 + I_{1i}^3) a_i + (I_{2i}^4 + I_{2i}^5 + I_{2i}^2) b_i] \\
 &\equiv m_0 a_0 + m_1 a_1 + m_2 a_2 + m_4 a_4 + m_5 a_5.
 \end{aligned}
 \tag{45}$$

By (4), the divergence of (1) at  $S_1$  and  $S_2$  is as follows:

$$\begin{aligned}
 d_1(\delta) &= \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) \Big|_{(1,2)} \\
 &= 2a_0 + 20a_1 + 150a_2 - 14a_4 \\
 &\quad - 70a_5 - 48b_4 + 240b_5, \\
 d_2(\delta) &= \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) \Big|_{(2,1)} \\
 &= 2a_0 + 20a_1 + 150a_2 - 14a_4 \\
 &\quad - 70a_5 + 48b_4 - 240b_5.
 \end{aligned}
 \tag{46}$$

Note that

$$\begin{aligned}
 \begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \Big|_{S_1} &= \begin{pmatrix} 0 & 24 \\ 6 & 0 \end{pmatrix}, \\
 \begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \Big|_{S_2} &= \begin{pmatrix} 0 & -6 \\ -24 & 0 \end{pmatrix},
 \end{aligned}
 \tag{47}$$

which yields

$$\lambda_1 = \lambda_2 = 12.
 \tag{48}$$

By Lemma 3, we have

$$\begin{aligned}
 c_1(\delta) &= e_1(\delta) = \sum_{i=1}^2 -\frac{1}{|\lambda_i|} \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) (S_i, \delta) \\
 &= -\frac{d_1(\delta)}{|\lambda_1|} - \frac{d_2(\delta)}{|\lambda_1|} \\
 &= -\frac{1}{3}a_0 - \frac{10}{3}a_1 - 25a_2 + \frac{7}{3}a_4 + \frac{35}{3}a_5.
 \end{aligned} \tag{49}$$

In the following by letting  $b_4 = 5b_5$ , then

$$d_1(\delta) = d_2(\delta), \quad c_1(S_1, \delta) = c_1(S_2, \delta). \tag{50}$$

Letting  $c_1(\delta) = 0$ , then

$$a_0 = -10a_1 - 75a_2 + 7a_4 + 35a_5. \tag{51}$$

Under  $c_1(\delta) = 0$ , we can apply Lemma 7 to calculate the coefficients  $c_2(\delta)$  and  $e_2(\delta)$ . For convenience, the following notations are introduced:  $h = dx/dt = 2y - 3y^3 + y^5$ . For  $j = 1, 2, 3, 4, 5$ ,  $h_j = (2y - 3y^3 + y^5)|_{y=y_j(x)}$ ,  $(dP_5/dx + dQ_5/dy) \equiv \sum_{j=0}^5 f_j a_j$ , and

$$\begin{aligned}
 f_0^j(x) &= \frac{f_0}{h} \Big|_{y=y_j(x)} = \frac{2}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)}, \\
 f_1^j(x) &= \frac{f_1}{h} \Big|_{y=y_j(x)} = \frac{4x^2 + 4y^2}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)}, \\
 f_2^j(x) &= \frac{f_2}{h} \Big|_{y=y_j(x)} = \frac{12y^2x^2 + 6x^4 + 6y^4}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)}, \\
 f_3^j(x) &= \frac{f_3}{h} \Big|_{y=y_j(x)} = \frac{0}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)} = 0, \\
 f_4^j(x) &= \frac{f_4}{h} \Big|_{y=y_j(x)} = \frac{-12y^2x^2 + 2x^4 + 2y^4}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)}, \\
 f_5^j(x) &= \frac{f_5}{h} \Big|_{y=y_j(x)} = \frac{-60y^2x^2 + 10x^4 + 10y^4}{2y - 3y^3 + y^5} \Big|_{y=y_j(x)}.
 \end{aligned} \tag{52}$$

By Lemma 7, we have

$$\begin{aligned}
 c_{21}(\delta) &= \int_{L_{12}} \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) dt = \sum_{i=0}^5 \int_{L_{12}} a_i \frac{f_i}{h} dx, \\
 c_{22}(\delta) &= \int_{L_{21}} \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) dt = \sum_{i=0}^5 \int_{L_{21}} a_i \frac{f_i}{h} dx, \\
 e_{21}(\delta) &= \int_{L_{81}} \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) dt = \sum_{i=0}^5 \int_{L_{81}} a_i \frac{f_i}{h} dx, \\
 e_{22}(\delta) &= \int_{L_{18}} \left( \frac{dP_5}{dx} + \frac{dQ_5}{dy} \right) dt = \sum_{i=0}^5 \int_{L_{18}} a_i \frac{f_i}{h} dx.
 \end{aligned} \tag{53}$$

