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Research Article

Nonlinear Decomposition of Doob-Meyer's Type for Continuous g-Supermartingale with Uniformly Continuous Coefficient

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We prove that a continuous g-supermartingale with uniformly continuous coeffcient g on finite or infinite horizon, is a g-supersolution of the corresponding backward stochastic differential equation. It is a new nonlinear Doob-Meyer decomposition theorem for the g-supermartingale with continuous trajectory.

1. Introduction

In 1990, Pardoux-Peng [1] proposed the following nonlinear backward stochastic differential equation (BSDE) driven by a Brownian motion:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\, ds - \int_t^T z_s dB_s, \quad t \in [0, T], \tag{1}$$

where the positive real number T, the random variable ξ, and the function g are called the time horizon, the terminal data, and the generator, respectively, and the pair of adapted processes $(y_t, z_t)_{t\in[0,T]}$ to be known is called the solution of the BSDE (1). In this paper, we study a more generalized BSDE with a given increasing process $(V_t)_{t\in[0,T]}$ with $V_0 = 0$:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\, ds - \int_t^T z_s\ dB_s + V_T - V_t, \quad t \in [0, T]. \tag{2}$$

If $(V_t)_{t\in[0,T]} \equiv 0$, the first component $(y_t)_{t\in[0,T]}$ of solution of (2) is called the g-solution of (1); otherwise, it is called the g-supersolution. Subsequently, Peng [2] introduced the nonlinear expectation and nonlinear martingale theories via BSDEs. In [3], Peng first obtained the monotonic limit theorem; that is, under some mild conditions, the limit of a monotonically increasing sequence of g-supersolutions is also a g-supersolution. And applying this result, he proved that a càdlàg g-martingale, which is right continuous with left limits, had a nonlinear decomposition of Doob-Meyer's type, corresponding to the classical martingale theory. Later, Lin [4, 5] extended Peng's result and got this decomposition for the g-supermartingale with respect to a general continuous filtration and that with jumps, respectively. It should be pointed out that, in Peng [3] and Lin [4, 5], the monotonic limit theorem for BSDEs plays a key role, and it is also useful in other problems. For example, in [6], Peng-Xu put forward a generalized version of monotonic limit theorem and proved that solving the reflected BSDE with a given lower barrier process was equivalent to finding the smallest g-supermartingale dominating the barrier. And Peng-Xu [7] used this technique to treat the problems of the BSDE with generalized constraints and solve the American option pricing problem in an incomplete market. On the other hand, motivated by the theories of the classical martingale and the nonlinear martingale, Chen-Wang [8] showed that the BSDEs on infinite time horizon were solvable, under the Lipschitz assumption on g, whose Lipschitzian coefficient is a function depending on t, and they obtained the convergence theorem of the nonlinear g-martingale. Afterward, Fan et al. [9] explored the BSDEs on finite or infinite horizon, without the Lipschitz assumption, and got an existence and uniqueness result and a comparison theorem.

Based on these results, a natural question is, under the generalized uniformly continuous assumption on the coefficient g, does the g-martingale still have a nonlinear decomposition of Doob-Meyer's type? Our answer is yes. We prove that if a g-supermartingale has a continuous trajectory on finite or infinite time interval, then it is a g-supersolution of the corresponding BSDE; that is, it has a nonlinear Doob-Meyer decomposition. It should be noted that our results are based on the conditions without the Lipschitz assumption on the coefficient g. And our results do not depend on the infinite time version of the monotonic limit but only on the penalization method.

The outline of this paper is as follows. Section 2 provides some assumptions, definitions, and the existence and uniqueness theorem and comparison theorem for a generalized BSDE with generalized uniformly continuous generator g. Then, Section 3 devotes to the main result a new version of nonlinear Doob-Meyer's decomposition theorem for the continuous g-supermartingale with the generalized uniformly continuous coefficient.

2. Preliminaries

Let T be a finite or infinite nonnegative extended real number, and let $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by this Brownian motion:

$$\mathcal{F}_t \triangleq \sigma\{B_s : 0 \leq s \leq t\} \vee \mathcal{N}, \quad \mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t, \tag{3}$$

where $\mathcal{N}$ is the set of all P-null subsets.

For simplicity of presentation, we use $|x|$ to denote the Euclidean norm of x in $\mathbb{R}$ or $\mathbb{R}^d$, and let $L^2(\Omega, \mathcal{F}_t, P)$ be the space of all the $\mathcal{F}_t$ measurable square integrable real valued random variables, and define the adapted process spaces as follows:

$\mathcal{H}^2(0,\tau;\mathbb{R}^d) := \{(\phi_t)_{t\in[0,\tau]}$ is a $\mathbb{R}^d$-valued process such that $\mathbb{E}[\int_0^\tau |\phi_t|^2 dt] < +\infty\}$;

$\mathcal{S}^2(0,\tau;\mathbb{R}) := \{(\phi_t)_{t\in[0,\tau]}$ is a càdlàg $\mathbb{R}$- valued process such that $\mathbb{E}[\sup_{0\leq t\leq\tau} |\phi_t|^2] < +\infty\}$;

$\mathcal{A}^2(0,\tau;\mathbb{R}) := \{(\phi_t)_{t\in[0,\tau]}$ is an increasing process in $\mathcal{S}^2(0,\tau;\mathbb{R})$ with $\phi(0) = 0\}$.

Clearly, all the above spaces of stochastic processes are completed Banach spaces.

Furthermore, we denote the set of linear increasing functions $\phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\phi(0) = 0$ by $\mathcal{X}$. Here the linear increasing means that, for any element $\phi \in \mathcal{X}$, there exists a pair of positive real numbers (a, b) depending on ϕ such that, for all $x \in \mathbb{R}_+$, $\phi(x) \leq ax + b$.

