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Research Article

Refinements of Aczél-Type Inequality and Their Applications

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We present some new sharpened versions of Aczél-type inequality. Moreover, as applications, some refinements of integral type of Aczél-type inequality are given.

1. Introduction

Let n be a positive integer, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$. Then, the famous Aczél inequality [1] can be stated as follows:

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \quad (1)$$

Aczél's inequality plays a very important role in the theory of functional equations in non-Euclidean geometry. Due to the importance of Aczél's inequality (1), it has received considerable attention by many authors and has motivated a large number of research papers giving it various generalizations, improvements, and applications (see [2–21] and the references therein).

In 1959, Popoviciu [10] first obtained an exponential extension of the Aczél inequality as follows.

Theorem B. *Let $p \ge q > 1$, $(1/p) + (1/q) = 1$, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \le a_1 b_1 - \sum_{i=2}^n a_i b_i. \quad (2)$$

Later, in 1982, Vasić and Pečarić [16] established the following reversed version of inequality (2).

Theorem C. *Let $q < 0$, $p > 0$, $(1/p) + (1/q) = 1$, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \ge a_1 b_1 - \sum_{i=2}^n a_i b_i. \quad (3)$$

In another paper, Vasić and Pečarić [15] generalized inequality (2) in the following form.

Theorem D. *Let $a_{rj} > 0$, $\beta_j > 0$, $a_{1j}^{\beta_j} - \sum_{r=2}^n a_{rj}^{\beta_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and let $\sum_{j=1}^m (1/\beta_j) \ge 1$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\beta_j} - \sum_{r=2}^n a_{rj}^{\beta_j}\right)^{1/\beta_j} \le \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \quad (4)$$

In 2012, Tian [13] presented the reversed version of inequality (4) as follows.

Theorem E. *Let $a_{rj} > 0$, $\beta_1 \ne 0$, $\beta_j < 0$ $(j = 2, 3, \ldots, m)$, $\sum_{j=1}^m (1/\beta_j) \le 1$, $a_{1j}^{\beta_j} - \sum_{r=2}^n a_{rj}^{\beta_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\beta_j} - \sum_{r=2}^n a_{rj}^{\beta_j}\right)^{1/\beta_j} \ge \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \quad (5)$$

Moreover, in [13] Tian established an integral type of inequality (5).

Theorem F. *Let $\beta_1 > 0$, $\beta_j < 0$ $(j = 2,3,\ldots,m)$, $\sum_{j=1}^{m}(1/\beta_j) = 1$, let $t_j > 0$ $(j = 1,2,\ldots,m)$, and let $f_j(x)$ $(j = 1,2,\ldots,m)$ be positive Riemann integrable functions on $[a,b]$ such that $t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x)dx > 0$. Then*

$$\prod_{j=1}^{m}\left(t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x)\,dx\right)^{1/\beta_j} \geq \prod_{j=1}^{m} t_j - \int_a^b \prod_{j=1}^{m} f_j(x)\,dx. \quad (6)$$

Remark 1. In fact, the integral form of inequality (4) is also valid; that is, one has the following.

Theorem G. *Let $\beta_j > 0$ $(j = 1,2,\ldots,m)$, $\sum_{j=1}^{m}(1/\beta_j) = 1$, let $t_j > 0$ $(j = 1,2,\ldots,m)$, and let $f_j(x)$ $(j = 1,2,\ldots,m)$ be positive Riemann integrable functions on $[a,b]$ such that $t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x)dx > 0$. Then*

$$\prod_{j=1}^{m}\left(t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x)\,dx\right)^{1/\beta_j} \leq \prod_{j=1}^{m} t_j - \int_a^b \prod_{j=1}^{m} f_j(x)\,dx. \quad (7)$$

The main purpose of this work is to give new refinements of inequalities (4) and (5). As applications, new refinements of inequalities (6) and (7) are also given.

2. Refinements of Aczél-Type Inequality

In order to present our main results, we need some lemmas as follows.

