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Research Article

Stability and Hopf Bifurcation of an *n*-Neuron Cohen-Grossberg Neural Network with Time Delays

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A Cohen-Grossberg neural network with discrete delays is investigated in this paper. Sufficient conditions for the existence of local Hopf bifurcation are obtained by analyzing the distribution of roots of characteristic equation. Moreover, the direction and stability of Hopf bifurcation are obtained by applying the normal form theory and the center manifold theorem. Numerical simulations are given to illustrate the obtained results.

1. Introduction

In recent years, more and more mathematicians, biologists, physicists, and computer scientists focus on artificial neural networks. It is well known that the analysis of the dynamical behaviors is a necessary step for practical design of neural networks since their applications heavily depend on the dynamical behaviors; many important results on dynamical behaviors of neural networks have been obtained [1–23]. The neural networks are large-scale and complex systems, and the dynamical behaviors of neural networks with delays are more complicated; in order to obtain a deep and clear understanding of the dynamics of complicated neural networks with time delays, researchers have focused on the studying of simple systems [12–22]. This is indeed very useful since the complexity found may be carried over to large neural networks.

The research on dynamical behaviors of neural networks involves not only the dynamic analysis of equilibrium but also that of periodic solution, bifurcation, and chaos; especially, the periodic oscillatory behavior of the neural networks is of great interest in many applications [2, 3]. Since periodic oscillatory can arise through the Hopf bifurcation in different system with or without time delays, it is very important to discuss the Hopf bifurcation of neural networks.

In 1983, Cohen-Grossberg [1] proposed a kind of neural networks, which are now called Cohen-Grossberg neural networks. The networks have been successfully applied to signal processing, pattern recognition, optimization, and associative memories. Recently, some results on the existence and globally asymptotical stability of periodic Cohen-Grossberg neural networks have been obtained [7–15]. However, up to now, to the best of the author's knowledge, bifurcation of Hopfield neural networks has been discussed by many researchers [12–19], but few results on the bifurcation of Cohen-Grossberg neural networks have been obtained. Zhao discussed the bifurcation of a two-neuron discrete-time Cohen-Grossberg neural network in [20] and the bifurcation of a two-neuron continuous-time Cohen-Grossberg neural network with distributed delays in which kernel function is $\alpha e^{-\alpha s}$ in [21]. We discussed the bifurcation of a two-neuron Cohen-Grossberg neural network with discrete delays in [22]. The objective of this paper is to study the following n-neuron continuous-time Cohen-Grossberg neural network with discrete delays and ring architecture:

$$\dot{x}_1(t) = -a_1(x_1(t))[b_1(x_1(t)) - c_1 f_1(x_n(t-\tau_n))],$$
$$\dot{x}_2(t) = -a_2(x_2(t))[b_2(x_2(t)) - c_2 f_2(x_1(t-\tau_1))],$$
$$\dot{x}_3(t) = -a_3(x_3(t))[b_3(x_3(t)) - c_3 f_3(x_2(t-\tau_2))],$$

$$\vdots$$

$$\dot{x}_n(t) = -a_n(x_n(t))\left[b_n(x_n(t)) - c_n f_n(x_{n-1}(t-\tau_{n-1}))\right], \tag{1}$$

where $x_i(t)$ denote the state variable of the ith neuron; $a_i(\cdot)$ represent amplification functions which are positive for R; $f_i(\cdot)$ denote the signal functions of the ith neuron; $b_i(\cdot)$ are appropriately behaved functions; c_i are connection weights of the neural networks; discrete delays τ_i correspond to the finite speed of the axonal signal transmission: $i = 1, 2, \ldots, n$, $n \geq 2$.

Ring architectures have been found in variety of neural structures, and they are investigated to gain insight into the mechanisms underlying the behaviors of recurrent neural networks [23].

The rest of this paper is organized as follows. Stability property and existence of Hopf bifurcation for system (1) are obtained in Section 2. Based on the normal form method and the center manifold, the formulas for the direction of Hopf bifurcation and stability of the bifurcating periodic solutions are derived in Section 3. An example is given in Section 4 to illustrate the main results, and conclusions are drawn in Section 5.

2. Stability Analysis and Existence of Local Bifurcation

Lemma 1 (see [11]). *Consider the exponential polynomial*

$$\begin{aligned} p&\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) \\ &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} \\ &\quad + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m}, \end{aligned} \tag{2}$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ and $p_j^{(i)}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \cdots, n)$ are constants. Then as $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the order of zeros of $p(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or across the imaginary axis.

In the following discussion, for convenience, we denote

$$\beta_i = a_i(0)\, c_i f_i'(0), \qquad \alpha_i = a_i(0)\, b_i'(0), \qquad \gamma = \beta_1\beta_2\cdots\beta_n. \tag{3}$$

Throughout this paper, we assume that

(H_1) $b_i(0) = 0$, $f_i(0) = 0$, and $i = 1, 2, \ldots, n$;

(H_2) there exist constants $\underline{a}_i, \overline{a}_i$ such that $0 < \underline{a}_i \leq a_i(\cdot) \leq \overline{a}_i$ for $i = 1, 2, \ldots, n$.

From assumption (H_1)-(H_2) that the origin $(0, 0, \ldots, 0)$ is an equilibrium of system (1).