Substituting (52) into (53), with  $f_0 = 2$  being considered, we have

$$\begin{aligned}
 c_{21}(\delta) &= \sum_{i=1}^5 \int_{L_{12}} a_i \frac{f_i}{h} dx \\
 &\quad + \int_{L_{12}} \frac{-20a_1 - 150a_2 + 14a_4 + 70a_5}{h} dx \\
 &= \sum_{i=1}^5 a_i J_i^1 + \sum_{i=1}^5 a_i J_i^2, \\
 c_{22}(\delta) &= \sum_{i=1}^5 \int_{L_{21}} a_i \frac{f_i}{h} dx \\
 &\quad + \int_{L_{21}} \frac{-20a_1 - 150a_2 + 14a_4 + 70a_5}{h} dx \\
 &= \sum_{i=1}^5 a_i J_i^3, \\
 e_{21}(\delta) &= \sum_{i=1}^5 \int_{L_{81}} a_i \frac{f_i}{h} dx \\
 &\quad + \int_{L_{81}} \frac{-20a_1 - 150a_2 + 14a_4 + 70a_5}{h} dx \\
 &= \sum_{i=1}^5 a_i J_i^4, \\
 e_{22}(\delta) &= \sum_{i=1}^5 \int_{L_{18}} a_i \frac{f_i}{h} dx \\
 &\quad + \int_{L_{18}} \frac{-20a_1 - 150a_2 + 14a_4 + 70a_5}{h} dx \\
 &= \sum_{i=1}^5 a_i J_i^5,
 \end{aligned} \tag{54}$$

where  $J_3^1 = J_3^2 = J_3^3 = J_3^4 = J_3^5 = 0$  and

$$\begin{aligned}
 J_1^1 &= \int_1^{\sqrt{22}/2} f_1^1(x) + \frac{-20}{h_1} dx, \\
 J_1^2 &= \int_{\sqrt{22}/2}^2 f_1^2(x) + \frac{-20}{h_2} dx, \\
 J_1^3 &= \int_2^1 f_1^3(x) + \frac{-20}{h_3} dx, \\
 J_2^1 &= \int_1^{\sqrt{22}/2} f_2^1(x) + \frac{-150}{h_1} dx,
 \end{aligned}$$

$$J_2^2 = \int_{\sqrt{22}/2}^2 f_2^2(x) + \frac{-150}{h_2} dx,$$

$$J_2^3 = \int_2^1 f_2^3(x) + \frac{-150}{h_3} dx,$$

$$J_4^1 = \int_1^{\sqrt{22}/2} f_4^1(x) + \frac{14}{h_1} dx,$$

$$J_4^2 = \int_{\sqrt{22}/2}^2 f_4^2(x) + \frac{14}{h_2} dx,$$

$$J_4^3 = \int_2^1 f_4^3(x) + \frac{14}{h_3} dx,$$

$$J_5^1 = \int_1^{\sqrt{22}/2} f_5^1(x) + \frac{70}{h_1} dx,$$

$$J_5^2 = \int_{\sqrt{22}/2}^2 f_5^2(x) + \frac{70}{h_2} dx,$$

$$J_5^3 = \int_2^1 f_5^3(x) + \frac{70}{h_3} dx,$$

$$J_1^4 = \int_{-1}^1 f_1^4(x) + \frac{-20}{h_4} dx,$$

$$J_1^5 = \int_1^{-1} f_1^5(x) + \frac{-20}{h_5} dx,$$

$$J_2^4 = \int_{-1}^1 f_2^4(x) + \frac{-150}{h_4} dx,$$

$$J_2^5 = \int_1^{-1} f_2^5(x) + \frac{-150}{h_5} dx,$$

$$J_4^4 = \int_{-1}^1 f_4^4(x) + \frac{14}{h_4} dx,$$

$$J_4^5 = \int_1^{-1} f_4^5(x) + \frac{14}{h_5} dx,$$

$$J_5^4 = \int_{-1}^1 f_5^4(x) + \frac{70}{h_4} dx,$$

$$J_5^5 = \int_1^{-1} f_5^5(x) + \frac{70}{h_5} dx.$$

Applying Lemma 7, we have

$$\begin{aligned} c_2(\delta) &= c_{21}(\delta) + c_{22}(\delta) \\ &= \sum_{i=0}^3 (J_i^1 + J_i^2 + J_i^3) a_i \\ &\equiv J_1 a_1 + J_2 a_2 + J_4 a_4 + J_5 a_5, \end{aligned}$$

$$\begin{aligned} e_2(\delta) &= e_{21}(\delta) + e_{22}(\delta) \\ &= \sum_{i=0}^3 (J_i^4 + J_i^5 + J_i^3) a_i \\ &\equiv R_1 a_1 + R_2 a_2 + R_4 a_4 + R_5 a_5. \end{aligned} \tag{56}$$

The integrals in (44) and (3) and the coefficients in (45) and (56) can be obtained by numeral computation on Maple 13; see Appendix.