Lemma 2 (see [6]). *Let a_i, x_i $(i = 1,2,\ldots,n)$ be real numbers such that $a_i \geq 0$ and $x_i > -1$. If $\sum_{i=1}^{n} a_i \leq 1$, then*

$$\prod_{i=1}^{n}(1 + x_i)^{a_i} \leq 1 + \sum_{i=1}^{n} a_i x_i. \quad (8)$$

If either $a_i \geq 1$ $(i = 1,2,\ldots,n)$ or $a_i \leq 0$ $(i = 1,2,\ldots,n)$ and if all x_i are positive or negative with $x_i > -1$, then the reverse inequality of (8) *holds.*

Lemma 3 (see [15]). *Let $a_{ij} > 0$ $(i = 1,2,\ldots,n,\ j = 1,2,\ldots,m)$.*

(a) *If $\lambda_j \geq 0$ and if $\sum_{j=1}^{m}\lambda_j \geq 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{\lambda_j} \leq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j}. \quad (9)$$

(b) *If $\lambda_j \leq 0$ $(j = 1,2,\ldots,m)$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{\lambda_j} \geq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j}. \quad (10)$$

(c) *If $\lambda_1 > 0$, $\lambda_j \leq 0$ $(j = 2,3,\ldots,m)$, and $\sum_{j=1}^{m}\lambda_j \leq 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{\lambda_j} \geq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j}. \quad (11)$$

Lemma 4 (see [18]). *Let $0 \leq x < 1$, $\alpha > 0$. Then*

$$(1 - x)^{1/\alpha} \leq 1 - \frac{x}{\max\{\alpha, 1\}}. \quad (12)$$

Lemma 5. *Let $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$, $\sum_{j=1}^{m}(1/\beta_j) \geq 1$, $m \geq 2$, let $0 < x_j < 1$ $(j = 1,2,\ldots,m)$, and let $\xi(m) = \begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd} \end{cases}$. Then*

$$\begin{aligned}&\prod_{j=1}^{m}\left(1 - x_j^{\beta_j}\right)^{1/\beta_j} + \prod_{j=1}^{m} x_j \\ &\quad\leq 1 - \frac{1}{\xi(m)} \times \sum_{j=1}^{\xi(m)}\left[\frac{1}{\max\{\beta_{2j}, 1\}}\left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right].\end{aligned} \quad (13)$$

Proof. From the assumptions we have that

$$\begin{aligned}&\frac{1}{\beta_1} \geq \frac{1}{\beta_2} \geq \cdots \geq \frac{1}{\beta_{m-1}} \geq \frac{1}{\beta_m} > 0, \\ &\frac{1}{\beta_j} - \frac{1}{\beta_{j+1}} \geq 0 \quad (j = 1,2,\ldots,m-1).\end{aligned} \quad (14)$$

Case (I) (let m be even). In view of $(1/\beta_1 - 1/\beta_2) + 1/\beta_2 + 1/\beta_2 + (1/\beta_3 - 1/\beta_4) + 1/\beta_4 + 1/\beta_4 + \cdots + (1/\beta_{m-1} - 1/\beta_m) + 1/\beta_m + 1/\beta_m = 1/\beta_1 + 1/\beta_2 + \cdots + 1/\beta_m \geq 1$ by using inequality (9), we get

$$\begin{aligned}&\prod_{j=1}^{m/2}\left[1 - \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}} \\ &\quad= \prod_{j=1}^{m/2}\left\{\left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right) + x_{2j}^{\beta_{2j}}\right]^{1/\beta_{2j}}\right. \\ &\qquad\times\left[\left(1 - x_{2j}^{\beta_{2j}}\right) + x_{2j-1}^{\beta_{2j-1}}\right]^{1/\beta_{2j}} \\ &\qquad\left.\times\left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right) + x_{2j-1}^{\beta_{2j}-1}\right]^{1/\beta_{2j-1} - 1/\beta_{2j}}\right\} \\ &\quad= \left[\left(1 - x_1^{\beta_1}\right) + x_2^{\beta_2}\right]^{1/\beta_2}\left[\left(1 - x_2^{\beta_2}\right) + x_1^{\beta_1}\right]^{1/\beta_2} \\ &\qquad\times\left[\left(1 - x_1^{\beta_1}\right) + x_1^{\beta_1}\right]^{1/\beta_1 - 1/\beta_2} \\ &\qquad\times\left[\left(1 - x_3^{\beta_3}\right) + x_4^{\beta_4}\right]^{1/\beta_4}\left[\left(1 - x_4^{\beta_4}\right) + x_3^{\beta_3}\right]^{1/\beta_4} \\ &\qquad\times\left[\left(1 - x_3^{\beta_3}\right) + x_3^{\beta_3}\right]^{1/\beta_3 - 1/\beta_4} \\ &\qquad\vdots \\ &\qquad\times\left[\left(1 - x_{m-1}^{\beta_{m-1}}\right) + x_m^{\beta_m}\right]^{1/\beta_m}\left[\left(1 - x_m^{\beta_m}\right) + x_{m-1}^{\beta_{m-1}}\right]^{1/\beta_m}\end{aligned}$$