Let

$$\begin{aligned} x_1(t) &= u_1(t - (\tau_2 + \tau_3 + \cdots + \tau_n)), \\ x_2(t) &= u_2(t - (\tau_3 + \tau_4 + \cdots + \tau_n)), \\ &\vdots \\ x_{n-1}(t) &= u_{n-1}(t - \tau_n), \\ x_n(t) &= u_n(t). \end{aligned} \tag{4}$$

System (1) can be transformed into the following equivalent system:

$$\begin{aligned} \dot{x}_1(t) &= -a_1(x_1(t))\left[b_1(x_1(t)) - c_1 f_1(x_n(t-\tau))\right], \\ \dot{x}_2(t) &= -a_2(x_2(t))\left[b_2(x_2(t)) - c_2 f_2(x_1(t))\right], \\ \dot{x}_3(t) &= -a_3(x_3(t))\left[b_3(x_3(t)) - c_3 f_3(x_2(t))\right], \\ &\vdots \\ \dot{x}_n(t) &= -a_n(x_n(t))\left[b_n(x_n(t)) - c_n f_n(x_{n-1}(t))\right], \end{aligned} \tag{5}$$

where $\tau = \tau_1 + \tau_2 + \cdots + \tau_n$.

The linear system of system (5) around the equilibrium $(0, 0, \ldots, 0)$ is given by

$$\begin{aligned} \dot{u}_1(t) &= -\alpha_1 u_1(t) + \beta_1 u_n(t-\tau), \\ \dot{u}_2(t) &= -\alpha_2 u_2(t) + \beta_2 u_1(t), \\ \dot{u}_3(t) &= -\alpha_3 u_3(t) + \beta_3 u_2(t), \\ &\vdots \\ \dot{u}_n(t) &= -\alpha_n u_n(t) + \beta_n u_{n-1}(t). \end{aligned} \tag{6}$$

The associated characteristic equation of system (5) is

$$(\lambda + \alpha_1)(\lambda + \alpha_2)\cdots(\lambda + \alpha_n)\, e^{\lambda\tau} = \gamma. \tag{7}$$

Suppose that $\lambda = i\omega (\omega > 0)$ is a root of the characteristic equation, where i is imaginary unit which satisfies $i^2 = -1$. Substituting $i\omega$ into (7), then we have

$$(\alpha_1 + i\omega)(\alpha_2 + i\omega)\cdots(\alpha_n + i\omega)(\cos\omega\tau + i\sin\omega\tau) = \gamma. \tag{8}$$

Separating the real and imaginary parts of (8), we have

$$\begin{aligned} \gamma &= \sqrt{(\alpha_1^2 + \omega^2)(\alpha_2^2 + \omega^2)\cdots(\alpha_n^2 + \omega^2)}\cos h(\omega) \\ &= \sqrt{\prod_{l=1}^{n}(\alpha_l^2 + \omega^2)}\cos h(\omega), \\ &\sin h(\omega) = 0, \end{aligned} \tag{9}$$

where

$$h(\omega) = \sum_{l=1}^{n}\cot^{-1}\left(\frac{\alpha_l}{\omega}\right) + \omega\tau \tag{10}$$

in which $\cot^{-1}$ denotes the inverse of the cotangent function.

Since

$$h'(\omega) = \sum_{l=1}^{n} \frac{\alpha_l}{\alpha_l^2 + \omega^2} + \tau > 0 \tag{11}$$

for $\omega \in R^+$ and

$$\lim_{\omega \to 0^+} h(\omega) = 0, \qquad \lim_{\omega \to +\infty} h(\omega) = +\infty. \tag{12}$$

Hence $h(\omega) : (0, R^+) \to (0, R^+)$ is an increasing bijective function.

We know from the second equation in (9) that $h(\omega) = j\pi$. Denote $\omega_j = h^{-1}(j\pi)$, $j = 1, 2, \ldots$; then we have from the first equation in (9) that

$$\gamma = \gamma_j = (-1)^j \sqrt{\prod_{l=1}^{n} \left(\alpha_l^2 + \omega_j^2\right)}, \quad j = 1, 2, \ldots. \tag{13}$$

When $\omega = 0$, $\gamma = \gamma_0 = \prod_{l=1}^{n} \alpha_l > 0$.

Furthermore, from the value γ_j given above, we have

$$\cdots < \gamma_5 < \gamma_3 < \gamma_1 < 0 < \gamma_0 < \gamma_2 < \gamma_4 \cdots. \tag{14}$$

Obviously, if ω is a root of equation (7), $-\omega$ is also the root of equation (7). This implies that $\pm i\omega$ is a pair of purely imaginary roots of equation (7). On the other hand, we have from (7) that

$$\left(\frac{d\lambda}{d\gamma}\right)^{-1}\Bigg|_{\gamma=\gamma_j} = b_j \left(\tau + \sum_{l=1}^{n} \frac{1}{\alpha_l + i\omega_j}\right), \tag{15}$$

where

$$\operatorname{Re}\left(\left(\frac{d\lambda}{d\gamma}\right)^{-1}\Bigg|_{\gamma=\gamma_j}\right) = \gamma_j \left(\tau + \sum_{l=1}^{n} \frac{\alpha_l}{\alpha_l^2 + \omega_j^2}\right). \tag{16}$$

Hence we have

$$\operatorname{sign}\left\{\frac{d\lambda}{d\gamma}\Bigg|_{\gamma=\gamma_j}\right\} = \operatorname{sign}\left\{\gamma_j\right\} = (-1)^j, \quad j = 0, 1, 2, \ldots. \tag{17}$$

Since the roots of the characteristic equation (7) are $-\alpha_i < 0$, $i = 1, 2, \ldots, n$ when $\gamma = 0$, so the equilibrium $(0, 0, \ldots, 0)$ of system (5) is asymptotically stable. As the parameter γ varies on the open right half plane can change only if a zero appears on or across the imaginary axis. According to Lemma 1 and (14), we obtain that the equilibrium $(0, 0, \ldots, 0)$ of system (5) is asymptotically stable if and only if $\gamma \in (\gamma_1, \gamma_0)$.

When $\gamma = \gamma_0 = \prod_{i=1}^{n} \alpha_i$, the characteristic equation of system (5) has a simple root $\lambda = 0$, and all the other roots have negative real parts. A pitchfork bifurcation may occur at the origin in system (5) [16].

