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Research Article

Iterative Schemes for Finite Families of Maximal Monotone Operators Based on Resolvents

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The purpose of this paper is to present two iterative schemes based on the relative resolvent and the generalized resolvent, respectively. And, it is shown that the iterative schemes converge weakly to common solutions for two finite families of maximal monotone operators in a real smooth and uniformly convex Banach space and one example is demonstrated to explain that some assumptions in the main results are meaningful, which extend the corresponding works by some authors.

1. Introduction and Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use " $\rightarrow$ " and "$\rightharpoonup$" to denote strong and weak convergence either in E or E^*, respectively. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \qquad x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1. \tag{1}$$

Also, E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2}$$

A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\} := S(E)$. In this case, the norm of E is said to be Gâteaux differentiable. The space E is said to have a uniformly Gâteaux differentiable norm if, for each $y \in S(E)$, the limit (3) is attained uniformly for $x \in S(E)$. The norm of E is said to be Frêchet differentiable if, for each $x \in S(E)$, the limit (3) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Frêchet differentiable if the limit (3) is attained uniformly for $x, y \in S(E)$.

The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$Jx := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad x \in E. \tag{4}$$

We call that J is weakly sequentially continuous if $\{x_n\}$ is a sequence in E which converges weakly to x it follows that $\{Jx_n\}$ converges in *weak** to Jx.

We know the following properties of J (see [1] for details):

(i) $Jx \neq \emptyset$ for each $x \in E$;

(ii) if E is smooth, then J is single-valued and strictly monotone;

(iii) if E is strictly convex, then J is one to one; that is, $x \neq y \Rightarrow Jx \bigcap Jy = \emptyset$;

(iv) if E has a uniformly Gâteaux differentiable norm, then J is norm to *weak** uniformly continuous on each bounded subset of E;

(v) if E is a smooth and uniformly convex Banach space, then $J^{-1} : E^* \rightarrow E$ is also a duality mapping and is uniformly continuous on each bounded subset of E^*.

An operator $A \subset E \times E^*$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$, for $\forall y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set

$A^{-1}0$ is closed and convex: moreover, if E is a real smooth and uniformly convex Banach space, then $G(A)$ is demiclosed; that is, $\forall \{x_n\} \subset D(A)$, $x_n \rightharpoonup x$, $(n \to \infty)$, $\forall y_n \in Ax_n$, $y_n \to y$, $(n \to \infty) \Rightarrow x \in D(A)$, and $y \in Ax$. If E is reflexive and strictly convex, then a monotone operator A is maximal if and only if $R(J+\lambda A) = E^*$, for each $\lambda > 0$ (see [2] for more details).

A mapping $\widehat{A} : D(\widehat{A}) \subset E \to E$ is said to be accretive (c.f. [3]) if $\|x_1 - x_2\| \le \|x_1 - x_2 + r(y_1 - y_2)\|$, for $\forall x_i \in D(\widehat{A})$, $y_i \in \widehat{A}x_i$, $i = 1, 2$, and $r > 0$. In a Hilbert space H, the m-accretive mapping is exactly the maximal monotone operator.

The Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows:

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{5}$$

It is obvious from the definition of Lyapunov functional that

$$(\|x\| - \|y\|)^2 \le \varphi(x, y) \le (\|x\| + \|y\|)^2, \tag{6}$$

for each $x, y \in E$.

We have the following well-known result.

Lemma 1 (see [4]). *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, $n \to \infty$, then $x_n - y_n \to 0$, $n \to \infty$.*

Definition 2 (see [5]). Let E be a real smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Then $\forall r > 0$, define $Q_r^A : E \to E$ by $Q_r^A x = (J + rA)^{-1}Jx$, which is called the relative resolvent.

We have the following property of the relative resolvent.

Lemma 3 (see [5]). *Let E be a real reflexive, strictly convex, and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Then $\forall x \in E$, $y \in A^{-1}0$, and $r > 0$, we have $\varphi(y, Q_r^A x) + \varphi(Q_r^A x, x) \le \varphi(y, x)$.*

Definition 4 (see [4]). Let E be a real reflexive, strictly convex, and smooth Banach space and let C be a nonempty closed and convex subset of E. Then $\forall x \in E$, there exists a unique element $x_0 \in C$ satisfying $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$. In this case, $\forall x \in E$, define $\Pi_C : E \to C$ by $\Pi_C x = x_0$, and then Π_C is called the generalized projection from E onto C.