The generator $g(t,\omega,y,z) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is a random function which is a progressively measurable stochastic process for any (y, z). We assume that it satisfies the following two assumptions, where (H2) is a generalized uniformly continuous condition; that is, its modulus of continuity may depend on t:

(H1) $\mathbb{E}[(\int_0^T |g(t,0,0)| dt)^2] < \infty$;

(H2) $|g(t,y,z) - g(t,y',z')| \leq u(t)\varphi(|y-y'|) + v(t)\psi(|z-z'|)$, where $u(\cdot)$ and $v(\cdot)$ are two positive functions mapping from $\mathbb{R}^+$ to $\mathbb{R}^+$, such that $\int_0^T [u(t) + v^2(t)]\, dt < \infty$; the functions φ and ψ belong to $\mathcal{X}$ and $\varphi(\cdot)$ is a concave function, with $\int_{0^+} \varphi(t)\, dt = +\infty$. And in addition, we assume that $\int_0^T v(t)\, dt < \infty$, if ψ cannot be dominated by a linear function; that is, we cannot find a real number a, such that $\phi(x) \leq ax$.

Remark 1. In (H2), $\varphi(\cdot)$ is a concave function which means that $\varphi(\lambda t_1 + (1-\lambda)t_2) \geq \lambda\varphi(t_1) + (1-\lambda)\varphi(t_2)$, for $\lambda \in [0,1]$ and $t_1, t_2 \in [0,T]$. And the equality $\int_{0^+} \varphi(t)\, dt = \infty$ means that the value of the integration $\int_0^\delta \varphi(t)\, dt$ will be infinite on any interval $[0,\delta]$ with $\delta > 0$. For simplicity, we also use u_s and v_s to denote $u(s)$ and $v(s)$, respectively, in the remaining of this paper.

Now, we consider the following problem. Suppose that the time horizon T, generator g, terminal data y_T , and the increasing càdlàg process $(V_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R})$ are given in advance; let us find a pair of processes $(y_t, z_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^d)$ satisfying

$$y_t = y_T + \int_t^T g(s, y_s, z_s)\, ds - \int_t^T z_s dB_s + V_T - V_t, \quad t \in [0,T]. \tag{4}$$

If $V_t \equiv 0$, the above equation (4) will be a classical BSDE on finite or infinite horizon; the existence and uniqueness result is already obtained, which is stated by Theorem 3 in Fan et al. [9]. Otherwise we can set $\overline{y}_t := y_t + V_t$ and treat the following BSDE as

$$\overline{y}_t = y_T + V_T + \int_t^T g(s, \overline{y}_s - V_s, z_s)\, ds - \int_t^T z_s dB_s. \tag{5}$$

It is a classical BSDE with the terminal data $\xi := y_T + V_T$ and the generator $\overline{g} := g(t, y - V_t, z)$. Since the assumptions (H1) and (H2) hold for generator g, it is easy to verify that $\overline{g}$ still satisfies the two conditions. So we have the following existence and uniqueness theorem.

Lemma 2 (existence and uniqueness). *One assumes that the generator g of the BSDE (4) satisfies the conditions (H1) and (H2). Then, for any random variable $y_T \in L^2(\Omega, \mathcal{F}_T, P)$, and a process $(V_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R})$, there exists a unique pair of processes $(y_t, z_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^d)$, which is a solution of the BSDE (4), such that $(y_t + V_t)_{t\in[0,T]}$ is continuous and*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y_t|^2\right] < \infty. \tag{6}$$

We can also have the following comparison theorem, which will be used in the latter part of this section and the next one.

Proposition 3 (comparison). *Suppose that the assumptions in Lemma 2 hold. Let $(\overline{y}_t, \overline{z}_t)_{t\in[0,T]}$ be the solution of another BSDE:*

$$\overline{y}_t = \overline{y}_T + \int_t^T \overline{g}_s ds - \int_t^T \overline{z}_s dB_s + \overline{V}_T - \overline{V}_t, \quad t \in [0,T], \quad (7)$$

where $(\overline{V}_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R})$, $\overline{y}_T \in L^2(\Omega, \mathcal{F}_T, P)$, and $(\overline{g}_t)_{t\in[0,T]}$ are given such that

(1) $\widehat{y}_T := y_T - \overline{y}_T \geq 0$;

(2) $\widehat{g}_t := g(t, \overline{y}_t, \overline{z}_t) - \overline{g}_t \geq 0$, $dP \times dt$*-a.e.;*

(3) $\widehat{V}_t := V_t - \overline{V}_t$ *is a càdlàg increasing process;*

(4) $\mathbb{E}[(\int_0^T |\overline{g}_t| dt)^2] < \infty$.

Then we have, P-a.s.,

$$y_t \geq \overline{y}_t, \quad \forall t \in [0,T]. \quad (8)$$

Proof. We sketch the proof as follows. Set $\widehat{y}_t = \overline{y}_t - y_t$ and $\widehat{z}_t = \overline{z}_t - z_t$; applying Itô-Meyer's formula to $(\overline{y}_t - y_t)^+$ leads to

$$\begin{aligned}\widehat{y}_t^+ \leq (\overline{y}_T - y_T)^+ &+ \int_t^T 1_{\{\widehat{y}_s>0\}} (\overline{g}_s - g(s, y_s, z_s))\, ds \\ &- \int_t^T 1_{\{\widehat{y}_s>0\}} \widehat{z}_s dB_s + \int_t^T 1_{\{\widehat{y}_{s-}>0\}} \left(d\overline{V}_s - dV_s\right).\end{aligned} \quad (9)$$

Since $\widehat{V}_t$ is an increasing process, we see that

$$\int_t^T 1_{\{\widehat{y}_{s-}>0\}} \left(d\overline{V}_s - dV_s\right) \leq -\int_t^T 1_{\{\widehat{y}_{s-}>0\}} d\widehat{V}_s \leq 0. \quad (10)$$