When $\gamma = \gamma_1 = -\sqrt{\prod_{i=1}^{n} (\alpha_i^2 + \omega_1^2)}$, $\omega_1 = h^{-1}(\pi)$, the characteristic equation of system (3) has a pair of purely imaginary roots $\pm i\omega$, and all the other roots have negative real parts. Note that $h(\omega) \ge \tau\omega$ due to $h'(\omega) \ge \tau$ according to (11), so, $\omega \le \pi/\tau$; that is, $\omega_1 \in (0, \pi/\tau)$. We also know from (17) that $(d\lambda/d\gamma)|_{\gamma=\gamma_1} < 0$. System (3)undergoes a Hopf bifurcation which occurs at the origin when $\gamma = \gamma_1$.

From the above discusses, Lemma 1, and the Hopf bifurcation theorem in [24] for functional differential equations, we have the following results.

Theorem 2. *Under assumptions (H_1)-(H_2), we have the following:*

(1) *if $\gamma \in (\gamma_1, \gamma_0)$, the equilibrium* $(0, 0, \ldots, 0)$ *of system* (1) *is asymptotically stable;*

(2) *if $\gamma \notin [\gamma_1, \gamma_0]$, the equilibrium* $(0, 0, \ldots, 0)$ *of system* (1) *is unstable;*

(3) *if $\gamma = \gamma_1$, a Hopf bifurcation occurs at the origin in system* (1),

where

$$\gamma_0 = \prod_{i=1}^{m} \alpha_i, \qquad \gamma_1 = -\sqrt{\prod_{l=1}^{n} \left(\alpha_l^2 + \omega_1^2\right)} \tag{18}$$

in which $\omega_1 \in (0, \pi/\tau)$ and satisfies the equation $\pi = \sum_{i=1}^{n} \cot^{-1}(\alpha_i/\omega) + \omega\tau$, $\tau = \tau_1 + \tau_2 + \cdots + \tau_n$.

3. Direction and Stability of Hopf Bifurcation for the Network

In this section, we will derive explicit formulas for determining the properties of the Hopf bifurcation at critical $\gamma = \gamma_1$ by using the normal form theory and the center manifold theorem [25], and we always make $\gamma = \beta_1\beta_2 \cdots \beta_n$, $\beta_i = \Pi_{i=1}^{n} a_i(0) c_i f_i'(0)$ vary with a parameter and the other $3n - 1$ ones are fixed.

We still discuss system (5). For the sake of generality, let $\overline{u}(s) = u(s\tau)$. Denote $\mu = \gamma$ and $t = s$, dropping the bars for simplification of notation; then system (5) can be written as functional differential equation in $C = C([-1, 0], R^n)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{19}$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$, $u_t(\theta) = u(t + \theta)$, and $L_\mu : C \to R, F : R \times C \to R$ are given, respectively, by

$$L_\mu(\phi) = B_1\phi(0) + B_2\phi(-1), \tag{20}$$

where

$$B_1 = \tau \begin{pmatrix} -\alpha_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta_2 & -\alpha_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta_3 & -\alpha_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{n-1} & -\alpha_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta_n & -\alpha_n \end{pmatrix},$$

$$B_2 = \tau \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$F(\mu, \phi) = \tau\left(F^{(1)}(\phi), F^{(2)}(\phi), \ldots, F^{(n)}(\phi)\right)^T, \tag{21}$$

where

$$\begin{aligned} F^{(1)}(\phi) = &-\left(a_1\frac{b_1''}{2} + a_1'b_1'\right)\phi_1^2(0) + a_1c_1\frac{f_n''}{2}\phi_n^2(-1) \\ &+ a_1'c_1f_n'\phi_1(0)\phi_n(-1) \\ &-\left(a_1\frac{b_1'''}{6} + a_1'\frac{b_1''}{2} + \frac{a_1''}{2}b_1'\right)\phi_1^3(0) \\ &+ a_1c_1\frac{f_n'''}{6}\phi_n^3(-1) + a_1'c_1\frac{f_n''}{2}\phi_1(0)\phi_n^2(-1) \\ &+ \frac{a_1''}{2}c_1f_n'\phi_1^2(0)\phi_n(-1) + \text{h.o.t.}, \\ F^{(i)}(\phi) = &-\left(a_i\frac{b_i''}{2} + a_i'b_i'\right)\phi_i^2(0) + a_ic_i\frac{f_{i-1}''}{2}\phi_{i-1}^2(0) \\ &+ a_i'c_if_{i-1}'\phi_{i-1}(0)\phi_i(0) \\ &-\left(a_i\frac{b_i'''}{6} + a_i'\frac{b_i''}{2} + \frac{a_i''}{2}b_i'\right)\phi_i^3(0) \\ &+ a_ic_i\frac{f_{i-1}'''}{6}\phi_{i-1}^3(0) + a_i'c_i\frac{f_{i-1}''}{2}\phi_i(0)\phi_{i-1}^2(0) \\ &+ \frac{a_i''}{2}c_if_{i-1}'\phi_{i-1}(0)\phi_i^2(0) + \text{h.o.t.} \end{aligned} \tag{22}$$

in which

$$\begin{gathered} a_i = a_i(0), \quad a_i' = a_i'(0), \quad a_i'' = a_i''(0), \\ b_i = b_i(0), \quad b_i' = b_i'(0), \quad b_i'' = b_i''(0), \\ b_i''' = b_i'''(0), \quad f_i = f_i(0), \quad f_i' = f_i'(0), \\ f_i'' = f_i''(0), \quad f_i''' = f_i'''(0), \\ i = 1, 2, \ldots, n. \end{gathered} \tag{23}$$

From the discussions in Section 2, we know that if $\mu = \gamma_1$, system (19) undergoes a Hopf bifurcation at the equilibrium $(0, 0, \ldots, 0)$, and the associated characteristic equation of system (19) has a pair simple imaginary roots $\pm i\omega_1$.