Lemma 5 (see [4]). *Let E be a real reflexive, strictly convex, and smooth Banach space and let C be a nonempty closed and convex subset of E. Then $\forall x \in E$, $\forall y \in C$,*

$$\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \le \varphi(y, x). \tag{7}$$

Lemma 6 (see [4]). *Let E be a real smooth Banach space and let C be a nonempty closed and convex subset of E. Let $x \in E$, and $x_0 \in C$. Then $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx\rangle \ge 0, \forall z \in C$.*

Let E be a smooth Banach space and let C be a nonempty closed and convex subset of E. A mapping $T : C \to C$ is said to be generalized nonexpansive (c.f. [5]) if $F(T) \neq \emptyset$ and $\varphi(Tx, y) \le \varphi(x, y)$, for $\forall x \in C$ and $y \in F(T)$, where $F(T)$ is a set of the fixed points of T; that is, $F(T) := \{x \in C : Tx = x\}$.

Let C be a nonempty, closed subset of E and let Q be a mapping of E onto C. Then Q is said to be sunny (c.f. [5]) if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \ge 0$. A mapping $Q : E \to C$ is said to be a retraction (c.f. [5]) if $Q(z) = z$ for every $z \in C$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely decided (c.f. [5]). Then, if E is smooth and strictly convex, a sunny generalized nonexpansive retraction of E onto C is denoted by R_C.

A subset C of E is said to be a sunny nonexpansive retract of E (c.f. [5]) if there exists a sunny nonexpansive retraction of E onto C and it is called a generalized nonexpansive retract of E if there exists a generalized nonexpansive retraction of E onto C.

Definition 7 (see [5]). Let E be a real reflexive, strictly convex, and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. Then $\forall r > 0$, define $R_r^B : E \to E$ by $R_r^B x = (I + rBJ)^{-1}x$, which is called the generalized resolvent.

Lemma 8 (see [5]). *Let E be a real reflexive and strictly Banach space with a Frêchet differential norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then* (i) $(BJ)^{-1}0 = F(R_r^B)$; (ii) $(BJ)^{-1}0$ *is closed;* (iii) R_r^B *is generalized nonexpansive, for $r > 0$.*

Lemma 9 (see [5]). *Let E be a real reflexive, smooth, and strictly Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then*

$$\begin{gathered} \varphi(x, R_r^B x) + \varphi(R_r^B x, u) \le \varphi(x, u), \\ \forall r > 0, \quad u \in (BJ)^{-1}0, \quad x \in E. \end{gathered} \tag{8}$$

Lemma 10 (see [6]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers and $a_{n+1} \le a_n + b_n$ for $\forall n \ge 1$. If $\sum_{n=0}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.*

Finding zeros of maximal monotone operators is a hot topic in applied mathematics since it has practical background. One classical method for studying the problem $0 \in Ax$, where A is a maximal monotone operator, is the following so-called proximal method (c.f. [7]), presented in a Hilbert space:

$$x_0 \in H, \quad x_{n+1} \approx J_{r_n}^A x_n, \quad n \ge 0, \tag{9}$$

where $J_{r_n}^A := (I + r_n A)^{-1}$. It was shown that the sequence generated by (9) converges weakly to a point in $A^{-1}0$ under some conditions.

In 2004, Kamimura et al. extended the study on zeros of maximal monotone operators to the following iterative scheme based on the relative resolvent $Q^A_{r_n}$ in Banach spaces (c.f. [8]):

$$x_1 \in E,$$
$$x_{n+1} = J^{-1}\left[\alpha_n J x_n + (1-\alpha_n) J Q^A_{r_n} x_n\right]. \tag{10}$$

And, they showed that $\{x_n\}$ generated by (10) converges weakly to a point in $A^{-1}0$, where $A \subset E \times E^*$ is a maximal monotone operator.