Recalling that $\widehat{g}_t := g(t, \overline{y}_t, \overline{z}_t) - \overline{g}_t \geq 0$, $dP \times dt$-a.e., and the assumption (H2), we can get

$$\begin{aligned}&1_{\{\widehat{y}_s>0\}} (\overline{g}_s - g(s, y_s, z_s)) \\ &\quad = 1_{\{\widehat{y}_s>0\}} (\overline{g}_s - g(s, \overline{y}_s, \overline{z}_s) + g(s, \overline{y}_s, \overline{z}_s) - g(s, y_s, z_s)) \\ &\quad \leq u_s \varphi(\widehat{y}_s^+) + 1_{\{\widehat{y}_s>0\}} v_s \psi(|\widehat{z}_s|).\end{aligned} \quad (11)$$

Thus, it follows that

$$\widehat{y}_t^+ \leq \int_t^T u_s \varphi(\widehat{y}_s^+) + 1_{\{\widehat{y}_s>0\}} v_s \psi(|\widehat{z}_s|)\, ds - \int_t^T 1_{\{\widehat{y}_s>0\}} \widehat{z}_s dB_s. \quad (12)$$

Now we are in the same position with Theorem 2 in Fan et al. [9]. Then we can prove that, for all $t \in [0,T]$, $\widehat{y}_t^+ \leq 0$, P-a.s. Therefore, for any $t \in [0,T]$, we have

$$\overline{y}_t \leq y_t, \quad \text{P-a.s.} \quad (13)$$

Observing that y_t and $\overline{y}_t$ are càdlàg processes, we can conclude that, P-a.s.,

$$\overline{y}_t \leq y_t, \quad \forall t \in [0,T]. \quad (14)$$

□

Remark 4. If we replace the deterministic terminal time T by a $\mathcal{F}_t$-stopping time $\tau \leq T$, then, by Lemma 2, existence and uniqueness theorem and the above comparison theorem still hold true.

For a given stopping time $\tau \leq T$, we now consider the following BSDE:

$$y_t = y_\tau + \int_{t\wedge\tau}^{\tau} g(s, y_s, z_s)\, ds - \int_{t\wedge\tau}^{\tau} z_s dB_s + A_\tau - A_{t\wedge\tau}, \quad (15)$$

where $y_\tau \in L^2(\Omega, \mathcal{F}_\tau, P)$ and $(A_t)_{t\in[0,T]} \in \mathcal{A}^2(0,\tau;\mathbb{R})$ is a given càdlàg increasing process with $A_0 = 0$.

Next, we introduce the conceptions of g-solution, g-supersolution, g-martingale, and g-supermartingale closely following Peng's definitions in [3].

Definition 5. If a process $(y_t)_{t\in[0,\tau]}$ can be written in the form of the BSDE (15) with the generator g, then one call it a g-supersolution on $[0,\tau]$. Particularly, if $(A_t)_{t\in[0,\tau]} \equiv 0$ on $[0,\tau]$, then one call $(y_t)_{t\in[0,\tau]}$ a g-solution on $[0,\tau]$.

Definition 6. A $\mathcal{F}_t$-progressively measurable real-valued process $(Y_t)_{t\in[0,\tau]}$ is called a g-supermartingale (resp. g-martingale), if for each stopping time $\tau \leq T$, $\mathbb{E}\,|Y_\tau|^2 < \infty$, and the g-solution $(y_t)_{t\in[0,\tau]}$ with terminal condition $y_\tau = Y_\tau$ satisfies $y_\sigma \leq Y_\sigma$ (resp. $y_\sigma = Y_\sigma$) for all stopping time $\sigma \leq \tau$. Indeed, a g-martingale on $[0,T]$ is a g-solution on $[0,T]$.

From Proposition 3, we know that a g-supersolution is a g-supermartingale. Conversely, a meaningful and interesting question follows immediately. Is a g-supermartingale a g-supersolution? If so, does the g-supermartingale, or g-supersolution, has a unique representation of the form (15)?

According to Proposition 1.6 in [3], we can assert that, given a g-supersolution $(y_t)_{t\in[0,\tau]}$ on $[0,\tau]$, there is a unique pair of processes $(z_t, A_t)_{t\in[0,\tau]} \in \mathcal{H}^2(0,\tau;\mathbb{R}^d) \times \mathcal{A}^2(0,\tau;\mathbb{R})$ on $[0,\tau]$ such that the triple $(y_t, z_t, A_t)_{t\in[0,\tau]}$ satisfies the BSDE (15). Now, we can propose the next conception as follows.

Definition 7. Provided that the process $(y_t)_{t\in[0,T]}$ is a g-supersolution and the triple of processes $(y_t, z_t, A_t)_{t\in[0,\tau]}$ satisfies the BSDE (15), one call $(z_t, A_t)_{t\in[0,\tau]}$ the unique decomposition of $(y_t)_{t\in[0,\tau]}$.

3. Nonlinear Doob-Meyer's Decomposition for g-Supermartingale with Uniformly Continuous Coefficient

In this section, we provide and prove the main result of this paper that a continuous g-supermartingale is a g-supersolution; that is, it has a unique decomposition in the sense of Definition 7.