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta, \mu)$ for $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{24}$$

In fact, we can choose

$$\eta(\theta, \mu) = B_1\delta(\theta) - B_2\delta(\theta + 1), \tag{25}$$

where $\delta(\theta)$ is the Dirac delta function and $\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases}$

For $\phi \in C^1([-1, 0], R^n)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^0 d\eta(\mu, s)\phi(s), & \theta = 0, \end{cases} \tag{26}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{27}$$

The system (19) can be transformed into the following operator equation form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{28}$$

where $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$.

Denote

$$\begin{gathered} A_{\mu=b_1} = A_0, \qquad R_{\mu=\gamma_1} = R_0, \qquad F(\mu, \phi)_{\mu=\gamma_1} = F_0(\phi), \\ L_{\mu=\gamma_1} = L_0, \qquad \eta(\theta, 0) = \eta(\theta). \end{gathered} \tag{29}$$

For $\psi \in C^1([0, 1], (R^n)^*)$, define

$$A_0^*\psi(s) = \begin{cases} -\dfrac{d\psi(\theta)}{ds}, & s \in (0, 1], \\ \displaystyle\int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases} \tag{30}$$

For $\phi \in C^1([-1, 0], R^n)$ and $\psi \in C^1([0, 1], (R^n)^*)$, we define a bilinear form

$$\langle\psi(s), \phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^0\int_{\xi=0}^\theta \overline{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi. \tag{31}$$

Then A_0 and A_0^* are adjoint operators. We know that $\pm i\omega_1\tau$ are eigenvalues of A_0, so $\pm i\omega_1\tau$ are also eigenvalues of A_0^*.

Now we compute the eigenvectors of A_0 and A_0^* corresponding to $i\omega_1\tau$ and $-i\omega_1\tau$.

Suppose that $q(\theta) = q_0 e^{i\omega_1\tau\theta}$ is the eigenvector of A_0 corresponding to $i\omega_0\tau$; then $A_0 q(\theta) = i\omega_1\tau q(\theta)$. From (25) and (26), we know

$$q_0 = \begin{pmatrix} 1 \\ \dfrac{d_2}{\alpha_2 + i\omega_1} \\ \dfrac{d_2 d_3}{(\alpha_2 + i\omega_1)(\alpha_3 + i\omega_1)} \\ \vdots \\ \dfrac{d_2 d_3 \cdots d_{n-1}}{(\alpha_2 + i\omega_1)(\alpha_3 + i\omega_1)\cdots(\alpha_{n-1} + i\omega_{n-1})} \\ \dfrac{d_2 d_3 \cdots d_n}{(\alpha_2 + i\omega_1)(\alpha_3 + i\omega_1)\cdots(\alpha_n + i\omega_n)} \end{pmatrix}. \tag{32}$$

Similarly, we know that $q^*(s) = \overline{D} q_0^* e^{i\omega_1\tau s}$ with

$$q_0^* = \begin{pmatrix} 1 \\ \dfrac{\alpha_1 - i\omega_1}{d_2} \\ \dfrac{(\alpha_1 - i\omega_1)(\alpha_2 - i\omega_1)}{d_2 d_3} \\ \vdots \\ \dfrac{(\alpha_1 - i\omega_1)(\alpha_2 - i\omega_1)\cdots(\alpha_{n-2} - i\omega_1)}{d_2 d_3 \cdots d_{n-1}} \\ \dfrac{(\alpha_1 - i\omega_1)(\alpha_2 - i\omega_1)\cdots(\alpha_{n-1} - i\omega_1)}{d_2 d_3 \cdots d_n} \end{pmatrix} \tag{33}$$

and $s \in [0, 1)$ is the eigenvector of A_0^* corresponding to $-i\omega_1\tau$, where

$$\overline{D} = (\alpha_1 + i\omega)\left[\sum_{i=1}^{n} \frac{1}{\alpha_i + i\omega} + \tau\right]^{-1}. \tag{34}$$

Moreover, $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

Using the same notations as Hassard et al. [25], we construct the coordinates to describe the center manifold C_0 at $\mu = \gamma_1$.

Define

$$\begin{aligned} z(t) &= \langle q^*, u_t\rangle, \\ w(t,\theta) &= u_t - 2\operatorname{Re}\{z(t)\, q(\theta)\}. \end{aligned} \tag{35}$$

On the center manifold C_0, we have

$$w(t,\theta) = w(z,\overline{z},\theta), \tag{36}$$

where

$$w(z,\overline{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)\, z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots, \tag{37}$$

z and $\overline{z}$ are local coordinates for the center manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that w is real if u_t is real. We only consider real solutions.

For solution $u_t \in C_0$ of (19), since $\mu = b_1$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega_1\tau z + \left\langle \overline{q^*}(\theta), F_0\left(w(z,\overline{z},\theta) + 2\operatorname{Re}\{zq(\theta)\}\right)\right\rangle \\ &= i\omega_1\tau z + \overline{q^*}(0)\, F_0\left(w(z,\overline{z},0) + 2\operatorname{Re}\{zq(0)\}\right) \\ &\overset{\text{def}}{=} i\omega_1\tau z + \overline{q^*}(0)\, G_0(z,\overline{z}). \end{aligned} \tag{38}$$

We rewrite this as

$$\dot{z}(t) = i\omega_1\tau z(t) + g(z,\overline{z}) \tag{39}$$

with

$$\begin{aligned} g(z,\overline{z}) = \overline{q^*}(0)\, G_0(z,\overline{z}) &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} \\ &\quad + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \tag{40}$$