In 2007, Ibaraki and Takahashi [9] studied the following iterative scheme based on the generalized resolvent $R^B_{r_n}$ in Banach spaces:

$$x_1 \in E,$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) R^B_{r_n} x_n. \tag{11}$$

And, they showed that $\{x_n\}$ generated by (11) converges weakly to a point in $(BJ)^{-1}0$, where $B \subset E^* \times E$ is a maximal monotone operator.

In 2010, Shehu and Ezeora, [10] presented the following iterative scheme for a family of m-accretive mappings $\{A_i\}_{i=1}^N$ in a real uniformly smooth and uniformly convex Banach space E:

$$x_1 \in E,$$
$$y_n = (1-\alpha_n) x_n, \tag{12}$$
$$x_{n+1} = (1-\beta_n) x_n + \beta_n S_N y_n, \quad n \geq 1,$$

where $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 1, 2, \ldots, N$. $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$, and $\sum_{k=0}^N a_k = 1$. Then $\{x_n\}$ converges strongly to the common point in $A_i^{-1}0$, where $i = 1, 2, \ldots, N$.

Can we extend the study on m-accretive mappings [10] to maximal monotone operators? Inspired by the work on (10)–(12), in Section 2, we will present the following iterative scheme based on the relative resolvent:

$$x_1 \in E,$$
$$u_n = J^{-1}\left[(1-\alpha_n) J x_n\right],$$
$$v_n = J^{-1}\left[(1-\beta_n) J x_n + \beta_n J U_n u_n\right], \tag{A}$$
$$x_{n+1} = J^{-1}\left[\gamma_n J x_n + (1-\gamma_n) J W_n v_n\right], \quad n \geq 1,$$

where $A_i, B_j \subset E \times E^*$ are maximal monotone operators, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Suppose $(\bigcap_{i=1}^N A_i^{-1}0) \bigcap (\bigcap_{j=1}^M B_j^{-1}0) \neq \emptyset$. $U_n := J^{-1}[a_0 J + a_1 J Q^{A_1}_{r_{n,1}} + a_2 J Q^{A_2}_{r_{n,2}} + \cdots + a_N J Q^{A_N}_{r_{n,N}}]$, and $Q^{A_i}_{r_{n,i}} = (J + r_{n,i} A_i)^{-1} J$ and $r_{n,i} > 0$, for $i = 1, 2, \ldots, N$ and $n \geq 1$. $W_n := J^{-1}[b_0 J + b_1 J Q^{B_1}_{s_{n,1}} + b_2 J Q^{B_2}_{s_{n,2}} Q^{B_1}_{s_{n,1}} + \cdots + b_M J Q^{B_M}_{s_{n,M}} Q^{B_{M-1}}_{s_{n,M-1}} \cdots Q^{B_1}_{s_{n,1}}]$, and $Q^{B_j}_{s_{n,j}} = (J + s_{n,j} B_j)^{-1} J$ and $s_{n,j} > 0$, for $j = 1, 2, \ldots, M$ and $n \geq 1$. $a_0, a_1, \ldots, a_N; b_0, b_1, \ldots, b_M$ are real numbers in $(0, 1)$ with $\sum_{i=0}^N a_i = 1$ and $\sum_{j=0}^M b_j = 1$.

In Section 3, we will study the following iterative scheme based on generalized resolvent:

$$x_1 \in E,$$
$$u_n = (1-\alpha_n) x_n,$$
$$v_n = (1-\beta_n) x_n + \beta_n S_n u_n, \tag{B}$$
$$x_{n+1} = \gamma_n x_n + (1-\gamma_n) T_n v_n, \quad n \geq 1,$$

