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Research Article

Existence of Nontrivial Solutions for Periodic Schrödinger Equations with New Nonlinearities

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We study the Schrödinger equation: $-\Delta u + V(x)u + f(x,u) = 0$, $u \in H^1(\mathbb{R}^N)$, where V is 1-periodic and f is 1-periodic in the x-variables; 0 is in a gap of the spectrum of the operator $-\Delta + V$. We prove that, under some new assumptions for f, this equation has a nontrivial solution. Our assumptions for the nonlinearity f are very weak and greatly different from the known assumptions in the literature.

1. Introduction and Statement of Results

In this paper, we consider the following Schrödinger equation:

$$-\Delta u + V(x)u + f(x,u) = 0, \quad u \in H^1\left(\mathbb{R}^N\right), \tag{1}$$

where $N \geq 1$. For V and f, we assume the following.

(**v**) $V \in C(\mathbb{R}^N)$ is 1-periodic in x_j for $j = 1, \ldots, N$, 0 is in a spectral gap $(-\mu_{-1}, \mu_1)$ of $-\Delta + V$, and $-\mu_{-1}$ and μ_1 lie in the essential spectrum of $-\Delta + V$.

Denote

$$\mu_0 := \min\{\mu_{-1}, \mu_1\}. \tag{2}$$

($\mathbf{f_1}$) $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in x_j for $j = 1, \ldots, N$. And there exist constants $C > 0$ and $2 < p < 2^*$ such that

$$\left|f(x,t)\right| \leq C\left(1 + |t|^{p-1}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3}$$

where

$$2^* := \begin{cases} \dfrac{2N}{(N-2)}, & N \geq 3 \\ \infty, & N = 1, 2. \end{cases} \tag{4}$$

($\mathbf{f_2}$) The limit $\lim_{t\to 0} f(x,t)/t = 0$ holds uniformly for $x \in \mathbb{R}^N$. And there exists $D > 0$ such that

$$\inf_{x\in\mathbb{R}^N, |t|\geq D} \frac{f(x,t)}{t} > \max_{\mathbb{R}^N} V_-, \tag{5}$$

where $V_{\pm}(x) = \max\{\pm V(x), 0\}$, $\forall x \in \mathbb{R}^N$.

($\mathbf{f_3}$) For any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $\widetilde{F}(x,t) \geq 0$, where

$$\widetilde{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t), \qquad F(x,t) = \int_0^t f(x,s)\,ds. \tag{6}$$

($\mathbf{f_4}$) There exist $0 < \kappa < D$ and $\nu \in (0, \mu_0)$ such that, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| < \kappa$,

$$\left|f(x,t)\right| \leq \nu |t| \tag{7}$$

and, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $\kappa \leq |t| \leq D$,

$$\widetilde{F}(x,t) > 0. \tag{8}$$

Remark 1. By the definitions of F and $\widetilde{F}$, it is easy to verify that, for all $(x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$,

$$\frac{\partial}{\partial t}\left(\frac{F(x,t)}{t^2}\right) = \frac{2\widetilde{F}(x,t)}{t^3}. \tag{9}$$

Together with $f(x,t) = o(t)$ as $|t| \to 0$ and $(\mathbf{f_3})$, this implies that

$$F(x,t) \geq 0 \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{10}$$

Remark 2. There are many functions satisfying $(\mathbf{f_1})$–$(\mathbf{f_4})$. We give several examples here.

Example 1. $D = 1 + \mu_0/2 + e^{1+\max_{\mathbb{R}^N} V_-}$, $\kappa = 1 + \mu_0/2$, $\nu = \mu_0/2$, and

$$f(x,t) = \begin{cases} 0, & |t| \leq 1, \\ t \ln |t|, & |t| > 1. \end{cases} \tag{11}$$

Example 2. $D = 3 + \mu_0/2 + 2\max_{\mathbb{R}^N} V_-$, $\kappa = 3/2$, $\nu = \mu_0/2$, and

$$f(x,t) = \begin{cases} 0, & |t| \leq 1, \\ D(t-1), & t > 1, \\ D(t+1), & t < -1. \end{cases} \tag{12}$$

Example 3. $D = \mu_0/2 + e^{1+\max_{\mathbb{R}^N} V_-}$, $\kappa = \nu = \mu_0/2$, and $f(x,t) = t\ln(1+|t|)$.

A solution u of (1) is called nontrivial if $u \not\equiv 0$. Our main results are as follows.

Theorem 3. *Suppose* $(\mathbf{v})$ *and* $(\mathbf{f_1})$–$(\mathbf{f_4})$ *are satisfied. Then* (1) *has a nontrivial solution.*

Note that

$(\mathbf{f_2'})$ the limits $\lim_{t\to 0} f(x,t)/t = 0$ and $\lim_{|t|\to\infty}(f(x,t)/t) = +\infty$ hold uniformly for $x \in \mathbb{R}^N$.

Implying $(\mathbf{f_2})$, we have the following corollary.

Corollary 4. *Suppose* $(\mathbf{v})$, $(\mathbf{f_1})$, $(\mathbf{f_2'})$, $(\mathbf{f_3})$, *and* $(\mathbf{f_4})$ *are satisfied. Then* (1) *has a nontrivial solution.*

It is easy to verify that the condition

$(\mathbf{f_4'})$ $\widetilde{F}(x,t) > 0$, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

And the assumption that $f(x,t)/t \to 0$ as $t \to 0$ uniformly for $x \in \mathbb{R}^N$ imply $(\mathbf{f_3})$ and $(\mathbf{f_4})$. Therefore, we have the following corollary.