From (29), (36), and (39), we have

$$\begin{aligned} \dot{w} &= \dot{u}_t - \dot{z}q + \dot{\overline{z}}\overline{q} \\ &= \begin{cases} Aw - 2\operatorname{Re}\left\{\overline{q^*}(0)\, F_0 q(\theta)\right\}, & \theta \in [-1, 0), \\ Aw - 2\operatorname{Re}\left\{\overline{q^*}(0)\, F_0 q(0)\right\} + F_0, & \theta = 0, \end{cases} \\ &\overset{\text{def}}{=} Aw + H(z,\overline{z},\theta), \end{aligned} \tag{41}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)\, z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{42}$$

Since

$$\begin{aligned} g(z,\overline{z}) &= \overline{q^*}(0)\, G_0(z,\overline{z}) = \overline{D} q_0^* F_0(z,\overline{z}) \\ &= \overline{D}\tau q_0^* \left(F^{(1)}(u_t), F^{(2)}(u_t), \ldots, F^{(n)}(u_t)\right)^T, \end{aligned} \tag{43}$$

where

$$F^{(1)}(u_t) = -\left(a_1\frac{b_1''}{2} + a_1'b_1'\right)u_{1t}^2(0) + a_1c_1\frac{f_n''}{2}u_{nt}^2(-1) + a_1'c_1f_n'u_{1t}(0)u_{nt}(-1) - \left(a_1\frac{b_1'''}{6} + a_1'\frac{b_1''}{2} + \frac{a_1''}{2}b_1'\right)u_{1t}^3(0) + a_1c_1\frac{f_n'''}{6}u_{nt}^3(-1) + a_1'c_1\frac{f_n''}{2}u_{1t}(0)u_{nt}^2(-1) + \frac{a_1''}{2}c_1f_n'u_{1t}^2(0)u_{nt}(-1) + \text{h.o.t.},$$
$$F^{(i)}(u_t) = -\left(a_i\frac{b_i''}{2} + a_i'b_i'\right)u_{it}^2(0) + a_ic_i\frac{f_{i-1}''}{2}u_{(i-1)t}^2(0) + a_i'c_if_{i-1}'u_{(i-1)t}(0)u_{it}(0) - \left(a_i\frac{b_i'''}{6} + a_i'\frac{b_i''}{2} + \frac{a_i''}{2}b_i'\right)u_{it}^3(0) + a_ic_i\frac{f_{i-1}'''}{6}u_{(i-1)t}^3(0) + a_i'c_i\frac{f_{i-1}''}{2}u_{it}(0)u_{(i-1)t}^2(0) + \frac{a_i''}{2}c_if_{i-1}'u_{(i-1)t}(0)u_{it}^2(0) + \text{h.o.t.} \tag{44}$$

in which $u_t = 2\,\mathrm{Re}\{z(t)q(\theta)\} + w(t,\theta) = w(z,\overline{z},\theta) + zq(\theta) + \overline{zq}(\theta)$, $i = 2,3,\ldots,n$.

Denote the ith element of $q(0)$ by q_i and the ith element of $w(z,\overline{z},\theta)$ by

$$W^{(i)}(z,\overline{z},\theta) = W_{20}^{(i)}\frac{z^2}{2} + W_{11}^{(i)}z\overline{z} + \cdots. \tag{45}$$

Then if follows that

$$u_{it}(0) = W^{(i)}(z,\overline{z},0) + zq_i + \overline{zq}_i, \quad i = 1,2,\ldots,n-1,$$
$$u_{nt}(-1) = W^{(i)}(z,\overline{z},-1) + zq_ne^{-i\tau\omega_1} + \overline{zq}_ne^{i\tau\omega_1}. \tag{46}$$

Substitute (46) into (43) and comparing the coefficients in (40) with those in (43), we have

$$g_{20} = \overline{D}\tau\left[-\left(a_1b_1'' + 2a_1'b_1'\right) + a_1c_1f_n''q_n^2e^{-i2\omega_1\tau} + 2a_1'c_1f_n'q_ne^{-i\omega_1\tau}\right] + \overline{D}\sum_{i=2}^{n}\frac{(\alpha_1 - i\omega_1)\cdots(\alpha_{i-1} - i\omega_1)}{d_2\cdots d_i}\tau \times\left[-\left(a_ib_i'' + 2a_i'b_i'\right)q_i^2 + a_ic_if_{i-1}''q_{i-1}^2 + 2a_i'c_if_{i-1}'q_iq_{i-1}\right],$$

$$g_{11} = \overline{D}\tau\left[-\left(a_1b_1'' + 2a_1'b_1'\right) + a_1c_1f_n''q_n\overline{q}_n + a_1'c_1f_n'\left(q_ne^{-i\omega_1\tau} + \overline{q}_ne^{i\omega_1\tau}\right)\right] + \overline{D}\sum_{i=2}^{n}\frac{(\alpha_1 - i\omega_1)\cdots(\alpha_{i-1} - i\omega_1)}{d_2\cdots d_i}\tau \times\left[-\left(a_ib_i'' + 2a_i'b_i'\right)q_i\overline{q}_i + a_ic_if_{i-1}''q_{i-1}\overline{q}_{i-1} + a_i'c_if_{i-1}'\left(q_{i-1}\overline{q}_i + \overline{q}_{i-1}q_i\right)\right],$$

$$g_{02} = \overline{D}\tau\left[-\left(a_1b_1'' + 2a_1'b_1'\right) + a_1c_1f_n''\overline{q}^2e^{i2\omega\tau} + 2a_1'c_1f_2'\overline{q}_ne^{i\omega\tau}\right] + \overline{D}\sum_{i=2}^{n}\frac{(\alpha_1 - i\omega_1)\cdots(\alpha_{i-1} - i\omega_1)}{d_2\cdots d_i}\tau \times\left[-\left(a_ib_i'' + 2a_i'b_i'\right)\overline{q}_i^2 + a_ic_if_{i-1}''\overline{q}_{i-1}^2 + 2a_i'c_if_{i-1}'\overline{q}_{i-1}\overline{q}_i\right],$$