In order to find the local coefficient  $c_3(S_1, \delta)$  at the saddle  $S_1(1, 2)$  we make a change of variables of the form  $u = (\sqrt{2}/2)(x - 1)$ ,  $v = \sqrt{2}(y - 2)$  and time rescaling  $\tau = kt$ ,  $k = 1$  so that the system (1) becomes

$$\frac{du}{d\tau} = \tilde{H}_v + \varepsilon \tilde{p}(u, v, \delta), \quad \frac{dv}{d\tau} = -\tilde{H}_u + \varepsilon \tilde{q}(u, v, \delta), \tag{57}$$

where

$$\begin{aligned} \tilde{H}(u, v) &= H(x, y)|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}} \\ &= 6v^2 - 6u^2 - \frac{10\sqrt{2}}{3}u^3 + \frac{25\sqrt{2}}{6}v^3 \\ &\quad + 5u^4 + \frac{35}{16}v^4 + 4\sqrt{2}u^5 + \frac{\sqrt{2}}{4}v^5 \\ &\quad + \frac{4}{3}u^6 + \frac{1}{48}v^6 - \frac{5}{12}, \end{aligned} \tag{58}$$

$$\tilde{p}(u, v) = \frac{\sqrt{2}}{2} P_5(x, y)|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}},$$

$$\tilde{q}(u, v) = \sqrt{2} Q_5(x, y)|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}}.$$

Writing functions  $\tilde{p}(u, v)$  and  $\tilde{q}(u, v)$  as the form

$$\tilde{p}(u, v) = \sum_{i+j=0}^5 \tilde{a}_{ij} u^i v^j, \tag{59}$$

$$\tilde{q}(u, v) = \sum_{i+j=0}^5 \tilde{b}_{ij} u^i v^j,$$

then the formula for the second local coefficient at the saddle in Lemma 2 can be applied directly; we have

$$c_3(S_1, \delta) = \frac{7}{216} a_1 + \frac{35}{72} a_2 + \frac{55}{216} a_4 + \frac{275}{216} a_5. \tag{60}$$

Similarly, we have

$$c_3(S_2, \delta) = \frac{7}{216} a_1 + \frac{35}{72} a_2 + \frac{55}{216} a_4 + \frac{275}{216} a_5. \tag{61}$$

Applying Lemma 7, we have

$$\begin{aligned} c_3(\delta) &= e_3(\delta) = c_3(S_1, \delta) + c_3(S_2, \delta) \\ &= \frac{7}{108} a_1 + \frac{35}{36} a_2 + \frac{55}{108} a_4 + \frac{275}{108} a_5. \end{aligned} \tag{62}$$

In order to find  $B_i(\delta)$ ,  $i = 0, 1, 2$ , we move the center  $C_1 = (2, 2)$  into the origin by letting  $u = 2\sqrt{6}(x - 2)$  and  $v = 2\sqrt{6}(y - 2)$ ; that is,  $x = (\sqrt{6}/12)u + 2$  and  $y = (\sqrt{6}/12)v + 2$  and make the time rescaling  $d\tau = 24dt$  so that the system (1) becomes

$$\frac{du}{d\tau} = \frac{dH_1^c}{dv} + \varepsilon p_1(u, v), \quad \frac{dv}{d\tau} = -\frac{dH_1^c}{du} + \varepsilon q_1(u, v), \tag{63}$$

where

$$\begin{aligned} H_1^c(u, v) &= H(x, y)|_{\{x=(\sqrt{6}/12)u+2, y=(\sqrt{6}/12)v+2\}} \\ &= -\frac{8}{3} + \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{25\sqrt{6}}{432}u^3 \\ &\quad + \frac{25\sqrt{6}}{432}v^3 + \frac{35}{2304}u^4 + \frac{35}{2304}v^4 + \frac{\sqrt{6}}{3456}u^5 \\ &\quad + \frac{\sqrt{6}}{3456}v^5 + \frac{1}{82944}u^6 + \frac{1}{82944}v^6, \end{aligned} \tag{64}$$

$$\begin{aligned} p_1(u, v) &= \frac{\sqrt{6}}{12} p(x, y)|_{\{x=(\sqrt{6}/12)u+2, y=(\sqrt{6}/12)v+2\}}, \\ q_1(u, v) &= \frac{\sqrt{6}}{12} q(x, y)|_{\{x=(\sqrt{6}/12)u+2, y=(\sqrt{6}/12)v+2\}}. \end{aligned}$$