Corollary 5. *Suppose* $(\mathbf{v})$, $(\mathbf{f_1})$, $(\mathbf{f_2})$, *and* $(\mathbf{f_4'})$ *are satisfied. Then* (1) *has a nontrivial solution.*

Semilinear Schrödinger equations with periodic coefficients have attracted much attention in recent years due to its numerous applications. One can see [1–24] and the references therein. In [2], the authors used the dual variational method to obtain a nontrivial solution of (1) with $f(x,t) = \pm W(x)|t|^{p-2}t$, where W is an asymptotically periodic function. In [20], Troestler and Willem firstly obtained nontrivial solutions for (1) with f being a C^1 function satisfying the Ambrosetti-Rabinowitz condition:

(AR) there exists $\alpha > 2$ such that, for every $u \neq 0$, $0 < \alpha G(x,u) \leq g(x,u)u$, where $g(x,u) = -f(x,u)$, $G(x,u) = -F(x,u)$, and

$$\left| \frac{\partial f(x,u)}{\partial u} \right| \leq C\left(|u|^{p-2} + |u|^{q-2}\right) \tag{13}$$

with $2 < p < q < 2^*$. Then, in [9], Kryszewski and Szulkin developed some infinite-dimensional linking theorems. Using these theorems, they improved Troestler and Willem's results and obtained nontrivial solutions for (1) with f only satisfying $(\mathbf{f_1})$ and the (AR) condition. These generalized linking theorems were also used by Li and Szulkin to obtain nontrivial solution for (1) under some asymptotically linear assumptions for f (see [11]). In [13] (see also [14]), existence of nontrivial solutions for (1) under $(\mathbf{f_1})$ and the (AR) condition was also obtained by Pankov and Pflüger through approximating (1) by a sequence of equations defined in bounded domains. In the celebrated paper [17], Schechter and Zou combined a generalized linking theorem with the monotonicity methods of Jeanjean (see [8]). They obtained a nontrivial solution of (1) when f exhibits the critical growth. A similar approach was applied by Szulkin and Zou to obtain homoclinic orbits of asymptotically linear Hamiltonian systems (see [19]). Moreover, in [5] (see also [6]), Ding and Lee obtained nontrivial solutions for (1) under some new superlinear assumptions on f different from the classical (AR) conditions.

Our assumptions on f are very weak and greatly different from the assumptions mentioned above. In fact, our assumptions $(\mathbf{f_1})$–$(\mathbf{f_4})$ do not involve the properties of f at infinity. It may be asymptotically linear growth at infinity, that is, $\limsup_{|t|\to\infty}(f(x,t)/t) < +\infty$, or superlinear growth at infinity as well, that is, $\liminf_{|t|\to\infty}(f(x,t)/t) = +\infty$. Moreover, the assumptions $(\mathbf{f_1})$–$(\mathbf{f_4})$ allow $f(x,t) \equiv 0$ in a neighborhood of $t = 0$ (see Remark 2).

In this paper, we use the generalized linking theorem for a class of parameter-dependent functionals (see [17, Theorem 2.1] or Proposition 8 in the present paper) to obtain a sequence of approximate solutions for (1). Then, we prove that these approximate solutions are bounded in $L^\infty(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ (see Lemmas 13 and 14). Finally, using the concentration-compactness principle, we obtain a nontrivial solution of (1).

Notation. $B_r(a)$ denotes the open ball of radius r and center a. For a Banach space E, we denote the dual space of E by E' and denote strong and weak convergence in E by $\to$ and $\rightharpoonup$, respectively. For $\varphi \in C^1(E;\mathbb{R})$, we denote the Fréchet derivative of φ at u by $\varphi'(u)$. The Gateaux derivative of φ is denoted by $\langle \varphi'(u), v\rangle$, $\forall u, v \in E$. $L^p(\mathbb{R}^N)$ denotes the standard L^p space $(1 \leq p \leq \infty)$, and $H^1(\mathbb{R}^N)$ denotes the standard Sobolev space with norm $\|u\|_{H^1} = (\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx)^{1/2}$. We use $O(h)$, $o(h)$ to mean $|O(h)| \le C|h|$, $o(h)/|h| \to 0$.

2. Existence of Approximate Solutions for (1)

Under the assumptions (**v**), ($\mathbf{f_1}$), and ($\mathbf{f_2}$), the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2 dx + \int_{\mathbb{R}^N}F(x,u)\,dx \tag{14}$$

is of class C^1 on $X := H^1(\mathbb{R}^N)$, and the critical points of Φ are weak solutions of (1).

There is a standard variational setting for the quadratic form $\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx$. For the reader's convenience, we state it here. One can consult [5] or [6] for more details.

Assume that (**v**) holds, and let $S = -\Delta + V$ be the self-adjoint operator acting on $L^2(\mathbb{R}^N)$ with domain $D(S) = H^2(\mathbb{R}^N)$. By virtue of (**v**), we have the orthogonal decomposition

$$L^2 = L^2(\mathbb{R}^N) = L^+ + L^- \tag{15}$$

such that S is negative (resp., positive) in L^- (resp., in L^+). As in [5, Section 2] (see also [6, Chapter 6.2]), let $X = D(|S|^{1/2})$ be equipped with the inner product

$$(u,v) = \left(|S|^{1/2}u, |S|^{1/2}v\right)_{L^2} \tag{16}$$

and norm $\|u\| = \||S|^{1/2}u\|_{L^2}$, where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. From (**v**),

$$X = H^1(\mathbb{R}^N) \tag{17}$$

with equivalent norms. Therefore, X continuously embeds in $L^q(\mathbb{R}^N)$ for all $2 \le q \le 2N/(N-2)$ if $N \ge 3$ and for all $q \ge 2$ if $N = 1, 2$. In addition, we have the decomposition

$$X = X^+ + X^-, \tag{18}$$

where $X^\pm = X \cap L^\pm$ is orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Therefore, for every $u \in X$, there is a unique decomposition

$$u = u^+ + u^-, \quad u^\pm \in X^\pm \tag{19}$$

with $(u^+, u^-) = 0$ and

$$\int_{\mathbb{R}^N}|\nabla u|^2 dx + \int_{\mathbb{R}^N}V(x)u^2 dx = \|u^+\|^2 - \|u^-\|^2, \quad u \in X. \tag{20}$$

Moreover,

$$\mu_{-1}\|u^-\|_{L^2}^2 \le \|u^-\|^2, \quad \forall u \in X, \tag{21}$$

$$\mu_1\|u^+\|_{L^2}^2 \le \|u^+\|^2, \quad \forall u \in X. \tag{22}$$

The functional Φ defined by (14) can be rewritten as

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \psi(u), \tag{23}$$

where

$$\psi(u) = \int_{\mathbb{R}^N}F(x,u)\,dx. \tag{24}$$

The above variational setting for the functional (14) is standard. One can consult [5] or [6] for more details.

Let $\{e_k^\pm\}$ be the total orthonormal sequence in $X^\pm$. Let $P : X \to X^-$, $Q : X \to X^+$ be the orthogonal projections. We define

$$|||u||| = \max\left\{\|Pu\|, \sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\left|(Qu, e_k^+)\right|\right\} \tag{25}$$

on X. The topology generated by $|||\cdot|||$ is denoted by τ, and all topological notation related to it will include this symbol.