where $A_i, B_j \subset E^* \times E$ are maximal monotone operators, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Suppose $(\bigcap_{i=1}^N A_i^{-1}0) \bigcap (\bigcap_{j=1}^M B_j^{-1}0) \neq \emptyset$. $S_n := a_0 I + a_1 R^{A_1}_{r_{n,1}} + a_2 R^{A_2}_{r_{n,2}} + \cdots + a_N R^{A_N}_{r_{n,N}}$, and $T_n := b_0 I + b_1 R^{B_1}_{s_{n,1}} + b_2 R^{B_2}_{s_{n,2}} R^{B_1}_{s_{n,1}} + \cdots + b_M R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}}$. For $i = 1, 2, \ldots, N$, $R^{A_i}_{r_{n,i}} = (I + r_{n,i} A_i J)^{-1}$. For $j = 1, 2, \ldots, M$, $R^{B_j}_{s_{n,j}} = (I + s_{n,j} B_j J)^{-1}$. $a_0, a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b_M$ are real numbers in $(0, 1)$ and $\sum_{i=0}^N a_i = 1, \sum_{j=0}^M b_j = 1$. $r_{n,i} > 0$, for $i = 1, 2, \ldots, N$, and $s_{n,j} > 0$, for $j = 1, 2, \ldots, M$ and $n \geq 1$.

In this paper, some weak convergence theorems are obtained, which can be regarded as the extension and complement of the work done in [7–10], and so forth. At the end of Section 3, one example is demonstrated to show that the assumption that $(\bigcap_{i=1}^N A_i^{-1}0) \bigcap (\bigcap_{j=1}^M B_j^{-1}0) \neq \emptyset$ in the discussions of (A) and (B) is meaningful.

2. Weak Convergence Theorems Based on the Relative Resolvent

Theorem 11. *Let E be a real smooth and uniformly convex Banach space. Let $A_i, B_j \subset E \times E^*$ be maximal monotone operators, where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Suppose that both $J : E \to E^*$ and $J^{-1} : E^* \to E$ are weakly sequentially continuous and $D := (\bigcap_{i=1}^N A_i^{-1}0) \bigcap (\bigcap_{j=1}^M B_j^{-1}0) \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative scheme* (A), *where $U_n := J^{-1}[a_0 J + a_1 J Q^{A_1}_{r_{n,1}} + a_2 J Q^{A_2}_{r_{n,2}} + \cdots + a_N J Q^{A_N}_{r_{n,N}}]$, and $Q^{A_i}_{r_{n,i}} = (J + r_{n,i} A_i)^{-1} J$, for $i = 1, 2, \ldots, N$, $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$, $\sum_{k=0}^N a_k = 1$. $W_n := J^{-1}[b_0 J + b_1 J Q^{B_1}_{s_{n,1}} + b_2 J Q^{B_2}_{s_{n,2}} Q^{B_1}_{s_{n,1}} + \cdots + b_M J Q^{B_M}_{s_{n,M}} Q^{B_{M-1}}_{s_{n,M-1}} \cdots Q^{B_1}_{s_{n,1}}]$, where $Q^{B_j}_{s,j} = (J + s_{n,j} B_j)^{-1} J$, for $j = 1, 2, \ldots, M$, $0 < b_k < 1$, for $k = 0, 1, 2, \ldots, M$, $\sum_{k=0}^M b_k = 1$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{r_{n,i}\}$, $\{s_{n,j}\} \subset (0, +\infty)$ satisfying the following conditions:*

(i) *$\alpha_n \to 0$, as $n \to \infty$;*

(ii) *$\sum_{n=1}^\infty \alpha_n \beta_n (1-\gamma_n) < +\infty$;*

(iii) *$\liminf_{n\to+\infty} \beta_n > 0$ and $\limsup_{n\to+\infty} \gamma_n < 1$;*

(iv) *$\liminf_{n\to\infty} r_{n,i} > 0$ and $\liminf_{n\to\infty} s_{n,j} > 0$, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$.*

Then $\{x_n\}$ converges weakly to the unique element $v_0 \in D$ which satisfies

$$v_0 = \lim_{n\to\infty} \Pi_D x_n. \tag{13}$$

Proof. We will split the proof into six steps.

Step 1. $\{x_n\}$ is bounded.