Let

$$\begin{aligned} \widetilde{M}_1^c(h, \delta) &= \oint_{H_1^c(u,v)=h} q_1 du - p_1 dv \\ &= \sum_{k \geq 0} \widetilde{b}_k \left( h + \frac{8}{3} \right)^{k+1} \end{aligned} \tag{65}$$

which is the Melnikov function of the new system (63). Applying the formula for the Hopf coefficients  $\widetilde{b}_0(\delta)$ ,  $\widetilde{b}_1(\delta)$ , and  $\widetilde{b}_2(\delta)$  directly in [20], we have

$$\begin{aligned} \widetilde{b}_0(\delta) &= \frac{1}{6}\pi a_0 + \frac{8}{3}\pi a_1 + 32\pi a_2 - \frac{32}{3}\pi a_4 - \frac{160}{3}\pi a_5, \\ \widetilde{b}_1(\delta)|_{\widetilde{b}_0=0} &= -\frac{11}{108}\pi a_1 - \frac{19}{9}\pi a_2 \\ &\quad + \frac{25}{27}\pi a_4 + \frac{125}{27}\pi a_5, \tag{66} \\ \widetilde{b}_2(\delta)|_{\widetilde{b}_0=0} &= -\frac{17545}{839808}\pi a_1 - \frac{28379}{69984}\pi a_2 \\ &\quad + \frac{37445}{209952}\pi a_4 + \frac{187225}{209952}\pi a_5. \end{aligned}$$

By Lemma 6,

$$M_1(h, \delta) = \widetilde{M}_1^c(h, \delta). \tag{67}$$

Therefore

$$\begin{aligned} B_0(\delta) &= \widetilde{b}_0(\delta) = \frac{1}{6}\pi a_0 + \frac{8}{3}\pi a_1 \\ &\quad + 32\pi a_2 - \frac{32}{3}\pi a_4 - \frac{160}{3}\pi a_5, \\ B_1(\delta)|_{B_0=0} &= \widetilde{b}_1(\delta)|_{\widetilde{b}_0=0} = -\frac{11}{108}\pi a_1 - \frac{19}{9}\pi a_2 \\ &\quad + \frac{25}{27}\pi a_4 + \frac{125}{27}\pi a_5, \\ B_2(\delta)|_{B_0=0} &= \widetilde{b}_2(\delta)|_{\widetilde{b}_0=0} = -\frac{17545}{839808}\pi a_1 - \frac{28379}{69984}\pi a_2 \\ &\quad + \frac{37445}{209952}\pi a_4 + \frac{187225}{209952}\pi a_5. \end{aligned} \tag{68}$$

Similarly, the expressions of  $d_k(\delta)$  are obtained as follows:

$$\begin{aligned} d_0(\delta) &= \frac{\sqrt{6}}{6}\pi a_0 + \frac{4\sqrt{6}}{3}\pi a_1 + 8\sqrt{6}\pi a_2 \\ &\quad + \frac{8\sqrt{6}}{3}\pi a_4 + \frac{40\sqrt{6}}{3}\pi a_5, \\ d_1(\delta)|_{d_0=0} &= -\frac{\sqrt{6}}{108}\pi a_1 + \frac{\sqrt{6}}{18}\pi a_2 \\ &\quad - \frac{35\sqrt{6}}{54}\pi a_4 - \frac{175\sqrt{6}}{54}\pi a_5, \\ d_2(\delta)|_{d_0=0} &= \frac{535}{104976}\pi\sqrt{6}a_1 + \frac{1949}{17496}\pi\sqrt{6}a_2 \\ &\quad - \frac{12055}{52488}\pi\sqrt{6}a_4 - \frac{60275}{52488}\pi\sqrt{6}a_5. \end{aligned} \tag{69}$$

We will use the coefficients  $c_0(\delta)$ ,  $c_1(\delta)$ ,  $c_2(\delta)$ ,  $c_3(\delta)$ ,  $B_0(\delta)$ ,  $B_1(\delta)$ ,  $B_2(\delta)$ ,  $e_0(\delta)$ ,  $e_1(\delta)$ ,  $e_2(\delta)$ ,  $e_3(\delta)$ ,  $d_0(\delta)$ ,  $d_1(\delta)$ , and  $d_2(\delta)$  obtained above to study the limit cycle bifurcation.