Lemma 6. *Suppose that* (**v**) *holds. Then*

(**a**) $\max_{\mathbb{R}^N}V_- \ge \mu_{-1}$, *where μ_{-1} is defined in* (**v**);

(**b**) *for any $C > \mu_{-1}$, there exists $u_0 \in X^-$ with $\|u_0\| = 1$ such that $C\|u_0\|_{L^2} > 1$.*

Proof. (**a**) We apply an indirect argument, and assume by contradiction that

$$\max_{\mathbb{R}^N}V_- < \mu_{-1}. \tag{26}$$

From assumption (**v**), $-\mu_{-1}$ is in the essential spectrum of the operator (with domain $D(L) = H^2(\mathbb{R}^N)$):

$$L = -\Delta + V : L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N). \tag{27}$$

Then, by Weyl's criterion (see, e.g., [25, Theorem VII.12] or [26, Theorem 7.2]), there exists a sequence $\{u_n\} \subset H^2(\mathbb{R}^N)$ with the properties that $\|u_n\|_{L^2} = 1$, $\forall n$ and $\|-\Delta u_n + Vu_n + \mu_{-1}u_n\|_{L^2} \to 0$.

Since $\mu_{-1} > \max_{\mathbb{R}^N}V_-$, we deduce that $-V_-(x) + \mu_{-1} > 0$ for all $x \in \mathbb{R}^N$. Together with the facts that V is a continuous periodic function and $V = V_+ - V_-$, this implies

$$\inf_{x\in\mathbb{R}^N}(V(x) + \mu_{-1}) > 0. \tag{28}$$

It follows that there exists a constant $C' > 0$ such that

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2 + (V(x) + \mu_{-1})u^2\right)dx \ge C'\|u\|^2, \quad \forall u \in X. \tag{29}$$

Note that

$$\int_{\mathbb{R}^N}(-\Delta u_n + V(x)u_n + \mu_{-1}u_n)u_n dx = \int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + (V(x) + \mu_{-1})u_n^2\right)dx. \tag{30}$$

Together with (29) and the fact that $\|-\Delta u_n + Vu_n + \mu_{-1}u_n\|_{L^2} \to 0$ and $\|u_n\|_{L^2} = 1$, this implies $\|u_n\| \to 0$. It contradicts $\|u_n\|_{L^2} = 1$, $\forall n$. Therefore, $\max_{\mathbb{R}^N}V_- \ge \mu_{-1}$.

(**b**) It suffices to prove that

$$\mu_{-1} = C_{-} := \inf\left\{\|u\|^2 \mid u \in X^-, \|u\|_{L^2} = 1\right\}. \tag{31}$$

From (21), we deduce that $\mu_{-1} \le C_-$. From assumption (**v**), $-\mu_{-1}$ is in the essential spectrum of L. By Weyl's criterion, there exists $\{u_n\} \subset H^2(\mathbb{R}^N)$ such that $\|u_n\|_{L^2} = 1$ and $\|-\Delta u_n + Vu_n + \mu_{-1}u_n\|_{L^2} \to 0$. Multiplying $-\Delta u_n + Vu_n + \mu_{-1}u_n$ by u_n^+ and then integrating it into $\mathbb{R}^N$, by (20) and (22), we get that

$$\begin{aligned}(\mu_1 + \mu_{-1})\|u_n^+\|_{L^2}^2 &\le \int_{\mathbb{R}^N}\left(|\nabla u_n^+|^2 + V(x)(u_n^+)^2 + \mu_{-1}(u_n^+)^2\right)dx \\ &= \int_{\mathbb{R}^N}(-\Delta u_n + V(x)u_n + \mu_{-1}u_n)u_n^+ dx \longrightarrow 0.\end{aligned} \tag{32}$$

It follows that $\|u_n^-\|_{L^2} \to 1$. Multiplying $-\Delta u_n + Vu_n + \mu_{-1}u_n$ by u_n^- and then integrating it into $\mathbb{R}^N$, we get that

$$\begin{aligned}-\|u_n^-\|^2 + \mu_{-1}\|u_n^-\|_{L^2}^2 &= \int_{\mathbb{R}^N}\left(|\nabla u_n^-|^2 + V(x)(u_n^-)^2 + \mu_{-1}(u_n^-)^2\right)dx \\ &= \int_{\mathbb{R}^N}(-\Delta u_n + Vu_n + \mu_{-1}u_n)u_n^- dx \longrightarrow 0.\end{aligned} \tag{33}$$

It implies that $\mu_{-1} \ge C_-$. This together with $\mu_{-1} \le C_-$ implies $\mu_{-1} = C_-$. □

Let $R > r > 0$ and

$$A := \inf_{x\in\mathbb{R}^N, |t|\ge D} \frac{f(x,t)}{t}. \tag{34}$$

From assumption (5), we have $A > \max_{\mathbb{R}^N} V_-$. Together with the result (**a**) of Lemma 6, this implies that $(1/2)(A + \mu_{-1}) > \mu_{-1}$. Choose

$$\gamma \in \left(\mu_{-1}, \frac{(A + \mu_{-1})}{2}\right). \tag{35}$$

Then by the result (**b**) of Lemma 6, there exists $u_0 \in X^-$ with $\|u_0\| = 1$ such that

$$\gamma\|u_0\|_{L^2} > 1. \tag{36}$$

Set

$$\begin{aligned}N &= \{u \in X^- \mid \|u\| = r\}, \\ M &= \{u \in X^+ \oplus \mathbb{R}^+ u_0 \mid \|u\| \le R\}.\end{aligned} \tag{37}$$

Then, M is a submanifold of $X^+ \oplus \mathbb{R}^+ u_0$ with boundary

$$\begin{aligned}\partial M = &\{u \in X^- \mid \|u\| \le R\} \\ &\cup \{u^- + tu_0 \mid u^- \in X^-, t > 0, \|u^- + tu_0\| = R\}.\end{aligned} \tag{38}$$

Definition 7. Let $\phi \in C^1(X;\mathbb{R})$. A sequence $\{u_n\} \subset X$ is called a Palais-Smale sequence at level c ($(PS)_c$-sequence for short) for ϕ, if $\phi(u_n) \to c$ and $\|\phi'(u_n)\|_{X'} \to 0$ as $n \to \infty$.

The following proposition is proved in [17] (see [17, Theorem 2.1]).