$$
\begin{aligned}
&\times \left[\left(1 - x_{m-1}^{\beta_{m-1}}\right) + x_{m-1}^{\beta_{m-1}}\right]^{1/\beta_{m-1} - 1/\beta_m} \\
&\geq \prod_{j=1}^{m/2} \left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j}} \left(1 - x_{2j}^{\beta_{2j}}\right)^{1/\beta_{2j}} \right. \\
&\qquad \left. \times \left(1 - x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1} - 1/\beta_{2j}}\right] \\
&\quad + \prod_{j=1}^{m/2} \left[\left(x_{2j}^{\beta_{2j}}\right)^{1/\beta_{2j}} \left(x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j}} \right. \\
&\qquad \left. \times \left(x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1} - 1/\beta_{2j}}\right] \\
&= \prod_{j=1}^{m/2} \left(1 - x_j^{\beta_j}\right)^{1/\beta_j} + \prod_{j=1}^{m} x_j.
\end{aligned} \tag{15}
$$

On the other hand, applying Lemma 4 and the arithmetic-geometric means inequality we obtain

$$
\begin{aligned}
&\prod_{j=1}^{m/2} \left[1 - \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}} \\
&\leq \prod_{j=1}^{m/2} \left[1 - \frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right] \\
&\leq \left\{\frac{2}{m} \sum_{j=1}^{m/2} \left[1 - \frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]\right\}^{m/2} \\
&= \left\{1 - \frac{2}{m} \sum_{j=1}^{m/2} \left[\frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]\right\}^{m/2}.
\end{aligned} \tag{16}
$$

Applying Lemma 4 again, we get

$$
\begin{aligned}
&\left\{1 - \frac{2}{m} \sum_{j=1}^{m/2} \left[\frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]\right\}^{m/2} \\
&\quad \leq 1 - \frac{2}{m} \sum_{j=1}^{m/2} \left[\frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right].
\end{aligned} \tag{17}
$$

Combining (15), (16), and (17) yields immediately inequality (13).

Case (II) (let m be odd). In view of $(1/\beta_1 - 1/\beta_2) + 1/\beta_2 + 1/\beta_2 + (1/\beta_3 - 1/\beta_4) + 1/\beta_4 + 1/\beta_4 + \cdots + (1/\beta_{m-2} - 1/\beta_{m-1}) + 1/\beta_{m-1} + 1/\beta_{m-1} + 1/\beta_m = 1/\beta_1 + 1/\beta_2 + \cdots + 1/\beta_m \geq 1$, by using inequality (9), we have