(1) Solving the equations

$$c_0(\delta) = c_1(\delta) = d_0(\delta) = d_1(\delta) = 0 \tag{70}$$

gives

$$\begin{aligned} a_0 &= \mu_1 a_1, & a_2 &= \mu_2 a_1, \\ a_4 &= \mu_3 a_1, & a_5 &= \mu_4 a_1, \end{aligned} \tag{71}$$

where

$$\begin{aligned} \mu_1 &= -\frac{232}{73}, & \mu_2 &= -\frac{163}{1752}, \\ \mu_3 &= \frac{1}{1752} \frac{39l_5 + 27840l_0 - 8760l_1 + 815l_2}{-l_5 + 5l_4}, \\ \mu_4 &= -\frac{1}{1752} \frac{5568l_0 - 1752l_1 + 163l_2 + 39l_4}{-l_5 + 5l_4}. \end{aligned} \tag{72}$$

Approximate computation using Maple.13 gives

$$\begin{aligned}
 a_0 &= \mu_1 a_1 = -3.1780821917808219178 a_1 \equiv a_0^*, \\
 a_2 &= \mu_2 a_1 \\
 &= -0.093036529680365296804 a_1 \equiv a_2^*, \\
 a_4 &= \mu_3 a_1 \\
 &= 1.3418001549779299848 \times 10^8 a_1 \equiv a_4^*, \\
 a_5 &= \mu_4 a_1 \\
 &= -2.6836003104010654490 \times 10^7 a_1 \equiv a_5^*.
 \end{aligned} \tag{73}$$

We can easily find that

$$\text{rank} \frac{\partial (c_0, c_1, d_0, d_1)}{\partial (a_0, a_2, a_4, a_5)} = 4. \tag{74}$$

We fix  $a_1 > 0$ ,  $a_3 \neq 0$ , and take  $\delta = (a_0, a_2, a_4, a_5)$ ,  $\delta_0 = (a_0^*, a_2^*, a_4^*, a_5^*)$ . Inserting  $\delta_0$  into  $c_2(\delta)$ ,  $d_2(\delta)$ ,  $e_0(\delta)$  and  $B_0(\delta)$  we have

$$\begin{aligned}
 c_2(\delta_0) &= -1.6154178146383561644 a_1, \\
 d_2(\delta_0) &= -0.000155025650574300610\pi\sqrt{6} a_1, \\
 e_0(\delta_0) &= 0.001040156096575342 a_1, \\
 B_0(\delta_0) &= -0.6027397260050228310\pi a_1.
 \end{aligned} \tag{75}$$

Taking  $h_1 = -5/12 - \varepsilon_1$ ,  $h_2 = -8/3 + \varepsilon_1$ ,  $h_3 = -5/12 - \varepsilon_1$ , and  $h_4 = -4/3 + \varepsilon_1$  with  $\varepsilon_1 > 0$  small enough, we have

$$\begin{aligned}
 M_1(h_1, \delta_0) &= c_2(\delta_0)(-\varepsilon_1) + O((-\varepsilon_1)^2 \ln |\varepsilon_1|) > 0, \\
 M_1(h_2, \delta_0) &= B_0(\delta_0)(\varepsilon_1) + O(\varepsilon_1^2) < 0, \\
 M_8(h_3, \delta_0) &= e_0(\delta_0) + O((\varepsilon_1)^2 \ln |\varepsilon_1|) > 0, \\
 M_8(h_4, \delta_0) &= d_2(\delta_0)(\varepsilon_1)^3 + O((\varepsilon_1)^4) < 0,
 \end{aligned} \tag{76}$$

which yield  $(1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2 = 1$  and  $(1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2 = 1$ . Hence, by Theorem 8, (1) can have 6 limit cycles inside the 2-polycycles  $L_1$  and  $L_8$ , 2 limit cycles near the 2-polycycle  $L_1$ , 2 limit cycles near the center  $C_8$ , 1 limit cycle between the center  $C_1$  and the polycycle  $L_1$ , and 1 limit cycle between the center  $C_8$  and the polycycle  $L_8$ . Considering that the system (1) is  $Z_4$ -equivariant, the system (1) can have at least 24 limit cycles. See Figure 2(a) for their distribution.

(2) Further let

$$c_0(\delta) = c_1(\delta) = B_0(\delta) = e_0(\delta) = 0, \tag{77}$$

to obtain

$$\begin{aligned}
 a_0 &= w_1 a_1, & a_2 &= w_2 a_1, \\
 a_4 &= w_3 a_1, & a_5 &= w_4 a_1,
 \end{aligned} \tag{78}$$

where  $w_1 = -16(55m_2l_4 - 11m_2l_5 - 55m_4l_2 + 15m_5l_4 + 11m_5l_2 - 15m_4l_5 + 360m_4l_1 - 72m_5l_1 - 360m_1l_4 + 72m_1l_5)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$ ,  $w_2 = -(-2m_4l_5 + 880m_4l_0 - 95m_4l_1 - 176m_5l_0 + 2m_5l_4 + 19m_5l_1 - 880m_0l_4 + 176m_0l_5 + 95m_1l_4 - 19m_1l_5)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$ ,  $w_3 = (240m_5l_0 - 2m_2l_5 + 2m_5l_2 - 240m_0l_5 - 39m_5l_1 + 39m_1l_5 - 5760l_0m_1 - 880l_2m_0 + 95l_2m_1 + 880l_0m_2 + 5760l_1m_0 - 95l_1m_2)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$ , and  $w_4 = (240m_0l_4 - 39m_1l_4 + 2m_2l_4 - 19l_2m_1 - 240m_4l_0 + 176l_2m_0 + 1152l_0m_1 - 2m_4l_2 - 176l_0m_2 + 39m_4l_1 + 19l_1m_2 - 1152l_1m_0)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$ . Approximate computation using Maple.13 gives