Theorem 8. *One assumes that g satisfies the conditions (H1) and (H2). Let $(Y_t)_{t\in[0,T]}$ be a continuous g-supermartingale on $[0,T]$ in $\mathcal{S}^2(0,T;\mathbb{R})$. Then $(Y_t)_{t\in[0,T]}$ is a g-supersolution on $[0,T]$ that is, there is a unique pair of* *processes* $(z_t, A_t)_{t\in[0,T]}$ *in* $\mathscr{H}^2(0,T;\mathbb{R}^d) \times \mathscr{A}^2(0,T;\mathbb{R})$, *such that* $(Y_t)_{t\in[0,T]}$ *coincides with the first component* $(y_t)_{t\in[0,T]}$ *of the solution for the following BSDE:*

$$y_t = Y_T + \int_t^T g(s, y_s, z_s)\,ds - \int_t^T z_s dB_s + A_T - A_t, \quad t \in [0,T]. \tag{16}$$

In order to prove this theorem, we consider the family of penalization BSDEs parameterized by $n = 1, 2, 3, \ldots$,

$$\begin{aligned} y_t^n = Y_T &+ \int_t^T g(s, y_s^n, z_s^n)\,ds - \int_t^T z_s^n dB_s \\ &+ n\int_t^T v_s^2 (Y_s - y_s^n)\,ds, \quad t \in [0,T], \end{aligned} \tag{17}$$

and set

$$A_t^n := n\int_0^t v_s^2 (Y_s - y_s^n)\,ds, \quad t \in [0,T]. \tag{18}$$

We first claim the next proposition.

Proposition 9. *For each* $n = 1, 2, \ldots$, *one has, P-a.s.,*

$$Y_t \ge y_t^n, \quad t \in [0,T]. \tag{19}$$

Proof. Using an argument similar to that in Lemma 3.4 in [3], one can carry out the proof by contradiction. We sketch it as follows.

Supposing that it is not the case, then there exist $\delta > 0$ and a positive integer n such that the measure of $\{(\omega, t) \mid y_t^n - Y_t - \delta \ge 0\} \subset \Omega \times [0,T]$ is nonzero; then we can define the following stopping times:

$$\begin{aligned} \sigma &:= \min\{T, \inf\{t \mid y_t^n \ge Y_t + \delta\}\}, \\ \tau &:= \inf\{t \ge \sigma \mid y_t^n \le Y_t\}. \end{aligned} \tag{20}$$

It is observed, from the above definition and the continuous of $(Y_t)_{t\in[0,T]}$, that $\sigma \le \tau \le T$ and $P(\tau > \sigma) > 0$. And furthermore, we have, P-a.s.,

$$\begin{aligned} &\text{(i)}\;\; y_\sigma^n \ge Y_\sigma + \delta; \\ &\text{(ii)}\;\; y_\tau^n \le Y_\tau. \end{aligned} \tag{21}$$

Now let $(y_t)_{t\in[0,\tau]}$ (resp. $(y_t')_{t\in[0,\tau]}$) be the g-solution on $[0,\tau]$ with terminal condition $y_\tau = y_\tau^n$ (resp. $y_\tau' = Y_\tau$). By Proposition 3, (21)-(ii) implies that $y_\sigma^n \le y_\sigma \le y_\sigma'$. On the other hand since $(Y_t)_{t\in[0,T]}$ is a g-supermartingale, thus we can get

$$Y_\sigma \ge y_\sigma^n, \quad \text{P-a.s.} \tag{22}$$

This is a contradiction to (21)-(i). Then by Fubini's theorem, we have, P-a.s.,

$$Y_t \ge y_t^n, \quad \text{dt-a.e.} \tag{23}$$

And the conclusion follows from the continuity of $(Y_t)_{t\in[0,T]}$ $(y_t^n)_{t\in[0,T]}$. The proof is completed. □

Now, we can get the following result; the boundedness of the triple of the processes $(y_t^n, z_t^n, A_t^n)_{t\in[0,T]}$ can be defined by the penalization BSDEs.

Proposition 10. *There exists a positive real number C such that for any positive integer n*

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_t^n|^2 + \int_0^T |z_t^n|^2 dt + |A_T^n|^2\right] \le C. \tag{24}$$

Proof. From BSDE (17), we have

$$\begin{aligned} A_T^n &= y_0^n - Y_T - \int_0^T g(s, y_s^n, z_s^n)\,ds + \int_0^T z_s^n dB_s \\ &\le |y_0^n| + |Y_T| + \int_0^T [|g(s,0,0)| + u_s \varphi(|y_s^n|) \\ &\quad + v_s \psi(|z_s^n|)]\,ds + \left|\int_0^T z_s^n dB_s\right| \\ &\le |y_0^n| + |Y_T| + \int_0^T [|g(s,0,0)| + u_s (a_\varphi |y_s^n| + b_\varphi) \\ &\quad + v_s (a_\psi |z_s^n| + b_\phi)]\,ds + \left|\int_0^T z_s^n dB_s\right|, \end{aligned} \tag{25}$$

where the real numbers a_φ, b_φ and a_ψ, b_ψ depend on the functions φ and ψ, respectively. From Proposition 9, we see that y_t^n is dominated by $|y_t^1| + |Y_t|$. Thus there exists a constant C_0 independent of n, such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_t^n|^2\right] \le C_0. \tag{26}$$

Now, noticing the boundedness of $(y_t^n)_{t\in[0,T]}$ in the above sense, from the basic algebraic inequality, Jensen's inequality and Hölder's inequality, we can get that there exists another constant C_1 such that

$$\begin{aligned} \mathbb{E}|A_T^n|^2 &\le 8\mathbb{E}\Bigg[|y_0^n|^2 + |Y_T|^2 + \left(\int_0^T |g(s,0,0)|\,ds\right)^2 \\ &\quad + a_\varphi^2 \left(\int_0^T u_s ds\right)^2 \sup_{0\le t\le T} |y_t^n|^2 \\ &\quad + b_\varphi^2 \left(\int_0^T u_s ds\right)^2 + a_\psi^2 \int_0^T v_s^2 ds \int_0^T |z_s^n|^2 ds \\ &\quad + b_\psi^2 \left(\int_0^T v_s ds\right)^2 + \int_0^T |z_s^n|^2 ds\Bigg] \\ &\le C_1 + C_1 \mathbb{E}\int_0^T |z_s^n|^2 ds. \end{aligned} \tag{27}$$