Proposition 8. *Let $0 < K < 1$. The family of C^1-functional $\{H_\lambda\}$ has the form*

$$H_\lambda(u) = \lambda I(u) - J(u), \quad u \in X, \ \lambda \in [K,1]. \tag{39}$$

Assume

(a) *$J(u) \ge 0$, $\forall u \in X$;*

(b) *$|I(u)| + J(u) \to +\infty$ as $\|u\| \to +\infty$;*

(c) *for all $\lambda \in [K,1]$, H_λ is τ-sequentially upper semicontinuous; that is, if $|||u_n - u||| \to 0$, then*

$$\limsup_{n\to\infty} H_\lambda(u_n) \le H_\lambda(u), \tag{40}$$

and H'_λ is weakly sequentially continuous. Moreover, H_λ maps bounded sets to bounded sets;

(d) *there exist $u_0 \in X^- \setminus \{0\}$ with $\|u_0\| = 1$ and $R > r > 0$ such that, for all $\lambda \in [K,1]$,*

$$\inf_N H_\lambda > \sup_{\partial M} H_\lambda. \tag{41}$$

Then there exists $E \subset [K,1]$ such that the Lebesgue measure of $[K,1] \setminus E$ is zero and, for every $\lambda \in E$, there exist c_λ and a bounded $(PS)_{c_\lambda}$-sequence for H_λ, where c_λ satisfies

$$\sup_M H_\lambda \ge c_\lambda \ge \inf_N H_\lambda. \tag{42}$$

For $0 < K < 1$ and $\lambda \in [K,1]$, let

$$\begin{aligned}\Psi_\lambda(u) = &\frac{\lambda}{2}\int_{\mathbb{R}^N} V_-(x)u^2 dx \\ &- \left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V_+(x)u^2\right)dx + \psi(u)\right), \quad u \in X.\end{aligned} \tag{43}$$

Then

$$\Psi_1 = -\Phi \tag{44}$$

and it is easy to verify that a critical point u of Ψ_λ is a weak solution of

$$-\Delta u + V_\lambda(x)u + f(x,u) = 0, \quad u \in X, \tag{45}$$

where

$$V_\lambda = V_+ - \lambda V_-. \tag{46}$$

Lemma 9. *Suppose that (**v**) and (**f$_1$**)–(**f$_3$**) hold. Then, there exist $0 < K_* < 1$ and $E \subset [K_*,1]$ such that the Lebesgue measure of $[K_*,1] \setminus E$ is zero and, for every $\lambda \in E$, there exist c_λ and a bounded $(PS)_{c_\lambda}$-sequence for Ψ_λ, where c_λ satisfies*

$$+\infty > \sup_{\lambda\in E} c_\lambda \ge \inf_{\lambda\in E} c_\lambda > 0. \tag{47}$$

Proof. For $u \in X$, let

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} V_-(x)\,u^2 dx,$$
$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V_+(x)\,u^2\right)dx + \psi(u). \tag{48}$$

Then, I and J satisfy assumptions (a) and (b) in Proposition 8, and, by (43), $\Psi_\lambda(u) = \lambda I(u) - J(u)$.

From (43) and (20), for any $u \in X$ and $\lambda \in [K,1]$, we have

$$\begin{aligned}\Psi_\lambda(u) &= \frac{\lambda-1}{2}\int_{\mathbb{R}^N} V_-(x)\,u^2 dx \\ &\quad - \left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)\,u^2\right)dx + \int_{\mathbb{R}^N} F(x,u)\,dx\right) \\ &= \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2 \\ &\quad - \frac{1-\lambda}{2}\int_{\mathbb{R}^N} V_-(x)\,u^2 dx - \int_{\mathbb{R}^N} F(x,u)\,dx.\end{aligned} \tag{49}$$

Let $u_* \in X$ and $\{u_n\} \subset X$ be such that $|||u_n - u_*||| \to 0$. It follows that $u_n^- \to u_*^-$, $u_n^+ \rightharpoonup u_*^+$, and $u_n \rightharpoonup u_*$. In addition, up to a subsequence, we can assume that $u_n \to u_*$ a.e. in $\mathbb{R}^N$. Then, we have

$$\|u_n^-\|^2 \longrightarrow \|u_*^-\|^2, \tag{50}$$

$$\begin{aligned}&\liminf_{n\to\infty}\int_{\mathbb{R}^N} V_-(x)\,u_n^2 dx \\ &\quad \geq \int_{\mathbb{R}^N} V_-(x)\,u_*^2 dx \quad \text{(by Fatou's lemma)}, \\ &\liminf_{n\to\infty}\|u_n^+\|^2 \geq \|u_*^+\|^2.\end{aligned} \tag{51}$$

By Remark 1, $F(x,t) \geq 0$ for all x and t. This together with Fatou's lemma implies

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n)\,dx \geq \int_{\mathbb{R}^N} F(x,u_*)\,dx. \tag{52}$$

Then, by (49), we obtain

$$\limsup_{n\to\infty}\Psi_\lambda(u_n) \leq \Psi_\lambda(u_*). \tag{53}$$

This implies that Ψ_λ is τ-sequentially upper semicontinuous.

If $u_n \rightharpoonup u_*$ in X, then, for any fixed $\varphi \in X$, as $n \to \infty$,

$$\begin{aligned}&\left\langle -\Psi_\lambda'(u_n),\varphi\right\rangle \\ &\quad = \int_{\mathbb{R}^N}\left(\nabla u_n \nabla\varphi + V_\lambda u_n \varphi\right)dx + \int_{\mathbb{R}^N} f(x,u_n)\,\varphi dx \\ &\quad \longrightarrow \int_{\mathbb{R}^N}\left(\nabla u_* \nabla\varphi + V_\lambda u_* \varphi\right)dx + \int_{\mathbb{R}^N} f(x,u_*)\,\varphi dx \\ &\quad = \left\langle -\Psi_\lambda'(u_*),\varphi\right\rangle.\end{aligned} \tag{54}$$

This implies that Ψ_λ' is weakly sequentially continuous. Moreover, it is easy to see that Ψ_λ maps bounded sets to bounded sets. Therefore, Ψ_λ satisfies assumption (c) in Proposition 8.

Finally, we will verify assumption (d) in Proposition 8 for Ψ_λ.