$$g_{21} = \overline{D}\tau\left[-\left(a_1b_1'' + 2a_1'b_1'\right)\left(w_{20}^{(1)}(0) + 2w_{11}^{(1)}(0)\right) + a_1c_1f_n'' \times\left(\overline{q}_ne^{i\omega_1\tau}w_{20}^{(n)}(-1) + 2q_ne^{-i\omega_1\tau}w_{11}^{(n)}(-1)\right) + a_1'c_1f_n'\left(\overline{q}_ne^{i\omega_1\tau}w_{20}^{(1)}(0) + 2q_ne^{-i\omega_1\tau}w_{11}^{(1)}(0) + w_{20}^{(n)}(-1) + 2w_{11}^{(n)}(-1)\right) - \left(a_1b_1''' + 3a_1'b_1'' + 3a_1''b_1'\right) + a_1c_1f_n'''\overline{q}_nq_n^2e^{-i\omega_1\tau} + a_1'd_1f_n''\left(q_n^2e^{-i2\omega_1\tau} + 2q_n\overline{q}_n\right) + a_1''c_1f_n'\left(\overline{q}_ne^{i\omega_1\tau} + 2q_ne^{-i\omega_1\tau}\right)\right] + \overline{D}\sum_{i=2}^{n}\frac{(\alpha_1 - i\omega_1)\cdots(\alpha_{i-1} - i\omega_1)}{d_2\cdots d_i}\tau \times\left[-\left(a_ib_i'' + 2a_i'b_i'\right)\left(\overline{q}_iw_{20}^{(i)}(0) + 2q_iw_{11}^{(i)}(0)\right)\right.$$

$$
\begin{aligned}
&+a_i c_i f_{i-1}''\left(w_{20}^{(i-1)}(0)\,q_{i-1}+2w_{11}^{(i-1)}(0)\,q_{i-1}\right)\\
&+a_i' c_i f_{i-1}'\left(w_{20}^{(i)}(0)\,\overline{q}_{i-1}+2w_{11}^{(i)}(0)\,q_{i-1}\right.\\
&\qquad\qquad\left.+w_{20}^{(i-1)}(0)\,\overline{q}_i+2w_{11}^{(i-1)}(0)\,q_i\right)\\
&+\left(a_i b_i'''+3a_i' b_i''+3a_i'' b_i'\right)q_i^2\overline{q}_i\\
&+a_i d_i f_{i-1}''' q_{i-1}^2\overline{q}_{i-1}\\
&+a_i' c_i f_{i-1}''\left(q_{i-1}^2\overline{q}_i+2q_{i-1}\overline{q}_{i-1}^2\right)\\
&+a_i'' c_i f_{i-1}'\left(q_i^2\overline{q}_i+2q_i q_{i-1}\overline{q}_i\right)\Big],
\end{aligned}
\tag{47}
$$

where q_i is ith element of $q(0)$ shown in (32).

Since there are $w_{20}(\theta)$ and $w_{11}(\theta)$ in g_{21}, we still need to figure them out. Note that on the center manifold C_0, we have

$$
\dot{w}=w_z\dot{z}+w_{\overline{z}}\dot{\overline{z}}. \tag{48}
$$

We have from (41), (42), and (48) that

$$
\left(A-2i\omega_1\tau\right)w_{20}(\theta)=-H_{20}(\theta),\qquad Aw_{11}(\theta)=-H_{11}(\theta). \tag{49}
$$

Equations (39) and (41) mean

$$
H(z,\overline{z},\theta)=-gq(\theta)-\overline{gq}(\theta) \tag{50}
$$

for $\theta\in[-1,0)$.

Comparing the coefficients with (42), we have

$$
\begin{aligned}
H_{20}(\theta)&=-g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta),\\
H_{11}(\theta)&=-g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta).
\end{aligned}
\tag{51}
$$

From (26), (49), and (50), we can obtain

$$
\dot{w}_{20}(\theta)=2i\omega_0\tau w_{20}(\theta)+g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta). \tag{52}
$$

So

$$
\begin{aligned}
w_{20}(\theta)=&\frac{i}{\omega_1\tau}g_{20}q(0)\,e^{i\omega_1\tau\theta}+\frac{i}{3\omega_1\tau}\overline{g}_{02}\overline{q}(0)\,e^{-i\omega_1\tau\theta}\\
&+E_1e^{2i\omega_1\tau\theta}
\end{aligned}
\tag{53}
$$

and similarly

$$
w_{11}(\theta)=-\frac{i}{\omega_1\tau}g_{11}q(0)\,e^{i\omega_1\tau\theta}+\frac{i}{\omega_1\tau}\overline{g}_{11}\overline{q}(0)\,e^{-i\omega_1\tau\theta}+E_2, \tag{54}
$$

where $E_1=(E_1^{(1)},E_1^{(2)},\ldots,E_1^{(n)})^T$, $E_2=(E_2^{(1)},E_2^{(2)},\ldots,E_2^{(n)})^T\in R^n$.

In the following, we focus on the computation of E_1 and E_2. From (26) and (49), we know

$$
\int_{-1}^{0}d\eta(\theta)\,w_{20}(\theta)=2i\omega_1\tau w_{20}(\theta)-H_{20}(0), \tag{55}
$$

$$
\int_{-1}^{0}d\eta(\theta)\,w_{11}(\theta)=-H_{11}(0) \tag{56}
$$

in which $\eta(\theta)=\eta(0,\theta)$.