On the other hand, in the light of (H2), applying Itô's formula to $|y_t^n|^2$ on $[0,T]$ will lead to

$$\begin{aligned} &|y_0^n|^2 + \mathbb{E}\int_0^T |z_s^n|^2\, ds \\ &= \mathbb{E}|Y_T|^2 + 2\mathbb{E}\int_0^T y_s^n g(s, y_s^n, z_s^n)\, ds + 2\mathbb{E}\int_0^T y_s^n dA_s^n \\ &\le \mathbb{E}|Y_T|^2 + 2\mathbb{E}\int_0^T [|y_s^n| (|g(s,0,0)| \\ &\qquad + u_s\varphi(|y_s^n|) + v_s\psi(|z_s^n|))]\, ds \\ &\quad + 2\mathbb{E}\int_0^T y_s^n dA_s^n. \end{aligned} \tag{28}$$

Then, the Hölder inequality and the inequality $ab \le \epsilon a^2 + 1/\epsilon b^2$, for all $a, b, \epsilon > 0$, imply that

$$\begin{aligned} &|y_0^n|^2 + \mathbb{E}\int_0^T |z_s^n|^2 ds \\ &\le \mathbb{E}|Y_T|^2 + \left(1 + b_\varphi + b_\psi + 2a_\varphi \int_0^T u_s ds \right. \\ &\qquad \left. + 2a_\psi^2 \int_0^T v_s^2 ds\right) \mathbb{E}\left[\sup_{0\le t\le T} |y_s^n|^2\right] \\ &\quad + \mathbb{E}\left[\left(\int_0^T |g(s,0,0)|\, ds\right)^2\right] + b_\varphi\left(\int_0^T u_s ds\right)^2 \\ &\quad + b_\psi\left(\int_0^T v_s ds\right)^2 + \frac{1}{2}\mathbb{E}\int_0^T |z_s^n|^2\, ds \\ &\quad + \frac{1}{4C_1}\mathbb{E}\left[|A_T^n|^2\right] + 4C_1\mathbb{E}\left[\sup_{0\le t\le T} |y_s^n|^2\right]. \end{aligned} \tag{29}$$

Thus, we can choose a constant C_2 satisfying

$$\mathbb{E}\int_0^T |z_s^n|^2 ds \le C_2 + \frac{1}{2C_1}\mathbb{E}|A_T^n|^2. \tag{30}$$

Combining the inequalities (27) and (30), we can conclude that $\mathbb{E}|A_T^n|^2 \le 2C_1(1+C_2)$ and $\mathbb{E}\int_0^T |z_s^n|^2\, ds \le 1 + 2C_2$. The proof is completed. □

Then, we give a proposition which plays a key role in the procedure to prove the main theorem.

Proposition 11.

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} (Y_t - y_t^n)^2\right] = 0. \tag{31}$$

Proof. Since the family of the processes $(y_t^n)_{t\in[0,T]}$ is increasing in n and dominated by the process $(Y_t)_{t\in[0,T]}$ from the above, we can define a process $(y_t)_{t\in[0,T]}$ pointwise by the limit of the processes sequence. Then we have, P-a.s.,

$$y_t := \lim_{n\to\infty} y_t^n, \quad \forall t \in [0,T]. \tag{32}$$

And according to Lemma 2, for any integer n, the following BSDE has a unique solution, denoted by $(\tilde{y}_t^n, \tilde{z}_t^n)_{t\in[0,T]}$:

$$\begin{aligned} \tilde{y}_t^n &= Y_T + \int_t^T g(s, y_s^n, y_s^n)\, ds \\ &\quad - \int_t^T \tilde{z}_s^n\, dB_s + n\int_t^T v_s^2 (Y_s - \tilde{y}_s^n)\, ds. \end{aligned} \tag{33}$$

Let τ be a stopping time such that $0 \le \tau \le T$; then we have

$$\begin{aligned} \tilde{y}_\tau^n = \mathbb{E}^{\mathcal{F}_\tau}&\left[e^{-n\int_\tau^T v_r^2 dr} Y_T + n\int_\tau^T v_s^2 e^{-n\int_\tau^s v_r^2 dr} Y_s ds \right. \\ &\left. + \int_\tau^T e^{-n\int_\tau^s v_r^2 dr} g(s, y_s^n, z_s^n)\, ds\right]. \end{aligned} \tag{34}$$

For the first two terms within the bracket on the right-hand side of (34), with the property of the vague convergence for the distribution functions, it is easily seen that

$$e^{-n\int_\tau^T v_r^2 dr} Y_T + n\int_\tau^T v_s^2 e^{-n\int_\tau^s v_r^2 dr} Y_s\, ds \longrightarrow Y_\tau, \quad \text{P-a.s.}, \tag{35}$$

and then, by dominated convergence, it converges in mean square; that is,

$$\mathbb{E}\left[\left(e^{-n\int_\tau^T v_r^2 dr} Y_T + n\int_\tau^T v_s^2 e^{-n\int_\tau^s v_r^2 dr} Y_s\, ds - Y_\tau\right)^2\right] \longrightarrow 0. \tag{36}$$

Now, we come to treat the third term. From the assumption (H2), we can deduce that

$$\begin{aligned} &\int_\tau^T e^{-n\int_\tau^s v_r^2 dr} |g(s, y_s^n, z_s^n)|\, ds \\ &\le a_\psi \int_\tau^T e^{-n\int_\tau^s v_r^2 dr} v_s |z_s^n|\, ds \\ &\quad + \int_\tau^T e^{-n\int_\tau^s v_r^2\, dr} \left(|g(s,0,0)| + a_\varphi u(s) |y_s^n| \right. \\ &\qquad \left. + b_\varphi u_s + b_\psi v_s\right) ds. \end{aligned} \tag{37}$$