From assumption $(\mathbf{f_1})$ and $f(x,t)/t \to 0$ as $t \to 0$ uniformly for $x \in \mathbb{R}^N$, we deduce that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$F(x,t) \leq \epsilon t^2 + C_\epsilon |t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{55}$$

From (49) and (55), we have, for $u \in N$,

$$\begin{aligned}\Psi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1-\lambda}{2}\int_{\mathbb{R}^N} V_-(x)\,u^2 dx \\ &\quad - \epsilon\int_{\mathbb{R}^N} u^2 dx - C_\epsilon\int_{\mathbb{R}^N} |u|^p dx.\end{aligned} \tag{56}$$

Then by the Sobolev inequality $\|u\|_{L^p(\mathbb{R}^N)} \leq C\|u\|$ and $\|u\|_{L^2} \leq C\|u\|$ (by (21) and (22)), we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned}\Psi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - C(1-\lambda)\max_{\mathbb{R}^N} V_-(x)\,\|u\|^2 \\ &\quad - \epsilon C\|u\|^2 - CC_\epsilon\|u\|^p.\end{aligned} \tag{57}$$

Choose $0 < K_* < 1$ and $\epsilon > 0$ such that $C(1-K_*)\max_{\mathbb{R}^N} V_-(x) < 1/4$ and $C_\epsilon = 1/8$. Then, for every $\lambda \in [K_*,1]$, we have

$$\Psi_\lambda(u) \geq \frac{1}{8}\|u\|^2 - CC_\epsilon\|u\|^p. \tag{58}$$

Let $r > 0$ be such that $r^{p-2}CC_\epsilon = 1/16$ and $\beta = r^2/16$. Then, from (58), we deduce that, for $N = \{u \in X^- \mid \|u\| = r\}$,

$$\inf_N \Psi_\lambda \geq \beta, \quad \forall \lambda \in [K_*,1]. \tag{59}$$

We will prove that $\sup_{K_* \leq \lambda \leq 1}\Psi_\lambda(u) \to -\infty$ as $\|u\| \to \infty$ and $u \in X^+ \oplus \mathbb{R}^+ u_0$. Arguing indirectly, assume that, for some sequences $\lambda_n \in [K_*,1]$ and $u_n \in X^+ \oplus \mathbb{R}^+ u_0$ with $\|u_n\| \to +\infty$, there is $\mathscr{L} > 0$ such that $\Psi_{\lambda_n}(u_n) \geq -\mathscr{L}$ for all n. Then, setting $w_n = u_n/\|u_n\|$, we have $\|w_n\| = 1$, and, up to a subsequence, $w_n \rightharpoonup w$, $w_n^- \to w^- \in X^-$ and $w_n^+ \rightharpoonup w^+ \in X^+$.

First, we consider the case $w \neq 0$. Dividing both sides of (49) by $\|u_n\|^2$, we get that

$$\begin{aligned}-\frac{\mathscr{L}}{\|u_n\|^2} &\leq \frac{\Psi_{\lambda_n}(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2}\|w_n^-\|^2 - \frac{1}{2}\|w_n^+\|^2 \\ &\quad - \frac{1-\lambda_n}{2}\int_{\mathbb{R}^N} V_-(x)\,w_n^2 dx - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^2}dx.\end{aligned} \tag{60}$$

From (5) and the result $(\mathbf{a})$ of Lemma 6, we deduce that

$$\liminf_{|t|\to\infty}\frac{F(x,t)}{t^2} \geq \frac{A}{2} > \frac{1}{2}\max_{\mathbb{R}^N} V_- \geq \frac{1}{2}\mu_{-1}, \tag{61}$$

where A is defined by (34). Note that, for $x \in \{x \in \mathbb{R}^N \mid w \neq 0\}$, we have $|u_n(x)| \to +\infty$. This implies that, when n is large enough,

$$\int_{\{x\in\mathbb{R}^N \mid w\neq 0\}} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \geq \frac{A+\mu_{-1}}{4} \int_{\{x\in\mathbb{R}^N \mid w\neq 0\}} w_n^2 dx. \quad (62)$$

By (10), we have, when n is large enough,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2} dx &= \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \\ &\geq \int_{\{x\in\mathbb{R}^N \mid w\neq 0\}} \frac{F(x,u_n)}{u_n^2} w_n^2 dx.\end{aligned} \quad (63)$$

Combining the above two inequalities yields

$$\begin{aligned}&\liminf_{n\to\infty}\Bigg(\frac{1}{2}\|w_n^-\|^2 - \frac{1}{2}\|w_n^+\|^2 \\ &\quad - \frac{1-\lambda_n}{2}\int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2} dx\Bigg) \\ &\leq \liminf_{n\to\infty}\Bigg(\frac{1}{2}\|w_n^-\|^2 - \frac{1}{2}\|w_n^+\|^2 \\ &\quad - \frac{A+\mu_{-1}}{4}\int_{\{x\in\mathbb{R}^N \mid w\neq 0\}} w_n^2 dx\Bigg) \\ &\leq \frac{1}{2}\|w^-\|^2 - \frac{1}{2}\|w^+\|^2 - \frac{A+\mu_{-1}}{4}\int_{\mathbb{R}^N} w^2 dx \\ &\leq \frac{1}{2}\|w^-\|^2 - \frac{1}{2}\|w^+\|^2 - \frac{A+\mu_{-1}}{4}\|w^-\|_{L^2}^2.\end{aligned} \quad (64)$$

We used the inequalities

$$\begin{gathered}\lim_{n\to\infty}\|w_n^-\|^2 = \|w^-\|^2, \\ \liminf_{n\to\infty}\|w_n^+\|^2 \geq \|w^+\|^2, \\ \liminf_{n\to\infty}\int_{\{x\in\mathbb{R}^N \mid w\neq 0\}} w_n^2 dx \geq \int_{\mathbb{R}^N} w^2 dx\end{gathered} \quad (65)$$

in the second inequality of (64).

Since $w^- = tu_0$ for some $t \in \mathbb{R}$, by (36), we get that

$$\frac{A+\mu_{-1}}{4}\|w^-\|_{L^2}^2 \geq \frac{A+\mu_{-1}}{4\gamma}\|w^-\|^2. \quad (66)$$

Note that, by the choice of γ (see (35)), we have $((A+\mu_{-1})/4\gamma) > 1/2$. Then by (64) and the fact that $w \neq 0$, we have that

$$\begin{aligned}&\liminf_{n\to\infty}\Bigg(\frac{1}{2}\|w_n^-\|^2 - \frac{1}{2}\|w_n^+\|^2 \\ &\quad - \frac{1-\lambda_n}{2}\int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2} dx\Bigg) \\ &\leq -\left(\frac{A+\mu_{-1}}{4\gamma} - \frac{1}{2}\right)\|w^-\|^2 - \frac{1}{2}\|w^+\|^2 < 0.\end{aligned} \quad (67)$$

It contradicts (60), since $-\mathscr{L}/\|u_n\|^2 \to 0$ as $n \to \infty$.