$$\begin{aligned}
 a_0 &= -9.4485863716681452021 a_1 \equiv a_0^*, \\
 a_2 &= 0.0030582821716100337153 a_1 \equiv a_2^*, \\
 a_4 &= 8655624.6835808229498 a_1 \equiv a_4^*, \\
 a_5 &= -1731124.9144080276985 a_1 \equiv a_5^*.
 \end{aligned} \tag{79}$$

As before, we have

$$\text{rank} \frac{\partial (c_0, c_1, B_0, e_0)}{\partial (a_0, a_2, a_4, a_5)} = 4, \tag{80}$$

fixing  $a_1 > 0$ ,  $a_3 \neq 0$ , and taking  $\delta = (a_0, a_2, a_4, a_5)$  and  $\delta_0 = (a_0^*, a_2^*, a_4^*, a_5^*)$ . Noting that

$$\begin{aligned}
 c_2(\delta_0) &= -0.45572159725036392404 a_1, \\
 B_1(\delta_0) &= -0.0050298137906989600656\pi a_1, \\
 e_2(\delta_0) &= -0.85071588097 a_1, \\
 d_0(\delta_0) &= 0.0804770206482136030\pi\sqrt{6} a_1,
 \end{aligned} \tag{81}$$

we have

$$\begin{aligned}
 M_1(h_1, \delta_0) &= c_2(\delta_0)(-\varepsilon_3) + O(-\varepsilon_3 \ln |\varepsilon_3|) > 0, \\
 M_1(h_2, \delta_0) &= B_1(\delta_0)(\varepsilon_3^2) + O(\varepsilon_3^3) < 0, \\
 M_8(h_3, \delta_0) &= e_2(\delta_0)(-\varepsilon_3) + O((-\varepsilon_3) \ln |\varepsilon_3|) > 0, \\
 M_8(h_4, \delta_0) &= d_0(\delta_0)\varepsilon_3 + O((-\varepsilon_3)^2) > 0,
 \end{aligned} \tag{82}$$

where  $h_1 = -5/12 - \varepsilon_3$ ,  $h_2 = -8/3 + \varepsilon_3$ ,  $h_3 = -5/12 - \varepsilon_3$ , and  $h_4 = -4/3 + \varepsilon_3$  with  $\varepsilon_3 > 0$  being small. Hence, noting that  $(1 - \text{sgn}(M_1(h_1, \delta_0)M_1(h_2, \delta_0)))/2 = 1$ ,  $(1 - \text{sgn}(M_8(h_3, \delta_0)M_8(h_4, \delta_0)))/2 = 0$ , and  $(1 + \text{sgn}(c_2(\delta_0)e_2(\delta_0)))/2 = 1$ , by Theorem 8 again, we can obtain 6 limit cycles inside the 2-polycycles  $L_1$  and  $L_8$ . By  $Z_4$ -equivariance, the system (1) can have 24 limit cycles. See Figure 2(b). Then Theorem 1 is proved.

### 4. Conclusion

In this paper, we proved that a  $Z_4$ -equivalent quintic near-Hamiltonian system can also have 24 limit cycles compared

to a  $z_6$  and  $z_3$ -equivalent quintic near-Hamiltonian system having 24 limit cycles. Certainly, the distributions of 24 limit cycles obtained in this paper are new. The method we use is the expansions of the corresponding Melnikov functions, which is different from the methods of detect function and normal form, which are the main methods of the previous work on  $z_q$ -equivalent quintic near-Hamiltonian system.

### Appendix

In this section, by numeral computation using Maple.13, we give the approximate values of the integrals in (44) and (3) and the coefficients in (45) and (56) as

$$\begin{aligned}
 I_{10}^1 &= 3.309003029, \\
 I_{10}^2 &= 1.730303665, \\
 I_{10}^3 &= -3.217505541, \\
 I_{11}^1 &= 28.51638338, \\
 I_{11}^2 &= 12.79094042, \\
 I_{11}^3 &= -16.08752770, \\
 I_{12}^1 &= 254.3849935, \\
 I_{12}^2 &= 97.6750154, \\
 I_{12}^3 &= -80.43763849, \\
 I_{13}^1 &= -13.03562364, \\
 I_{13}^2 &= 1.03562364, \quad I_{13}^3 = 12, \\
 I_{14}^1 &= -141.0435372, \\
 I_{14}^2 &= -0.5116118346, \\
 I_{14}^3 &= 60, \quad I_{15}^1 = -317.8639878, \\
 I_{15}^2 &= -149.9117575, \\
 I_{15}^3 &= 60, \quad I_{10}^4 = 4.636476089, \\
 I_{10}^5 &= -3.114725692, \\
 I_{11}^4 &= 23.18238044, \\
 I_{11}^5 &= -10.86431923, \\
 I_{12}^4 &= 115.9119022, \\
 I_{12}^5 &= -39.42284767,
 \end{aligned}$$