$$
\begin{aligned}
&\prod_{j=1}^{(m-1)/2} \left[1 - \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}} \\
&= \left\{\prod_{j=1}^{(m-1)/2} \left[1 - \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}\right\} \\
&\quad \times \left[\left(1 - x_m^{\beta_m}\right) + x_m^{\beta_m}\right]^{1/\beta_m} \\
&= \left\{\prod_{j=1}^{(m-1)/2} \left\{\left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right) + x_{2j}^{\beta_{2j}}\right]^{1/\beta_{2j}} \right.\right. \\
&\qquad \times \left[\left(1 - x_{2j}^{\beta_{2j}}\right) + x_{2j-1}^{\beta_{2j-1}}\right]^{1/\beta_{2j}} \\
&\qquad \left.\left. \times \left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right) + x_{2j-1}^{\beta_{2j-1}}\right]^{1/\beta_{2j-1} - 1/\beta_{2j}}\right\}\right\} \\
&\quad \times \left[\left(1 - x_m^{\beta_m}\right) + x_m^{\beta_m}\right]^{1/\beta_m} \\
&\geq \left\{\prod_{j=1}^{(m-1)/2} \left[\left(1 - x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j}} \left(1 - x_{2j}^{\beta_{2j}}\right)^{1/\beta_{2j}} \right.\right. \\
&\qquad \left.\left. \left(1 - x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1} - 1/\beta_{2j}}\right]\right\} \left(1 - x_m^{\beta_m}\right)^{1/\beta_m} \\
&\quad + \left\{\prod_{j=1}^{(m-1)/2} \left[\left(x_{2j}^{\beta_{2j}}\right)^{1/\beta_{2j}} \left(x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j}} \right.\right. \\
&\qquad \left.\left. \times \left(x_{2j-1}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1} - 1/\beta_{2j}}\right]\right\} \left(x_m^{\beta_m}\right)^{1/\beta_m} \\
&= \prod_{j=1}^{m} \left(1 - x_j^{\beta_j}\right)^{1/\beta_j} + \prod_{j=1}^{m} x_j.
\end{aligned} \tag{18}
$$

On the other hand, applying Lemma 4 and the arithmetic-geometric means inequality we obtain

$$
\begin{aligned}
&\prod_{j=1}^{(m-1)/2} \left[1 - \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}} \\
&\quad \leq \prod_{j=1}^{(m-1)/2} \left[1 - \frac{1}{\max\left\{\beta_{2j}, 1\right\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}}\right)^2\right]
\end{aligned}
$$

$$\leq \left\{ \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[1 - \frac{1}{\max\{\beta_{2j}, 1\}} \times \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2 \right] \right\}^{(m-1)/2}$$

$$= \left\{ 1 - \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[\frac{1}{\max\{\beta_{2j}, 1\}} \times \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2 \right] \right\}^{(m-1)/2}. \tag{19}$$

Applying Lemma 4 again, we have

$$\left\{ 1 - \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[\frac{1}{\max\{\beta_{2j}, 1\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2 \right] \right\}^{(m-1)/2}$$

$$\leq 1 - \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[\frac{1}{\max\{\beta_{2j}, 1\}} \left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2 \right]. \tag{20}$$

Hence, combining (18), (19), and (20) yields immediately inequality (13). □

Similar to the proof of Lemma 5 but using Lemma 2 in place of Lemma 4, we immediately obtain the following result.

Lemma 6. *Let* $\beta_1 > 0$, $0 > \beta_2 \geq \beta_3 \geq \cdots \geq \beta_m$, $\sum_{j=1}^{m}(1/\beta_j) \leq 1$, $m \geq 2$, *let* $0 < x_1 < 1$, $x_j > 1$ $(j = 2, 3, \ldots, m)$, *and let* $\xi(m) = \begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd} \end{cases}$.

Then

$$\prod_{j=1}^{m} \left(1 - x_j^{\beta_j} \right)^{1/\beta_j} + \prod_{j=1}^{m} x_j \geq 1 - \sum_{j=1}^{\xi(m)} \frac{\left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \tag{21}$$

Using the same methods as in Lemma 6, we get the following Lemma.

Lemma 7. *Let* $0 > \beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$, $m \geq 2$, *let* $x_j > 1$ $(j = 1, 2, \ldots, m)$, *and let* $\xi(m) = \begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd} \end{cases}$.

Then

$$\prod_{j=1}^{m} \left(1 - x_j^{\beta_j} \right)^{1/\beta_j} + \prod_{j=1}^{m} x_j \geq 1 - \sum_{j=1}^{\xi(m)} \frac{\left(x_{2j}^{\beta_{2j}} - x_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \tag{22}$$

Now, we present some new refinements of inequalities (4) and (5).