For $\forall p \in D$, noticing the definition of the Lyapunov functional and by using Lemma 3 repeatedly, we have

$$\begin{aligned}
&\varphi(p, x_{n+1}) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)\varphi(p, W_n v_n) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n) \\
&\qquad \times \left[b_0 \varphi(p, v_n) + b_1 \varphi\left(p, Q_{s_{n,1}}^{B_1} v_n\right)\right. \\
&\qquad\qquad + b_2 \varphi\left(p, Q_{s_{n,2}}^{B_2} Q_{s_{n,1}}^{B_1} v_n\right) \\
&\qquad\qquad \left. + \cdots + b_M \varphi\left(p, Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n\right)\right] \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)\varphi(p, v_n) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n) \\
&\qquad \times \left[(1-\beta_n)\varphi(p, x_n) + \beta_n \varphi(p, U_n u_n)\right] \\
&\quad \le \left[1 - \beta_n(1-\gamma_n)\right]\varphi(p, x_n) \\
&\qquad + (1-\gamma_n)\beta_n \left[a_0 \varphi(p, u_n) + \sum_{i=1}^{N} a_i \varphi\left(p, Q_{r_{n,i}}^{A_i} u_n\right)\right] \\
&\quad \le \left[1 - \beta_n(1-\gamma_n)\right]\varphi(p, x_n) + (1-\gamma_n)\beta_n \varphi(p, u_n) \\
&\quad \le \left[1 - \alpha_n\beta_n(1-\gamma_n)\right]\varphi(p, x_n) + (1-\gamma_n)\alpha_n\beta_n \|p\|^2.
\end{aligned} \tag{14}$$

Lemma 10 ensures that $\lim_{n\to\infty}\varphi(p, x_n)$ exists, which implies that $\{x_n\}$ is bounded in view of (6).

Then from iterative scheme (A), $\{u_n\}$ is bounded. Since $\varphi(p, U_n u_n) \le \varphi(p, u_n)$, for $\forall p \in D$, then $\{U_n u_n\}$ is bounded, which ensures that $\{v_n\}$ is bounded. For $\forall p \in D$, $\varphi(p, Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,1}}^{B_1} v_n) \le \varphi(p, v_n)$, then we know that $\{Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,1}}^{B_1} v_n\}$ is bounded.

Step 2. $\omega(x_n) \subset D$, where $\omega(x_n)$ is the set of the weak limit points of all of the weakly convergent subsequences of $\{x_n\}$.

Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. And, there exists a subsequence of $\{x_n\}$; for simplicity, we still denote it by $\{x_n\}$ such that $x_n \rightharpoonup x$, $n \to \infty$.

For $\forall p \in D$, using Lemma 3 again, we have the following:

$$\begin{aligned}
&\varphi(p, x_{n+1}) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)\varphi(p, W_n v_n) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n) \\
&\qquad \times \left[(b_0 + b_1 + \cdots + b_{M-1})\varphi(p, v_n)\right. \\
&\qquad\qquad \left. + b_M \varphi\left(p, Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n\right)\right] \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)(1-b_M)\varphi(p, v_n) \\
&\qquad + b_M(1-\gamma_n)\varphi\left(p, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right) \\
&\qquad - b_M(1-\gamma_n) \\
&\qquad \times \varphi\left(Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n) \\
&\qquad \times \left[(1-\beta_n)\varphi(p, x_n) + \beta_n \varphi(p, U_n u_n)\right] - b_M(1-\gamma_n) \\
&\qquad \times \varphi\left(Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right) \\
&\quad \le \left[1 - \alpha_n\beta_n(1-\gamma_n)\right]\varphi(p, x_n) + (1-\gamma_n)\alpha_n\beta_n \|p\|^2 \\
&\qquad - b_M(1-\gamma_n) \\
&\qquad \times \varphi\left(Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right).
\end{aligned} \tag{15}$$

Then (15) implies that

$$\begin{aligned}
&b_M(1-\gamma_n)\varphi\left(Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right) \\
&\quad \le \left[1 - \alpha_n\beta_n(1-\gamma_n)\right]\varphi(p, x_n) \\
&\qquad + (1-\gamma_n)\alpha_n\beta_n \|p\|^2 - \varphi(p, x_{n+1}).
\end{aligned} \tag{16}$$