Second, we consider the case $w = 0$. In this case, $\lim_{n\to\infty}\|w_n^-\| = 0$. It follows that

$$\liminf_{n\to\infty}\|w_n^+\| \geq 1, \quad (68)$$

since $\|w_n\| = 1$ and $w_n = w_n^+ + w_n^-$. Therefore, the right hand side of (60) is less than $-1/4$ when n is large enough. However, as $n \to \infty$, the left hand side of (60) converges to zero. It induces a contradiction.

Therefore, there exists $R > r$ such that

$$\sup_{\lambda\in[K_*,1]}\sup_{\partial M}\Psi_\lambda \leq 0. \quad (69)$$

This implies that Ψ_λ satisfies assumption (d) in Proposition 8 if $\lambda \in [K_*, 1]$. Finally, it is easy to see that

$$\sup_{\lambda\in[K_*,1]}\sup_{M}\Psi_\lambda < +\infty. \quad (70)$$

Then, the results of this lemma follow immediately from Proposition 8. □

Lemma 10. *Suppose that* (**v**) *and* ($\mathbf{f_1}$)–($\mathbf{f_3}$) *are satisfied. Let* $\lambda \in [K_*, 1]$ *be fixed, where* K_* *is the constant in Lemma 9. If* $\{v_n\}$ *is a bounded* $(PS)_c$*-sequence for* Ψ_λ *with* $c \neq 0$*, then, for every* $n \in \mathbb{N}$*, there exists* $a_n \in \mathbb{Z}^N$ *such that, up to a subsequence,* $u_n := v_n(\cdot + a_n)$ *satisfies*

$$u_n \rightharpoonup u_\lambda \neq 0, \quad \Psi_\lambda(u_\lambda) \leq c, \quad \Psi_\lambda'(u_\lambda) = 0. \quad (71)$$

Proof. The proof of this lemma is inspired by the proof of Lemma 3.7 in [19]. Because $\{v_n\}$ is a bounded sequence in X, up to a subsequence, either

(a) $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n|^2 dx = 0$ or

(b) there exist $\varrho > 0$ and $a_n \in \mathbb{Z}^N$ such that $\int_{B_1(a_n)}|v_n|^2 dx \geq \varrho$.

If (a) occurs, using the Lions lemma (see, e.g., [21, Lemma 1.21]), a similar argument as for the proof of [19, Lemma 3.6] shows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,v_n)\,dx = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,v_n)\,v_n^{\pm} dx = 0. \quad (72)$$

It follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(2F(x,v_n)-f(x,v_n)v_n\right)dx=0. \tag{73}$$

On the other hand, as $\{v_n\}$ is a $(PS)_c$-sequence of Ψ_λ, we have $\langle\Psi_\lambda'(v_n),v_n\rangle \to 0$ and $\Psi_\lambda(v_n) \to c\neq 0$. It follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(f(x,v_n)v_n-2F(x,v_n)\right)dx\\&\quad=2\Psi_\lambda(v_n)-\left\langle\Psi_\lambda'(v_n),v_n\right\rangle\longrightarrow 2c\neq 0,\quad n\longrightarrow\infty.\end{aligned} \tag{74}$$

This contradicts (73). Therefore, case (a) cannot occur.

If case (b) occurs, let $u_n=v_n(\cdot+a_n)$. For every n,

$$\int_{B_1(0)}|u_n|^2dx\geq\varrho. \tag{75}$$

Because V and $F(x,t)$ are 1-periodic in every x_j, $\{u_n\}$ is still bounded in X,

$$\lim_{n\to\infty}\Psi_\lambda(u_n)\leq c,\quad \Psi_\lambda'(u_n)\rightharpoonup 0,\quad n\longrightarrow\infty. \tag{76}$$

Up to a subsequence, we assume that $u_n\rightharpoonup u_\lambda$ in X as $n\to\infty$. Since $u_n\to u_\lambda$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, it follows from (75) that $u_\lambda\neq 0$. Recall that $\Psi_\lambda'(u_n)$ is weakly sequentially continuous. Therefore, $\Psi_\lambda'(u_n)\rightharpoonup\Psi_\lambda'(u_\lambda)$ and, by (76), $\Psi_\lambda'(u_\lambda)=0$.

Finally, by $(\mathbf{f_3})$ and Fatou's lemma

$$\begin{aligned}c&=\lim_{n\to\infty}\left(\Psi_\lambda(u_n)-\frac{1}{2}\left\langle\Psi_\lambda'(u_n),u_n\right\rangle\right)\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}\widetilde{F}(x,u_n)\geq\int_{\mathbb{R}^N}\widetilde{F}(x,u_\lambda)=\Psi_\lambda(u_\lambda).\end{aligned} \tag{77}$$

□

Lemma 11. *There exist $0<K_{**}<1$ and $\eta>0$ such that, for any $\lambda\in[K_{**},1]$, if $u\neq 0$ satisfies $\Psi_\lambda'(u)=0$, then $\|u\|\geq\eta$.*

Proof. We adapt the arguments of Yang [23, p. 2626] and Liu [12, Lemma 2.2]. Note that, by $(\mathbf{f_1})$ and $(\mathbf{f_2})$, for any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$|f(x,t)|\leq\epsilon|t|+C_\epsilon|t|^{p-1}. \tag{78}$$

Let $u\neq 0$ be a critical point of Ψ_λ. Then u is a solution of

$$-\Delta u+V_\lambda u+f(x,u)=0,\quad u\in X. \tag{79}$$

Multiplying both sides of this equation by $u^\pm$, respectively, and then integrating into $\mathbb{R}^N$, we get that

$$\begin{aligned}0=&\pm\|u^\pm\|^2+(1-\lambda)\int_{\mathbb{R}^N}V_-(x)u_nu^\pm dx\\&+\int_{\mathbb{R}^N}f(x,u)u^\pm dx.\end{aligned} \tag{80}$$