For the integrand of the second integration term on the right hand of (37), it is dominated by

$$P_t := |g(t,0,0)| + a_\varphi u(t) \left(|y_t^1| + |Y_t|\right) + b_\varphi u_t + b_\psi v_t. \tag{38}$$

Combining the assumption (H1), and the fact that $(y_t^1)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ belong to the space $\mathcal{S}^2(0,T;R)$, we can obtain that this term converges to zero almost surely with respect to probability P, by dominated convergence theorem, and then

$$\begin{aligned} \mathbb{E}&\left[\left(\int_\tau^T e^{-n\int_\tau^s v_r^2 dr} \left(|g(s,0,0)| + a_\varphi u(s) |y_s^n| \right.\right.\right. \\ &\left.\left.\left. + b_\varphi u_s + b_\psi v_s\right) ds\right)^2\right] \longrightarrow 0. \end{aligned} \tag{39}$$

Applying Hölder's inequality to the first term on the right hand of (37), we can get

$$\int_{\tau}^{T} e^{-n\int_{\tau}^{s} v_r^2 dr} v_s \left|z_s^n\right| \, ds \leq \frac{1}{\sqrt{2n}} \left(\int_{\tau}^{T} \left|z_s^n\right|^2 ds \right)^{1/2}. \quad (40)$$

Thus, from Proposition 10, it is easy to obtain the following convergence:

$$\mathbb{E}\left[\left(\int_{\tau}^{T} e^{-n\int_{\tau}^{s} v_r^2 dr} v_s \left|z_s^n\right| ds \right)^2 \right] \leq \frac{1}{2n} C \longrightarrow 0, \quad (41)$$

and then

$$\mathbb{E}\left[\left(\int_{\tau}^{T} e^{-n\int_{\tau}^{s} v_r^2 dr} \left|g\left(s, y_s^n, z_s^n\right)\right| ds \right)^2 \right] \longrightarrow 0. \quad (42)$$

Consequently, using Jensen's inequality and the property of conditional expectation, we have $\mathbb{E}[(\tilde{y}_\tau^n - Y_\tau)^2] \to 0$.

According to the uniqueness of the solutions for BSDE (17) and the definition (32), we can obtain $y_t^n = \tilde{y}_t^n$, for all $t \in [0,T]$,P-a.s., and $y_\tau = Y_\tau$. By section theorem, we have, P-a.s.,

$$y_t = Y_t, \quad \forall t \in [0,T]. \quad (43)$$

Therefore, if $T < \infty$, that $Y_t - y_t^n$ uniformly converges to zero in t almost surely with respect to probability P, is the immediate result of Dini's theorem. Otherwise $T = \infty$, since the increasing sequence of the continuous process $(Y_t - y_t^n)_{t\in[0,T]}$ has the same value 0 at T; then almost surely, for any n and $\epsilon > 0$, we can choose a real number M, which may depend only on ϵ and ω, such that if $t > M$, then

$$\left|Y_t - y_t^n\right| \leq \left|Y_t - y_t^1\right| \leq \epsilon. \quad (44)$$

On the other hand, by Dini's theorem, $(Y_t - y_t^n)_{t\in[0,T]}$ converges uniformly to zero almost surely on the interval $[0,M]$. So we can choose a number N depending only on ϵ and ω such that if $n > N$, then

$$\left|Y_t - y_t^n\right| \leq \epsilon, \quad \forall t \in [0,M]. \quad (45)$$

Thus, $(Y_t - y_t^n)_{t\in[0,T]}$ uniformly converges to zero on the whole interval $[0,T]$ almost surely with respect to probability P. Noticing the fact that $|Y_t - y_t^n| \leq |Y_t| + |y_t^1|$, we can obtain the desired result by dominated convergence theorem. The proof is completed. □

After that, we can get the following proposition about the two sequences of $(z_t^n)_{t\in[0,T]}$ and $(A_t^n)_{t\in[0,T]}$ parameterized by n.

Proposition 12. *The processes $(z_t^n)_{t\in[0,T]}$ and $(A_t^n)_{t\in[0,T]}$, at least their subsequences, are the Cauchy sequences in $\mathscr{H}^2(0,T;\mathbb{R}^d)$ and $\mathcal{S}^2(0,T;\mathbb{R})$, respectively.*

Proof. Applying Itô's formula to $|y_t^n - y_t^m|^2$ on $[0,T]$, we can obtain

$$\begin{aligned}
&\left|y_0^n - y_0^m\right|^2 + \int_0^T \left|z_s^n - z_s^m\right|^2 ds \\
&\quad = 2\int_0^T \left(y_s^n - y_s^m\right)\left(g\left(s, y_s^n, z_s^n\right) - g\left(s, y_s^m, z_s^m\right)\right) ds \\
&\qquad + 2\int_0^T \left(y_s^n - y_s^m\right) d\left(A_s^n - A_s^m\right) \\
&\qquad - 2\int_0^T \left(y_s^n - y_s^m\right)\left(z_s^n - z_s^m\right) dB_s \\
&\quad \leq 2\int_0^T \left(y_s^n - y_s^m\right)\left(g\left(s, y_s^n, z_s^n\right) - g\left(s, y_s^m, z_s^m\right)\right) ds \\
&\qquad + 2\int_0^T \left(Y_s - y_s^n\right) dA_s^m \\
&\qquad + 2\int_0^T \left(Y_s - y_s^m\right) dA_s^n - 2\int_0^T \left(y_s^n - y_s^m\right)\left(z_s^n - z_s^m\right) dB_s.
\end{aligned} \quad (46)$$