Since $\lim_{n\to\infty}\varphi(p, x_n)$ exists and $\{Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,1}}^{B_1} v_n\}$ is bounded, then, using Lemma 1, we know that

$$Q_{s_{n,M}}^{B_M} Q_{s_{n,M-1}}^{B_{M-1}} \cdots Q_{s_{n,1}}^{B_1} v_n - Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n \longrightarrow 0, \tag{17}$$

as $n \to \infty$. Revise (14) in the following way:

$$\begin{aligned}
&\varphi(p, x_{n+1}) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)\varphi(p, W_n v_n) \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n) \\
&\qquad \times \left[(b_0 + b_1 + \cdots + b_{M-2} + b_M)\varphi(p, v_n)\right. \\
&\qquad\qquad \left. + b_{M-1}\varphi\left(p, Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n\right)\right] \\
&\quad \le \gamma_n \varphi(p, x_n) + (1-\gamma_n)(1-b_{M-1})\varphi(p, v_n) \\
&\qquad + b_{M-1}(1-\gamma_n)\varphi\left(p, Q_{s_{n,M-2}}^{B_{M-2}} Q_{s_{n,M-3}}^{B_{M-3}} \cdots Q_{s_{n,1}}^{B_1} v_n\right) \\
&\qquad - b_{M-1}(1-\gamma_n) \\
&\qquad \times \varphi\left(Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n, Q_{s_{n,M-2}}^{B_{M-2}} Q_{s_{n,M-3}}^{B_{M-3}} \cdots Q_{s_{n,1}}^{B_1} v_n\right).
\end{aligned} \tag{18}$$

Then repeating the above process, we have

$$Q_{s_{n,M-1}}^{B_{M-1}} Q_{s_{n,M-2}}^{B_{M-2}} \cdots Q_{s_{n,1}}^{B_1} v_n - Q_{s_{n,M-2}}^{B_{M-2}} Q_{s_{n,M-3}}^{B_{M-3}} \cdots Q_{s_{n,1}}^{B_1} v_n \longrightarrow 0, \tag{19}$$

as $n \to \infty$. Similarly, we have

$$Q^{B_{M-2}}_{s_{n,M-2}} Q^{B_{M-3}}_{s_{n,M-3}} \cdots Q^{B_1}_{s_{n,1}} v_n - Q^{B_{M-3}}_{s_{n,M-3}} Q^{B_{M-4}}_{s_{n,M-4}} \cdots Q^{B_1}_{s_{n,1}} v_n \longrightarrow 0,$$
$$Q^{B_{M-3}}_{s_{n,M-3}} Q^{B_{M-4}}_{s_{n,M-4}} \cdots Q^{B_1}_{s_{n,1}} v_n - Q^{B_{M-4}}_{s_{n,M-4}} Q^{B_{M-5}}_{s_{n,M-5}} \cdots Q^{B_1}_{s_{n,1}} v_n \longrightarrow 0,$$
$$\vdots$$
$$Q^{B_1}_{s_{n,1}} v_n - v_n \longrightarrow 0. \tag{20}$$

On the other hand, noticing (14) and using Lemma 3, we have

$$\begin{aligned}
&\varphi(p, x_{n+1}) \\
&\le [1 - \beta_n(1-\gamma_n)]\varphi(p, x_n) \\
&\quad + (1-\gamma_n)\beta_n \left[a_0\varphi(p, u_n) + \sum_{i=1}^{N} a_i \varphi\left(p, Q^{A_i}_{r_{n,i}} u_n\right)\right] \\
&\le [1 - \beta_n(1-\gamma_n)]\varphi(p, x_n) + (1-\gamma_n) \\
&\quad \times \beta_n \left\{a_0\varphi(p, u_n) + \sum_{i=1}^{N} a_i \left[\varphi(p, u_n) - \varphi\left(Q^{A_i}_{r_{n,i}} u_n, u_n\right)\right]\right\} \\
&\le [1 - \alpha_n\beta_n(1-\gamma_n)]\varphi(p, x_n) + (1-\gamma_n)\alpha_n\beta_n\|p\|^2 \\
&\quad - (1-\gamma_n)\beta_n \sum_{i=1}^{N} a_i \varphi\left(Q^{A_i}_{r_{n,i}} u_n, u_n\right).
\end{aligned} \tag{21}$$

Similar to the discussion of (20), we have

$$Q^{A_i}_{r_{n,i}} u_n - u_n \longrightarrow 0, \tag{22}$$

as $n \to \infty$, $i = 1, 2, \ldots, N$.