We know from (41) and (43) that

$$
H_{20}(0)=-g_{20}q(0)-\overline{g}_{02}\overline{q}(0)+\tau\begin{pmatrix}h_1\\h_2\\\vdots\\h_n\end{pmatrix} \tag{57}
$$

in which

$$
\begin{aligned}
h_1=&-\left(a_1b_1''+2a_1'b_1'\right)+a_1c_1f_n''q_n^2e^{-i2\omega_1\tau}\\
&+2a_1'c_1f_n'q_ne^{-i\omega_1\tau},\\
h_i=&\frac{(\alpha_1-i\omega_1)\cdots(\alpha_{i-1}-i\omega_1)}{d_2\cdots d_i}\tau\\
&\times\left[-\left(a_ib_i''+2a_i'b_i'\right)q_i^2+a_ic_if_{i-1}''\right.\\
&\left.+2a_i'c_if_{i-1}'q_iq_{i-1}\right],\quad 2\le i\le n,
\end{aligned}
\tag{58}
$$

$$
H_{11}(0)=-g_{11}q(0)-\overline{g}_{11}\overline{q}(0)+\tau\begin{pmatrix}l_1\\l_2\\\vdots\\l_n\end{pmatrix}
$$

in which

$$
\begin{aligned}
l_1=&-\left(a_1b_1''+2a_1'b_1'\right)+a_1d_1f_n''q_n\overline{q}_n\\
&+a_1'c_1f_n'\left(q_ne^{-i\omega_1\tau}+\overline{q}_ne^{i\omega_1\tau}\right),\\
l_i=&\frac{(\alpha_1-i\omega_1)\cdots(\alpha_{i-1}-i\omega_1)}{d_2\cdots d_i}\tau\\
&\times\left[-\left(a_ib_i''+2a_i'b_i'\right)q_i\overline{q}_i+a_ic_if_{i-1}''q_{i-1}\overline{q}_{i-1}\right.\\
&\left.+a_i'd_if_{i-1}'\left(q_{i-1}\overline{q}_i+\overline{q}_{i-1}q_i\right)\right],\quad 2\le i\le n.
\end{aligned}
\tag{59}
$$

Substituting (53) and (57) into (55), we have

$$\begin{pmatrix} 2i\omega_1+\alpha_1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\beta_1 \\ -\beta_2 & 2i\omega_1+\alpha_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\beta_3 & 2i\omega_1+\alpha_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\beta_{n-1} & 2i\omega_1+\alpha_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\beta_n & 2i\omega_1+\alpha_n \end{pmatrix} E_1 = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}. \quad (60)$$

Solving this we can obtain $E_1 = (E_1^{(1)}, E_1^{(2)}, \ldots, E_1^{(n)})^T$.

Similarly, we can obtain $E_2 = (E_2^{(1)}, E_2^{(2)}, \ldots, E_2^{(n)})^T$ from

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\beta_1 \\ -\beta_2 & \alpha_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\beta_3 & \alpha_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\beta_{n-1} & \alpha_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\beta_n & \alpha_n \end{pmatrix} E_2 = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix}. \quad (61)$$

Based on the above analysis, we can see that each g_{ij} in (47) is determined by the parameters and delays for (1). Thus, we can compute the following quantities:

$$\begin{gathered} C_1(0) = \frac{i}{2\omega_1\tau}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\gamma_1)\}}, \\ \beta_2 = 2\operatorname{Re}\{C_1(0)\}, \\ T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\gamma_1)\}}{\tau\omega_1}. \end{gathered} \quad (62)$$

It is known that μ_2 determines the direction of the Hopf bifurcation and β_2 determines the stability of the bifurcating periodic solutions. Since $\operatorname{Re}\{\lambda'(\gamma_1)\} < 0$, we know if $\operatorname{Re}\{C_1(0)\} < 0$ ($\operatorname{Re}\{C_1(0)\} > 0$); then the Hopf Bifurcation is supercritical (subcritical), the bifurcating periodic solutions exist for $\gamma < \gamma_1$, and the bifurcating periodic solutions are stable (unstable). T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

Under some conditions, the equilibrium $(0, 0, \ldots, 0)$ of system (1) is globally asymptotically stable. The following result can be directly obtained from Corollary 2 in [5].

Theorem 3. *Under assumptions (H_1)–(H_3), the equilibrium $(0, 0, \ldots, 0)$ of system (1) is globally asymptotically stable if the following conditions hold.*

(H_3) *There exist constants $\underline{b}_i$ such that $b_i'(\cdot) \geq \underline{b}_i > 0$ for $i = 1, 2, \ldots, n$.*

(H_4) *There exist positive constants L_i such that $|f_i'(\cdot)| \leq L_i$ for $i = 1, 2, \ldots, n$.*

(H_5) *The following matrix A is an M-matrix:*

$$A = \begin{pmatrix} \underline{b}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & -|c_1| L_1 \\ -|c_2| L_2 & \underline{b}_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -|c_3| L_3 & \underline{b}_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -|c_{n-1}| L_{n-1} & \underline{b}_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -|c_n| L_n & \underline{b}_n \end{pmatrix}. \quad (63)$$

Note that the conditions in Theorem 3 have more restrictions than those in Theorem 2. Since A is an M-matrix, we have $|A| > 0$ [26]; that is, $\prod_{i=1}^{n} \underline{b}_i > \prod_{i=1}^{n} |c_i L_i|$; it yields $\prod_{i=1}^{n} a_i(0)\underline{b}_i > \prod_{i=1}^{n} |a_i(0) c_i L_i|$, which, together with

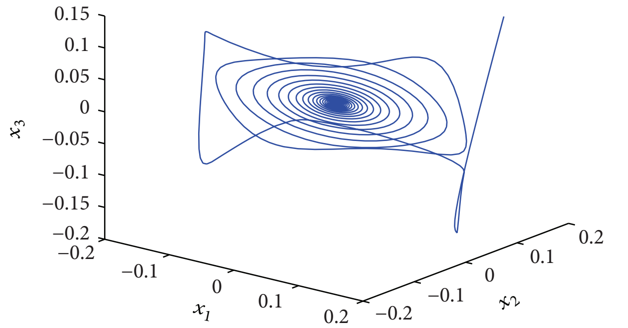

FIGURE 1: Phase plot in space (x_1, x_2, x_3) for system (64) with $c_1 = 0.8$.