Due to the fact that the part of Itô integration is uniformly integrable martingale, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|z_s^n - z_s^m\right|^2 ds \\
&\quad \leq 2\mathbb{E}\int_0^T \left(y_s^n - y_s^m\right)\left(g\left(s, y_s^n, z_s^n\right) - g\left(s, y_s^m, z_s^m\right)\right) ds \\
&\qquad + 2\mathbb{E}\int_0^T \left(Y_s - y_s^n\right) dA_s^m + 2\mathbb{E}\int_0^T \left(Y_s - y_s^m\right) dA_s^n.
\end{aligned} \quad (47)$$

As for the last two terms of the above inequality, Propositions 10 and 11 lead to the fact that if $m, n \to \infty$, then

$$\mathbb{E}\int_0^T \left(Y_s - y_s^n\right) dA_s^m + \mathbb{E}\int_0^T \left(Y_s - y_s^m\right) dA_s^n \longrightarrow 0. \quad (48)$$

Next, we will show that, as $m, n \to \infty$,

$$\mathbb{E}\int_0^T \left(y_s^n - y_s^m\right)\left(g\left(s, y_s^n, z_s^n\right) - g\left(s, y_s^m, z_s^m\right)\right) ds \longrightarrow 0. \quad (49)$$

Because the generator g satisfies the assumption (H2), by Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|\left(y_s^n - y_s^m\right)\left(g\left(s, y_s^n, z_s^n\right) - g\left(s, y_s^m, z_s^m\right)\right)\right| ds \\
&\quad \leq \mathbb{E}\int_0^T \left|y_s^n - y_s^m\right| \left(a_\varphi u_s \left|y_s^n - y_s^m\right| + b_\varphi u_s \right. \\
&\qquad \left. + a_\psi v_s \left|z_s^n - z_s^m\right| + b_\psi v_s\right) ds
\end{aligned}$$

$$\leq a_\psi\left(\mathbb{E}\left[\int_0^T |z_s^n - z_s^m|^2\, ds\right]\right)^{1/2}$$
$$\times\left(\mathbb{E}\left[\int_0^T |y_s^n - y_s^m|^2 v_s^2\, ds\right]\right)^{1/2}$$
$$+ b_\psi \mathbb{E}\int_0^T |y_s^n - y_s^m| v_s ds$$
$$+ a_\varphi \mathbb{E}\left[\sup_{0\leq s\leq T}|y_s^n - y_s^m|^2\right]\int_0^T u_s ds$$
$$+ b_\varphi \mathbb{E}\int_0^T |y_s^n - y_s^m| u_s ds. \quad (50)$$

According to Proposition 10 and the algebraic inequality, we can conclude

$$\mathbb{E}\int_0^T |(y_s^n - y_s^m)(g(s, y_s^n, z_s^n) - g(s, y_s^m, z_s^m))| ds$$
$$\leq 2C^2\left(\mathbb{E}\left[\int_0^T |y_s^n - y_s^m|^2 v_s^2 ds\right]\right)^{1/2}$$
$$+ b_\varphi \mathbb{E}\int_0^T |y_s^n - y_s^m| u_s ds \quad (51)$$
$$+ a_\varphi \mathbb{E}\left[\sup_{0\leq s\leq T}|y_s^n - y_s^m|^2\right]\int_0^T u_s ds$$
$$+ b_\psi \mathbb{E}\int_0^T |y_s^n - y_s^m| v_s ds.$$

Now set $G_s = |y_s^1| + |Y_s|, H_s = 4G_s^2 v_s^2$, and $F_s = 2G_s u_s$; then, for any $m, n \geq 1$, we have

$$v_s^2 |y_s^n - y_s^m|^2 \leq H_s, \quad u_s |y_s^n - y_s^m| \leq F_s, \quad s \in [0, T];$$
$$\mathbb{E}\left[\int_0^T H_s ds\right] \leq 4\mathbb{E}\left[\sup_{0\leq s\leq T}|G_s|^2\right]\int_0^T v_s^2 ds < +\infty; \quad (52)$$
$$\mathbb{E}\left[\int_0^T F_s ds\right] \leq 2\left(\mathbb{E}\left[\sup_{0\leq s\leq T}|G_s|^2\right]\right)^{1/2}\int_0^T u_s ds < +\infty.$$

The first two terms of the right-hand side of (51) converge to zero by the Lebesgue dominated theorem. And Proposition 11 implies that $(y_t^n)_{t\in[0,T]}$ is a Cauchy sequence in $\mathcal{S}^2(0, T; \mathbb{R})$; then the third term converges to zero. The convergence of the last term can be proved in a similar way to the second one.

Now, coming back to the inequality (47), we can conclude that $\mathbb{E}\int_0^T |z_s^n - z_s^m|^2\, ds \to 0$. This meansthat $(z_t^n)_{t\in[0,T]}$ is a Cauchy sequence in $\mathcal{H}^2(0, T; \mathbb{R}^d)$, and we denoted its limit by $(z_t)_{t\in[0,T]}$.