Theorem 8. *Let* $a_{rj} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $m \geq 2, n \geq 2$, *let* $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$, $\sum_{j=1}^{m}(1/\beta_j) \geq 1$, $a_{1j}^{\beta_j} - \sum_{r=2}^{n} a_{rj}^{\beta_j} > 0$, *and let* $\xi(m) = \begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd} \end{cases}$.

Then

$$\prod_{j=1}^{m} \left(a_{1j}^{\beta_j} - \sum_{r=2}^{n} a_{rj}^{\beta_j} \right)^{1/\beta_j} \leq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} - \frac{a_{11} a_{12} \ldots a_{1m}}{\xi(m)} \times \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\max\{\beta_{2j}, 1\}} \times \left[\sum_{r=2}^{n} \left(\frac{a_{r(2j)}^{\beta_{2j}}}{a_{1(2j)}^{\beta_{2j}}} - \frac{a_{r(2j-1)}^{\beta_{2j-1}}}{a_{1(2j-1)}^{\beta_{2j-1}}} \right) \right]^2 \right\}. \tag{23}$$

Proof. From the assumptions we find that

$$0 < \frac{\left(a_{1j}^{\beta_j} - \sum_{r=2}^{n} a_{rj}^{\beta_j} \right)^{1/\beta_j}}{\left(a_{1j}^{\beta_j} \right)^{1/\beta_j}} < 1 \quad (j = 1, 2, \ldots, m). \tag{24}$$

Thus, by using Lemma 5 with a substitution $x_j \to ((a_{1j}^{\beta_j} - \sum_{r=2}^{n} a_{rj}^{\beta_j})/a_{1j}^{\beta_j})^{1/\beta_j}$ $(j = 1, 2, \ldots, m)$ in (13), we obtain

$$\prod_{j=1}^{m} \left(\frac{\sum_{r=2}^{n} a_{rj}^{\beta_j}}{a_{1j}^{\beta_j}} \right)^{1/\beta_j} + \prod_{j=1}^{m} \left(\frac{a_{1j}^{\beta_j} - \sum_{r=2}^{n} a_{rj}^{\beta_j}}{a_{1j}^{\beta_j}} \right)^{1/\beta_j}$$

$$\leq 1 - \frac{1}{\xi(m)} \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\max\{\beta_{2j}, 1\}} \times \left[\left(1 - \frac{\sum_{r=2}^{n} a_{r(2j)}^{\beta_{2j}}}{a_{1(2j)}^{\beta_{2j}}} \right) - \left(1 - \frac{\sum_{r=2}^{n} a_{r(2j-1)}^{\beta_{2j-1}}}{a_{1(2j-1)}^{\beta_{2j-1}}} \right) \right]^2 \right\}$$

$$= 1 - \frac{1}{\xi(m)} \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\max\{\beta_{2j}, 1\}} \times \left[\sum_{r=2}^{n} \left(\frac{a_{r(2j)}^{\beta_{2j}}}{a_{1(2j)}^{\beta_{2j}}} - \frac{a_{r(2j-1)}^{\beta_{2j-1}}}{a_{1(2j-1)}^{\beta_{2j-1}}} \right) \right]^2 \right\}, \tag{25}$$

which implies

$$\prod_{j=1}^{m}\left(a_{1j}^{\beta_j}-\sum_{r=2}^{n}a_{rj}^{\beta_j}\right)^{1/\beta_j} \leq \prod_{j=1}^{m}a_{1j}-\prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{rj}^{\beta_j}\right)^{1/\beta_j} - \frac{a_{11}a_{12}\dots a_{1m}}{\xi(m)} \times \sum_{j=1}^{\xi(m)}\left\{\frac{1}{\max\{\beta_{2j},1\}} \times \left[\sum_{r=2}^{n}\left(\frac{a_{r(2j)}^{\beta_{2j}}}{a_{1(2j)}^{\beta_{2j}}}-\frac{a_{r(2j-1)}^{\beta_{2j-1}}}{a_{1(2j-1)}^{\beta_{2j-1}}}\right)\right]^2\right\}. \tag{26}$$

On the other hand, we get from Lemma 3 that

$$\prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{rj}^{\beta_j}\right)^{1/\beta_j} \geq \sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}. \tag{27}$$