It follows that

$$\begin{aligned}\|u^\pm\|^2=&\mp(1-\lambda)\int_{\mathbb{R}^N}V_-(x)uu^\pm dx\mp\int_{\mathbb{R}^N}f(x,u)u^\pm dx\\\leq&(1-\lambda)\sup_{\mathbb{R}^N}V_-\int_{\mathbb{R}^N}|u|\cdot|u^\pm|dx\\&+\epsilon\int_{\mathbb{R}^N}|u|\cdot|u^\pm|dx+C_\epsilon\int_{\mathbb{R}^N}|u|^{p-1}|u^\pm|dx\\\leq&C_1((1-\lambda)+\epsilon)\|u\|\cdot\|u^\pm\|+C_2\|u\|^{p-1}\|u^\pm\|,\end{aligned} \tag{81}$$

where C_1 and C_2 are positive constants related to the Sobolev inequalities and $\sup_{\mathbb{R}^N}V_-$. From the above two inequalities, we obtain

$$\|u\|^2=\|u^+\|^2+\|u^-\|^2\leq 2C_1((1-\lambda)+\epsilon)\|u\|^2+2C_2\|u\|^p. \tag{82}$$

Because $p>2$, this implies that $\|u\|\geq\eta$ for some $\eta>0$ if $\epsilon>0$ and $1-K_{**}>0$ are small enough and $\lambda\in[K_{**},1]$. The desired result follows. □

Let $K=\max\{K_*,K_{**}\}$, where K_* and K_{**} are the constants that appeared in Lemmas 9 and 11, respectively. Combining Lemmas 9–11, we obtain the following lemma.

Lemma 12. *Suppose $(\mathbf{v})$ and $(\mathbf{f_1})$–$(\mathbf{f_3})$ are satisfied. Then, there exist $\eta>0$, $\{\lambda_n\}\subset[K,1]$, and $\{u_n\}\subset X$ such that $\lambda_n\to 1$,*

$$\sup_n\Psi_{\lambda_n}(u_n)<+\infty,\quad\|u_n\|\geq\eta,\quad\Psi_{\lambda_n}'(u_n)=0. \tag{83}$$

3. A Priori Bound of Approximate Solutions and Proof of the Main Theorem

In this section, we give a priori bound for the sequence of approximate solutions $\{u_n\}$ obtained in Lemma 12. We then give the proofs of Theorem 3.

Lemma 13. *Suppose $(\mathbf{v})$ and $(\mathbf{f_1})$–$(\mathbf{f_3})$ are satisfied. Let $\{u_n\}$ be the sequence obtained in Lemma 12. Then, $\{u_n\}\subset L^\infty(\mathbb{R}^N)$ and*

$$\sup_n\|u_n\|_{L^\infty(\mathbb{R}^N)}\leq D. \tag{84}$$

Proof. From $\Psi_{\lambda_n}'(u_n)=0$, we deduce that u_n is a weak solution of (45) with $\lambda=\lambda_n$; that is,

$$-\Delta u_n+V_{\lambda_n}(x)u_n+f(x,u_n)=0\quad\text{in }\mathbb{R}^N. \tag{85}$$

By assumption $(\mathbf{f_1})$ and the bootstrap argument of elliptic equations, we deduce that $u_n\in L^\infty(\mathbb{R}^N)$.

Multiplying both sides of (85) by $v_n=(u_n-D)^+:=\max\{u_n-D,0\}$ and integrating into $\mathbb{R}^N$, we get that

$$\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\int_{u_n\geq D}\left(V_{\lambda_n}(x)u_n+f(x,u_n)\right)v_ndx=0. \tag{86}$$

Recall that $V_{\lambda_n} = V_+ - \lambda_n V_-$ and $\lambda_n \le 1$. Then by (5), we get that

$$\begin{aligned}&\int_{u_n \ge D} \left(V_{\lambda_n}(x) u_n + f(x,u_n)\right) v_n dx \\ &= \int_{u_n \ge D} \left(V_{\lambda_n}(x) + \frac{f(x,u_n)}{u_n}\right) u_n v_n dx \ge 0.\end{aligned} \tag{87}$$

This together with (86) yields $v_n = 0$. It follows that $u_n(x) \le D$ on $\mathbb{R}^N$.

Similarly, multiplying both sides of (85) by $w_n = (u_n + D)^- := \max\{-(u_n + D), 0\}$ and integrating into $\mathbb{R}^N$, we can get that $u_n \ge -D$ on $\mathbb{R}^N$. Therefore, for all n, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \le D$. □

Lemma 14. *Suppose that* (**v**), ($\mathbf{f_1}$), ($\mathbf{f_2}$), ($\mathbf{f_3}$), *and* ($\mathbf{f_4}$) *are satisfied. Let* $\{u_n\}$ *be the sequence obtained in Lemma 12. Then*

$$0 < \inf_n \|u_n\| \le \sup_n \|u_n\| < +\infty. \tag{88}$$

Proof. As $\Psi'_{\lambda_n}(u_n) = 0$ and $u_n \ne 0$, Lemma 11 implies that $\inf_n \|u_n\| > 0$.

To prove $\sup_n \|u_n\| < +\infty$, we apply an indirect argument and assume by contradiction that $\|u_n\| \to +\infty$.

Since $\Psi'_{\lambda_n}(u_n) = 0$, by (81), we get that

$$\begin{aligned}\|u_n^\pm\|^2 &= \mp (1-\lambda_n) \int_{\mathbb{R}^N} V_-(x) u_n u_n^\pm dx \mp \int_{\mathbb{R}^N} f(x,u_n) u_n^\pm dx \\ &= \mp \int_{\mathbb{R}^N} f(x,u_n) u_n^\pm dx + (1-\lambda_n) O\left(\|u_n\|^2\right).\end{aligned} \tag{89}$$

It follows that

$$\begin{aligned}&\|u_n\|^2 + \int_{\mathbb{R}^N} f(x,u_n)(u_n^+ - u_n^-) dx \\ &= \|u_n^+\|^2 + \|u_n^-\|^2 \\ &\quad + \int_{\mathbb{R}^N} f(x,u_n)(u_n^+ - u_n^-) dx \\ &= (1-\lambda_n) O\left(\|u_n\|^2\right).\end{aligned} \tag{90}$$

Set $w_n = u_n/\|u_n\|$. Then, by (90),

$$\begin{aligned}&\|u_n\|^2 \left(1 + \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right) \\ &= (1-\lambda_n) O\left(\|u_n\|^2\right).\end{aligned} \tag{91}$$

Then, by $\lambda_n \to 1$ as $n \to \infty$, we have that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} (w_n^+ - w_n^-) w_n dx \longrightarrow -1, \quad n \longrightarrow \infty. \tag{92}$$