$$\begin{aligned}
 I_{13}^4 &= 12, \quad I_{13}^5 = -12.00000000, \\
 I_{14}^4 &= 60, \quad I_{14}^5 = -43.62567211, \\
 I_{15}^4 &= 60, \quad I_{15}^5 = 21.87163945, \\
 I_{20}^1 &= \frac{9}{4}, \quad I_{20}^2 = -\frac{9}{4}, \quad I_{20}^3 = 0, \\
 I_{20}^4 &= 0, \quad I_{20}^5 = 0, \quad I_{21}^1 = \frac{261}{16}, \\
 I_{21}^2 &= -\frac{261}{16}, \quad I_{21}^3 = 0, \\
 I_{21}^4 &= 0, \quad I_{21}^5 = 0, \\
 I_{22}^1 &= \frac{1953}{16}, \quad I_{22}^2 = -\frac{1953}{16}, \quad I_{22}^3 = 0, \\
 I_{22}^4 &= 0, \quad I_{22}^5 = 0, \quad I_{23}^1 = -\frac{315}{16}, \\
 I_{23}^2 &= \frac{315}{16}, \quad I_{23}^3 = 0, \quad I_{23}^4 = 0, \\
 I_{23}^5 &= 0, \quad I_{24}^1 = -\frac{2385}{16}, \quad I_{24}^2 = \frac{2385}{16}, \\
 I_{24}^3 &= 0, \quad I_{24}^4 = 0, \quad I_{24}^5 = 0, \\
 I_{25}^1 &= -\frac{549}{16}, \quad I_{25}^2 = \frac{549}{16}, \quad I_{25}^3 = 0, \\
 I_{25}^4 &= 0, \quad I_{25}^5 = 0, \\
 J_1^1 &= 1.831700353, \\
 J_1^2 &= 1.100027834, \quad J_1^3 = 0, \\
 J_2^1 &= 36.46355425, \\
 J_2^2 &= 20.04837845, \\
 J_2^3 &= 0, \quad J_4^1 = -14.15186229, \\
 J_4^2 &= -7.471228579, \\
 J_4^3 &= -10.29601773, \\
 J_5^1 &= -70.75931146, \\
 J_5^2 &= -37.35614289, \\
 J_5^3 &= -51.48008865, \\
 J_1^4 &= 0, \quad J_1^5 = -3.964132938, \\
 J_2^4 &= 0, \quad J_2^5 = -49.77854225, \\
 J_4^4 &= 14.83672348, \\
 J_4^5 &= 14.44096924, \\
 J_5^4 &= 74.18361742,
 \end{aligned}$$

$$J_5^5 = 72.20484619, \quad (\text{A.1})$$

and  $l_3 = m_3 = 0$ ,

$$\begin{aligned} l_0 &= 1.821801153, \\ l_1 &= 25.21979610, \\ l_2 &= 271.6223704, \\ l_4 &= -81.55514903, \\ l_5 &= -407.7757453, \\ m_0 &= 1.521750397, \\ m_1 &= 12.31806121, \\ m_2 &= 76.48905453, \\ m_4 &= 16.37432789, \\ m_5 &= 81.87163945, \\ J_1 &= 2.931728187, \\ J_2 &= 56.51193270, \\ J_3 &= 0, \quad J_4 = -31.91910860, \\ J_5 &= -159.5955430, \\ R_1 &= -3.964132938, \\ R_2 &= -49.77854225, \\ R_3 &= 0, \quad R_4 = 29.27769272, \\ R_5 &= 146.3884636. \end{aligned} \quad (\text{A.2})$$

## Conflict of Interests

The authors declare that there is no conflict of interests for any of the authors in this paper.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (NSFC) (11161051), the Natural Science foundation of Guangxi Province in China (2012GXNSFDA 276040) supervised by Professor Hongjian Xi, and the Research Project of GUFU (2013YB216, 2012C08). The authors thank Professor Zhuoen Nong and Professor Qingguang He at Department of Information and Statistics of GUFU for their encouragement in research of differential equations.