Combining (26) and (27) yields immediately the desired inequality (23). □

Theorem 9. *Let $a_{rj}>0$, $0>\beta_1\geq\beta_2\geq\cdots\geq\beta_m$, $a_{1j}^{\beta_j}-\sum_{r=2}^{n}a_{rj}^{\beta_j}>0$, $r=1,2,\dots,n$, $j=1,2,\dots,m$, let $m\geq 2, n\geq 2$, and let $\xi(m)=\begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd}\end{cases}$.*
Then

$$\prod_{j=1}^{m}\left(a_{1j}^{\beta_j}-\sum_{r=2}^{n}a_{rj}^{\beta_j}\right)^{1/\beta_j} \geq \prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj} - a_{11}a_{12},\dots,a_{1m} \times \sum_{j=1}^{\xi(m)}\left\{\frac{1}{\beta_{2j}}\left[\sum_{r=2}^{n}\left(\frac{a_{r(2j)}^{\beta_{2j}}}{a_{1(2j)}^{\beta_{2j}}}-\frac{a_{r(2j-1)}^{\beta_{2j-1}}}{a_{1(2j-1)}^{\beta_{2j-1}}}\right)\right]^2\right\}. \tag{28}$$

Inequality (28) *is also valid for $\beta_1>0$, $0>\beta_2\geq\beta_3\geq\cdots\geq\beta_m$, $\sum_{j=1}^{m}(1/\beta_j)\leq 1$.*

Proof. The proof of Theorem 9 is similar to the one of Theorem 8, and we omit it. □

3. Applications

In this section, we show two applications of the inequalities newly obtained in Section 2.

Firstly, we present a new refinement of inequality (6) by using Theorem 9.

Theorem 10. *Let $t_j>0$ $(j=1,2,\dots,m)$, $\beta_1>0$, $0>\beta_2\geq\beta_3\geq\cdots\geq\beta_m$, $\sum_{j=1}^{m}(1/\beta_j)=1$, let $f_j(x)$ $(j=1,2,\dots,m)$ be positive integrable functions defined on $[a,b]$ with $t_j^{\beta_j}-\int_a^b f_j^{\beta_j}(x)dx>0$, and let $\xi(m)=\begin{cases} m/2 & \text{if } m \text{ even} \\ (m-1)/2 & \text{if } m \text{ odd}\end{cases}$.*
Then

$$\prod_{j=1}^{m}\left(t_j^{\beta_j}-\int_a^b f_j^{\beta_j}(x)\,dx\right)^{1/\beta_j} \geq \prod_{j=1}^{m}t_j-\int_a^b\prod_{j=1}^{m}f_j(x)\,dx - t_1t_2,\dots,t_m \times \sum_{j=1}^{\xi(m)}\left[\frac{1}{\beta_{2j}}\int_a^b\left(\frac{f_{2j}^{\beta_{2j}}(x)}{t_{2j}^{\beta_{2j}}}-\frac{f_{2j-1}^{\beta_{2j-1}}(x)}{t_{2j-1}^{\beta_{2j-1}}}\right)dx\right]^2. \tag{29}$$

Proof. For any positive integer n, we choose an equidistant partition of $[a,b]$ as

$$a<a+\frac{b-a}{n}<\cdots<a+\frac{b-a}{n}k <\cdots<a+\frac{b-a}{n}(n-1)<b, \tag{30}$$

$$x_i=a+\frac{b-a}{n}i,\quad i=0,1,\dots,n,$$
$$\Delta x_k=\frac{b-a}{n},\quad k=1,2,\dots,n. \tag{31}$$

Since $t_j^{\beta_j}-\int_a^b f_j^{\beta_j}(x)dx>0$ $(j=1,2,\dots,m)$, it follows that

$$t_j^{\beta_j}-\lim_{n\to\infty}\sum_{k=1}^{n}f_j^{\beta_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}>0 \quad (j=1,2,\dots,m). \tag{32}$$

Therefore, there exists a positive integer N such that

$$t_j^{\beta_j}-\sum_{k=1}^{n}f_j^{\beta_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}>0, \tag{33}$$

for all $n>N$ and $j=1,2,\dots,m$.