From Lemma 12,

$$C_0 := \sup_n \Psi_{\lambda_n}(u_n) < +\infty. \tag{93}$$

Then, by $\Psi'_{\lambda_n}(u_n) = 0$, we obtain

$$2C_0 \ge 2\Psi_{\lambda_n}(u_n) - \left\langle \Psi'_{\lambda_n}(u_n), u_n \right\rangle = 2\int_{\mathbb{R}^N} \widetilde{F}(x,u_n) dx. \tag{94}$$

From ($\mathbf{f_3}$), we have

$$2C_0 \ge 2\int_{\mathbb{R}^N} \widetilde{F}(x,u_n) dx \ge 2\int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} \widetilde{F}(x,u_n) dx, \tag{95}$$

where κ is the constant in ($\mathbf{f_4}$). As the continuous function $\widetilde{F}$ is 1-periodic in every x_j variable, we deduce from (8) that there exists a constant $C' > 0$ such that

$$\begin{aligned}&\widetilde{F}(x,t) \ge C' t^2, \\ &\quad \text{for every } (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ with } \kappa \le |t| \le D.\end{aligned} \tag{96}$$

Combining (95) and (96) leads to

$$C_0 \ge C' \int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} u_n^2 dx. \tag{97}$$

Dividing both sides of this inequality by $\|u_n\|^2$ and sending $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} w_n^2 dx = 0. \tag{98}$$

From (7), (21), and (22), we have that

$$\begin{aligned}&\int_{\{x \mid |u_n(x)| < \kappa\}} \left| \frac{f(x,u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\ &\le \nu \int_{\{x \mid |u_n(x)| < \kappa\}} \left|(w_n^+ - w_n^-) w_n\right| dx \\ &\le \nu \int_{\mathbb{R}^N} \left|(w_n^+ - w_n^-) w_n\right| dx \\ &\le \nu \|w_n\|_{L^2}^2 \le \frac{\nu}{\mu_0} \|w_n\|^2 = \frac{\nu}{\mu_0} < 1,\end{aligned} \tag{99}$$

where μ_0 is the constant defined in (**v**).

Since $f \in C(\mathbb{R}^N \times \mathbb{R})$ and $\lim_{t\to 0} f(x,t)/t = 0$, we deduce that there exists $C > 0$ such that, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \le D$,

$$|f(x,t)| \le C|t|. \tag{100}$$

This together with (98) gives

$$\begin{aligned}&\int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} \left| \frac{f(x,u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\ &\le C \int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} \left|(w_n^+ - w_n^-) w_n\right| dx \\ &\le C \|w_n^+ - w_n^-\|_{L^2} \left(\int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} w_n^2 dx \right)^{1/2} \\ &\le 2C \|w_n\|_{L^2} \left(\int_{\{x \mid D \ge |u_n(x)| \ge \kappa\}} w_n^2 dx \right)^{1/2} \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{101}$$

Combining (99) and (101) yields

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left|\frac{f(x,u_n)}{u_n}(w_n^+-w_n^-)\,w_n\right|dx \\
&\leq \limsup_{n\to\infty}\int_{\{x||u_n(x)|<\kappa\}}\left|\frac{f(x,u_n)}{u_n}(w_n^+-w_n^-)\,w_n\right|dx \\
&\quad+\limsup_{n\to\infty}\int_{\{x|D\geq|u_n(x)|\geq\kappa\}}\left|\frac{f(x,u_n)}{u_n}(w_n^+-w_n^-)\,w_n\right|dx<1.
\end{aligned} \tag{102}$$

This contradicts (92). Therefore, $\{u_n\}$ is bounded in X. □

Proof of Theorem 3. Let $\{u_n\}$ be the sequence obtained in Lemma 12. From Lemma 14, $\{u_n\}$ is bounded in X. Therefore, up to a subsequence, either

(a) $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx=0$ or

(b) there exist $\varrho>0$ and $y_n\in\mathbb{Z}^N$ such that $\int_{B_1(y_n)}|u_n|^2dx\geq\varrho$.

According to (72), if case (a) occurs,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)\,u_n^{\pm}dx=0. \tag{103}$$

Then, by (81) and $\lambda_n\to 1$, we have

$$\begin{aligned}
\|u_n^{\pm}\|^2 &= \mp(1-\lambda_n)\int_{\mathbb{R}^N}V_-(x)\,u_nu_n^{\pm}dx \\
&\quad\mp\int_{\mathbb{R}^N}f(x,u_n)\,u_n^{\pm}dx \\
&\leq C(1-\lambda_n)\|u_n\|_{L^2}^2+\left|\int_{\mathbb{R}^N}f(x,u_n)\,u_n^{\pm}dx\right|\longrightarrow 0.
\end{aligned} \tag{104}$$

This contradicts $\inf_n\|u_n\|>0$ (see (88)). Therefore, case (a) cannot occur. As case (b) therefore occurs, $w_n=u_n(\cdot+y_n)$ satisfies $w_n\rightharpoonup u_0\neq 0$. From (14) and (43), we have that

$$\Psi_\lambda(u)=-\Phi(u)+\frac{\lambda-1}{2}\int_{\mathbb{R}^N}V_-u^2dx,\quad\forall u\in X. \tag{105}$$

It follows that

$$\left\langle\Psi_\lambda'(u),\varphi\right\rangle=-\left\langle\Phi'(u),\varphi\right\rangle+(\lambda-1)\int_{\mathbb{R}^N}V_-u\varphi dx,\quad\forall u,\varphi\in X. \tag{106}$$

By $\Psi_{\lambda_n}'(u_n)=0$ (see Lemma 12), we have $\Psi'_{\lambda_n}(w_n)=0$. From (106), we have that, for any $\varphi\in X$,

$$\left\langle\Psi'_{\lambda_n}(w_n),\varphi\right\rangle=-\left\langle\Phi'(w_n),\varphi\right\rangle+(\lambda_n-1)\times\int_{\mathbb{R}^N}V_-(x)\,w_n\varphi dx. \tag{107}$$

Together with $\Psi_{\lambda_n}'(w_n)=0$ and $\lambda_n\to 1$, this yields

$$\left\langle\Phi'(w_n),\varphi\right\rangle\longrightarrow 0,\quad\forall\varphi\in X. \tag{108}$$

Finally, by $w_n\rightharpoonup u_0\neq 0$ and the weakly sequential continuity of Φ', we have that $\Phi'(u_0)=0$. Therefore, u_0 is a nontrivial solution of (1). This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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