## References

- [1] D. Hilbert, "Mathematical problems," *Bulletin of the American Mathematical Society*, vol. 8, no. 10, pp. 437–479, 1902.
- [2] J. Li, *Chaos and Melnikov Method*, Chongqin University Press, Chongqing, China, 1989, (Chinese).
- [3] J. Li, "Hilbert's 16th problem and bifurcations of planar polynomial vector fields," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 13, no. 1, pp. 47–106, 2003.
- [4] C. Li, "Abelian integrals and limit cycles," *Qualitative Theory of Dynamical Systems*, vol. 11, no. 1, pp. 111–128, 2012.
- [5] M. Han and J. Li, "Lower bounds for the Hilbert number of polynomial systems," *Journal of Differential Equations*, vol. 252, no. 4, pp. 3278–3304, 2012.
- [6] C. Li, C. Liu, and J. Yang, "A cubic system with thirteen limit cycles," *Journal of Differential Equations*, vol. 246, no. 9, pp. 3609–3619, 2009.
- [7] L. Zhao, "Some new distribution of 13 limit cycles of a cubic system," *Journal of Beijing Normal University*, vol. 48, pp. 231–234, 2012.
- [8] J. Li and Y. Liu, "New results on the study of  $Z_q$ -equivariant planar polynomial vector fields," *Qualitative Theory of Dynamical Systems*, vol. 9, no. 1-2, pp. 167–219, 2010.
- [9] T. Zhang, M. Han, H. Zang, and X. Meng, "Bifurcations of limit cycles for a cubic Hamiltonian system under quartic perturbations," *Chaos, Solitons & Fractals*, vol. 22, no. 5, pp. 1127–1138, 2004.
- [10] C. Christopher, "Estimating limit cycle bifurcations from center," in *Differential Equations with Symbolic Computation*, Trends in Mathematics, pp. 23–35, Birkhäuser, Basel, Switzerland, 2005.
- [11] W. Xu and M. Han, "On the number of limit cycles of a  $Z_4$ -equivariant quintic polynomial system," *Applied Mathematics and Computation*, vol. 216, no. 10, pp. 3022–3034, 2010.
- [12] J. Li, H. S. Y. Chan, and K. W. Chung, "Investigations of bifurcations of limit cycles in  $Z_2$ -equivariant planar vector fields of degree 5," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 10, pp. 2137–2157, 2002.
- [13] Y. Wu, L. Tian, and Y. Hu, "On the limit cycles of a Hamiltonian under  $Z_4$ -equivariant quintic perturbation," *Chaos, Solitons and Fractals*, vol. 33, no. 1, pp. 298–307, 2007.
- [14] J. Li, H. S. Y. Chan, and K. W. Chung, "Bifurcations of limit cycles in a  $Z_6$ -equivariant planar vector field of degree 5," *Science in China. Series A*, vol. 45, no. 7, pp. 817–826, 2002.
- [15] W. H. Yao and P. Yu, "Bifurcation of small limit cycles in  $Z_5$ -equivariant planar vector fields of order 5," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 1, pp. 400–413, 2007.
- [16] Y. H. Wu, X. D. Wang, and L. X. Tian, "Bifurcations of limit cycles in a  $Z_4$ -equivariant quintic planar vector field," *Acta Mathematica Sinica. English Series*, vol. 26, no. 4, pp. 779–798, 2010.
- [17] Y. Wu and M. Han, "New configurations of 24 limit cycles in a quintic system," *Computers & Mathematics with Applications*, vol. 55, no. 9, pp. 2064–2075, 2008.
- [18] M. Han, "Asymptotic expansions of Melnikov functions and limit cycle bifurcations," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 22, Article ID 1250296, 30 pages, 2012.
- [19] M. Han, J. Yang, A.-A. Tarța, and Y. Gao, "Limit cycles near homoclinic and heteroclinic loops," *Journal of Dynamics and Differential Equations*, vol. 20, no. 4, pp. 923–944, 2008.
- [20] Y. Hou and M. Han, "Melnikov functions for planar near-Hamiltonian systems and Hopf bifurcations," *Journal of Shanghai Normal University*, vol. 35, no. 1, pp. 1–10, 2006.

- [21] J. Yang, "On the limit cycles of a kind of Liénard system with a nilpotent center under perturbations," *The Journal of Applied Analysis and Computation*, vol. 2, no. 3, pp. 325–339, 2012.
- [22] R. Kazemi and H. R. Z. Zangeneh, "Bifurcation of limit cycles in small perturbations of a hyper-elliptic Hamiltonian system with two nilpotent saddles," *The Journal of Applied Analysis and Computation*, vol. 2, no. 4, pp. 395–413, 2012.
- [23] X. Sun, "Bifurcation of limit cycles from a Liénard system with a heteroclinic loop connecting two nilpotent saddles," *Nonlinear Dynamics*, vol. 73, no. 1-2, pp. 869–880, 2013.
- [24] M. Han and X. Sun, "General form of reversible equivariant system and its limit cycles," *Journal of Shanghai Normal University*, vol. 40, no. 1, pp. 1–14, 2011.