Moreover, for any $n > N$, it follows from Theorem 9 that

$$\begin{aligned}
&\prod_{j=1}^{m}\left[t_j^{\beta_j}-\sum_{k=1}^{n}f_j^{\beta_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/\beta_j}\\
&\quad\geq\prod_{j=1}^{m}t_j^{\beta_j}-\sum_{k=1}^{n}\left(\prod_{j=1}^{m}f_j\left(a+\frac{k(b-a)}{n}\right)\right)\\
&\qquad\times\left(\frac{b-a}{n}\right)^{1/\beta_1+1/\beta_2+\cdots+1/\beta_m}-t_1t_2,\ldots,t_m\sum_{j=1}^{\xi(m)}\frac{1}{\beta_{2j}}\\
&\qquad\times\left[\sum_{k=1}^{n}\left(\frac{1}{t_{2j}^{\beta_{2j}}}f_{2j}^{\beta_{2j}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{t_{2j-1}^{\beta_{2j-1}}}f_{2j-1}^{\beta_{2j-1}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned}\tag{34}$$

Noting that

$$\sum_{j=1}^{m}\frac{1}{\beta_j}=1,\tag{35}$$

we get

$$\begin{aligned}
&\prod_{j=1}^{m}\left[t_j^{\beta_j}-\sum_{k=1}^{n}f_j^{\beta_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/\beta_j}\\
&\quad\geq\prod_{j=1}^{m}t_j^{\beta_j}-\sum_{k=1}^{n}\left(\prod_{j=1}^{m}f_j\left(a+\frac{k(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right)\\
&\qquad-t_1t_2,\ldots,t_m\sum_{j=1}^{\xi(m)}\frac{1}{\beta_{2j}}\\
&\qquad\times\left[\sum_{k=1}^{n}\left(\frac{1}{t_{2j}^{\beta_{2j}}}f_{2j}^{\beta_{2j}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{t_{2j-1}^{\beta_{2j-1}}}f_{2j-1}^{\beta_{2j-1}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned}\tag{36}$$

In view of the assumption that $f_j(x)$ $(j=1,2,\ldots,m)$ are positive Riemann integrable functions on $[a,b]$, we find that $\prod_{j=1}^{m}f_j(x)$ and $f_j^{\lambda_j}(x)$ are also integrable on $[a,b]$. Letting $n\to\infty$ on both sides of inequality (36), we get the desired inequality (29). □

Next, we give a new refinement of inequality (7) by using Theorem 8.

Theorem 11. *Let $t_j>0$ $(j=1,2,\ldots,m)$, $0<\beta_1\leq\beta_2\leq\cdots\leq\beta_m$, $\sum_{j=1}^{m}(1/\beta_j)=1$, $m\geq 2$, and let $f_j(x)$ $(j=1,2,\ldots,m)$ be positive integrable functions defined on $[a,b]$ with $t_j^{\beta_j}-\int_a^b f_j^{\beta_j}(x)dx>0$, and let $\xi(m)=\begin{cases} m/2 & \text{if } m \text{ even}\\ (m-1)/2 & \text{if } m \text{ odd}\end{cases}$. Then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(t_j^{\beta_j}-\int_a^b f_j^{\beta_j}(x)\,dx\right)^{1/\beta_j}\leq\prod_{j=1}^{m}t_j-\int_a^b\prod_{j=1}^{m}f_j(x)\,dx\\
&\quad-\frac{t_1t_2,\ldots,t_m}{\xi(m)}\\
&\quad\times\sum_{j=1}^{\xi(m)}\left\{\frac{1}{\beta_{2j}}\right.\\
&\qquad\left.\times\left[\int_a^b\left(\frac{f_{2j}^{\beta_{2j}}(x)}{t_{2j}^{\beta_{2j}}}-\frac{f_{2j-1}^{\beta_{2j-1}}(x)}{t_{2j-1}^{\beta_{2j-1}}}\right)dx\right]^2\right\}.
\end{aligned}\tag{37}$$

Proof. The proof of Theorem 11 is similar to the one of Theorem 10, and we omit it. □

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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