From (17), we know that

$$A_t^n - A_t^m = y_0^n - y_0^m - \int_0^t (g(s, y_s^n, z_s^n) - g(s, y_s^m, z_s^m))\, ds$$
$$+ \int_0^t (z_s^n - z_s^m)\, dB_s, \quad (53)$$

and then, from the basic algebraic inequality and BDG's inequality, we can get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|A_t^n - A_t^m|^2\right]$$
$$\leq 3\mathbb{E}|y_0^n - y_0^m|^2 + 3\mathbb{E}\int_0^T |z_s^n - z_s^m|^2 ds$$
$$+ 3\mathbb{E}\left[\left(\int_0^T |g(s, y_s^n, z_s^n) - g(s, y_s^m, z_s^m)|\, ds\right)^2\right]. \quad (54)$$

In order to show that, when $m, n \to \infty$, the limit of the third term of the right-hand side of (54) is zero, we only need to show that if $n \to \infty$, then

$$\mathbb{E}\left[\left(\int_0^T |g(s, y_s^n, z_s^n) - g(s, y_s, z_s)|\, ds\right)^2\right] \longrightarrow 0. \quad (55)$$

Because $(z_t^n)_{t\in[0,T]}$ is a Cauchy sequence in $\mathcal{H}^2(0, T; \mathbb{R}^d)$, there is at least a subsequence $(z_t^{n_k})_{t\in[0,T]}$ such that $dP \times dt$-a.e., $z_t^{n_k} \to z_t$, and $\check{z}_t := \sup_{k\geq 1}|z_t^{n_k}| \in \mathcal{H}^2(0, \mathrm{T}; \mathbb{R}^d)$. For convenience, we denote the subsequence by $(z_t^n)_{t\in[0,T]}$ itself. According to the assumption (H2), we can deduce that

$$|g(s, y_s^n, z_s^n) - g(s, y_s, z_s)|$$
$$\leq a_\varphi u_s |y_s^n - y_s| + b_\varphi u_s + a_\psi v_s |z_s^n - z_s| + b_\psi v_s. \quad (56)$$

The right-hand side of the above inequality is dominated by

$$R_s := a_\varphi u_s G_s + b_\varphi u_s + a_\psi v_s(\check{z}_s + z_s) + b_\psi v_s. \quad (57)$$

It is easy to check that $\mathbb{E}[(\int_0^T R_s\, ds)^2] < \infty$. Then the convergence of (55) is a direct consequence of the Lebesgue dominated convergence theorem.

From the above argument and Proposition 11, we can assert that $(A_t^n)_{t\in[0,T]}$ is also a Cauchy sequence in $\mathcal{S}^2(0, T; \mathbb{R})$with a unique limit $(A_t)_{t\in[0,T]}$. The proof is completed. □

Proof of Theorem 8. From the procedure of the proof of Proposition 12, we know that

$$\int_t^T g(s, y_s^n, z_s^n)\, ds \longrightarrow \int_t^T g(s, y_s, z_s)\, ds, \quad (58)$$

uniformly on [0,T], in mean square, that is,

$$\mathbb{E}\left[\left(\sup_{0\le t\le T}\left|\int_t^T g(s,y_s^n,z_s^n)\,ds-\int_t^T g(s,y_s,z_s)\,ds\right|\right)^2\right] \longrightarrow 0. \tag{59}$$

And, by the property of Itô's integration, BDG's inequality, and Proposition 12, we also have

$$\int_t^T z_s^n dB_s \longrightarrow \int_t^T z_s\, dB_s, \tag{60}$$

uniformly on [0,T], in mean square, that is,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T z_s^n\, dB_s-\int_t^T z_s\, dB_s\right|^2\right] \le C\mathbb{E}\left[\int_0^T \left|z_s^n-z_s\right|^2 ds\right]\longrightarrow 0. \tag{61}$$

Then combining the above convergence and the fact that the sequences $(y_t^n)_{t\in[0,T]}$ and $(A_t^n)_{t\in[0,T]}$ themselves or their subsequences converge to $(y_t)_{t\in[0,T]}$ and $(A_t)_{t\in[0,T]}$ uniformly on $[0,T]$, in mean square, respectively, we can obtain the following equation:

$$y_t = Y_T+\int_t^T g(s,y_s,z_s)\,ds-\int_t^T z_s dB_s + A_T - A_t, \quad t\in[0,T]. \tag{62}$$

Notice that $(A_t^n)_{t\in[0,T]}$ is an increasing process with $A_0^n=0$; then its limit $(A_t)_{t\in[0,T]}$ will preserve this property. In fact, we have proved the first part of Theorem 8, because, according to (43), the g-supermartingale $(Y_t)_{t\in[0,T]}$ coincides with the first component $(y_t)_{t\in[0,T]}$ of the solution for the BSDE (62). And finally, the uniqueness of the decomposition of g-supermartingale follows from the uniqueness of the decomposition of g-supersolution. The proof is completed. □

Now, in addition, if we assume that g is independent of y, then we can write the decomposition of Doob-Meyer's type for g-supermartingale in a more clear sense like the classical martingale theory.

Corollary 13. *Let g be independent of y and satisfy the conditions (H1) and (H2). If $(X_t)_{t\in[0,T]}$ is a continuous g-supermartingale in $\mathcal{S}^2(0,T;\mathbb{R})$, then it has the following decomposition:*

$$X_t = M_t - A_t, \tag{63}$$

where $(M_t)_{t\in[0,T]}$ is a g-martingale and $(A_t)_{t\in[0,T]}$ is an increasing process which belongs to $\mathcal{A}^2(0,T;\mathbb{R})$.

Proof. By Theorem 8, a g-supermartingale $(X_t)_{t\in[0,T]}$ on $[0,T]$ has the following form. There exists a pair of processes $(z_t,A_t)_{t\in[0,T]}$ such that

$$X_t = X_T+\int_t^T g(s,z_s)\,ds-\int_t^T z_s dB_s + A_T - A_t, \quad t\in[0,T]. \tag{64}$$

We set $M_t = X_t + A_t$; then

$$M_t = X_T + A_T + \int_t^T g(s,z_s)\,ds-\int_t^T z_s dB_s, \quad t\in[0,T]. \tag{65}$$

Obviously, the pair of the processes $(M_t,z_t)_{t\in[0,T]}$ is a solution of the BSDE with the terminal data X_T+A_T and the generator g. Definition 6 implies that M_t is a g-martingale. The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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