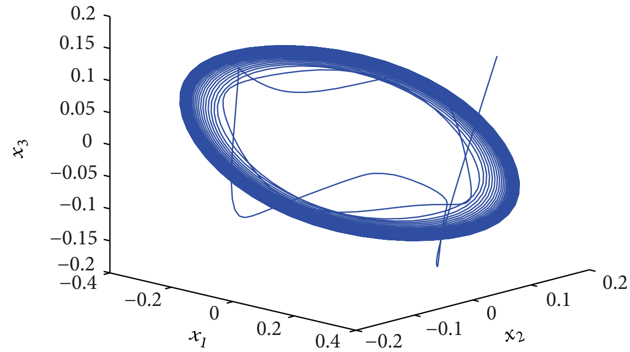

FIGURE 2: Phase plot in space (x_1, x_2, x_3) for system (64) with $c_1 = 0.9$.

conditions (H_3) and (H_4), implies that $|\gamma| \le \gamma_0$; moreover, $\gamma \in (-\gamma_1, \gamma_0)$ due to $|\gamma_1| > \gamma_0$. Hence, conditions (H_3)–(H_5) imply that the condition $\gamma \in (-\gamma_1, \gamma_0)$ in Theorem 2 holds.

4. A Numerical Example

Example 1. Consider the following Cohen-Grossberg neural network with discrete delays:

$$\begin{aligned} \dot{x}_1(t) &= -\left[0.6x_1(t) - c_1 \tanh\left(x_3(t-3)\right)\right], \\ \dot{x}_1(t) &= -\left(2+\cos\left(x_2(t)\right)\right)\left[x_2(t) - 0.8\tanh\left(x_1(t-2)\right)\right], \\ \dot{x}_1(t) &= -\left(2+\cos\left(x_3(t)\right)\right)\left[x_3(t) + \tanh\left(x_2(t-2)\right)\right]. \end{aligned} \tag{64}$$

We can obtain that $\omega_1 = 0.3423$ and furthermore we obtain that $\gamma_1 = -6.2979$ in view of bisection method by using MATLAB. It is easy to know $\gamma_0 = 5.4$. We also know from (3) that $\gamma = -7.2c_1$.

According to Theorem 2, the zero solution of system (64) is asymptotically stable when $\gamma \in (-6.2979, 5.4)$, and when $\gamma = \gamma_1$, the Hopf bifurcation occurs at the origin.

Case 1. Let $c_1 = 0.8$. $\gamma = -7.2 \times 0.8 = -5.66 \in (-6.2979, 5.4)$; then the zero solution of system (64) is asymptotically stable. Figure 1 shows the dynamic behaviors of system (64) with initial condition $(0.1, 0.2, 0.1)$.

Case 2. Let $c_1 = 0.9$, and $\gamma = -7.2 \times 0.9 = -6.48 < \gamma_1$. We know from Theorem 2 that the Hopf bifurcation occur at the origin; furthermore, we can obtain $\mathrm{Re}\{C_1(0)\} = -4.031$, so the bifurcating periodic solutions are supercritical and asymptotically stable. Figures 2 and 3 show the dynamic behaviors of system (64) with initial conditions $(0.1, 0.2, 0.1)$ and $(0.5, 1, 0.5)$, respectively.

The presented numerical simulations illustrate the theoretical results.

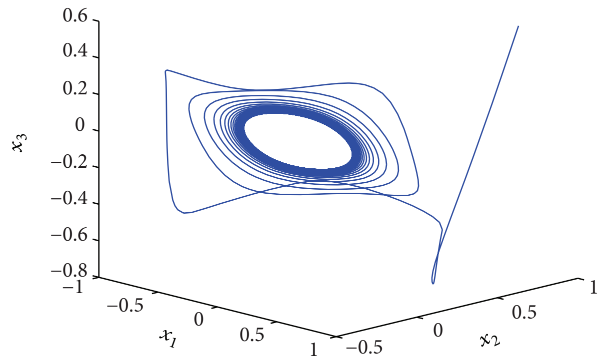

FIGURE 3: Phase plot in space (x_1, x_2, x_3) for system (64) with $c_1 = 0.9$.

5. Conclusions

An n-neuron Cohen-Grossberg neural network with discrete delays and ring architecture is analyzed in this paper. By using $\gamma = \prod_{l=1}^{n} \beta_i = \prod_{l=1}^{n} a_i(0) c_i f_i'(0)$ as a bifurcation parameter, we show that this system undergoes a Hopf bifurcations at a critical parameter:

$$\gamma_0 = \prod_{l=1}^{n} \alpha_l, \qquad \gamma_1 = -\sqrt{\prod_{l=1}^{n} \left(\alpha_l^2 + \omega_1^2\right)}, \tag{65}$$

where $\omega_1 \in (0, \pi/\tau)$ and satisfies the equation $\pi = \sum_{l=1}^{n} \cot^{-1}(\alpha_l/\omega) + \omega\tau$, $\tau = \tau_1 + \tau_2 + \cdots + \tau_n$. The direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are investigated by applying the normal form theory and the center manifold theorem for continuous time system. The phenomena of bifurcating periodic solutions for Cohen-Grossberg neural networks coincide with the fact that learning usually requires repetition [2], and periodic sequences of neural impulse are also of fundamental significance for the control of dynamic functions of the body such as heart beat which occurs with great regularity and breathing [19]. In this paper, we extend the results about the existence of local Hopf bifurcation in [22] to the case of a discrete-time n-neuron Cohen-Grossberg system with discrete delays. In the future, the problem for the existence of global Hopf bifurcation will be expected to be solved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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