Since both J and J^{-1} are weakly sequentially continuous, $Ju_n = (1-\alpha_n)Jx_n$, and $x_n \rightharpoonup x$, then $u_n \rightharpoonup x$, as $n \to \infty$. Now, (22) implies that $Q^{A_i}_{r_{n,i}} u_n \rightharpoonup x$. If we set $z_{n,i} = Q^{A_i}_{r_{n,i}} u_n$, then from (22) and the fact that J is uniformly norm to norm continuous on each bounded subset of E, we have $A_i z_{n,i} = (Ju_n - Jz_{n,i})/r_{n,i} \to 0$, as $n \to \infty$, for $i = 1, 2, \ldots, N$. Since $G(A_i)$ is demiclosed, then $x \in \bigcap_{i=1}^{N} A_i^{-1}0$.

Now, from $Q^{A_i}_{r_{n,i}} u_n \rightharpoonup x$, we have $U_n u_n \rightharpoonup x$, and then $JU_n u_n \rightharpoonup Jx$, which implies that $v_n \rightharpoonup x$, as $n \to \infty$, since $Jv_n = (1-\beta_n)Jx_n + \beta_n JU_n u_n$. Thus (20) implies that $Q^{B_1}_{s_{n,1}} v_n \rightharpoonup x$. In the same way as the proof of $x \in A_i^{-1}0$, we have $x \in B_1^{-1}0$.

From the fact that $Q^{B_1}_{s_{n,1}} v_n \rightharpoonup x$ and (20), we have $Q^{B_2}_{s_{n,2}} Q^{B_1}_{s_{n,1}} v_n \rightharpoonup x$, and $Q^{B_2}_{s_{n,2}} Q^{B_1}_{s_{n,1}} v_n - Q^{B_1}_{s_{n,1}} v_n \to 0$, as $n \to \infty$. Then, if we set $w_{n,2} = Q^{B_2}_{s_{n,2}} Q^{B_1}_{s_{n,1}} v_n = Q^{B_2}_{s_{n,2}} w_{n,1}$, we have $B_2 w_{n,2} = (Jw_{n,1} - Jw_{n,2})/s_{n,2} \to 0$, which ensures that $x \in B_2^{-1}0$. By induction, using (20) repeatedly, we know that $x \in \bigcap_{j=1}^{M} B_j^{-1}0$.

Therefore, $x \in D$, and then $\omega(x_n) \subset D$.

Step 3. There exists a unique element $v_0 \in D$ such that

$$\lim_{n\to\infty} \varphi(v_0, x_n) = \min_{y \in D} \lim_{n\to\infty} \varphi(y, x_n). \tag{23}$$

In fact, let $h(y) = \lim_{n\to\infty}\varphi(y, x_n)$, $\forall y \in D$. Then $h : D \to R^+$ is proper, convex, and lower-semicontinuous and $h(y) \to +\infty$, as $\|y\| \to +\infty$. Thus there exists $v_0 \in D$ such that $h(v_0) = \min_{y\in D} h(y)$. Since h is strictly convex, then v_0 is unique.

Step 4. $\lim_{n\to\infty}\varphi(\Pi_D x_n, x_n)$ exists.

From the definition of Π_D, we have $\varphi(\Pi_D x_{n+1}, x_{n+1}) \le \varphi(\Pi_D x_n, x_{n+1})$.

Using (14), we have

$$\begin{aligned}
\varphi(\Pi_D x_n, x_{n+1}) &\le [1 - \alpha_n\beta_n(1-\gamma_n)]\varphi(\Pi_D x_n, x_n) \\
&\quad + (1-\gamma_n)\alpha_n\beta_n\|p\|^2 \\
&\le \varphi(\Pi_D x_n, x_n) + (1-\gamma_n)\alpha_n\beta_n\|p\|^2.
\end{aligned} \tag{24}$$

Thus

$$\varphi(\Pi_D x_{n+1}, x_{n+1}) \le \varphi(\Pi_D x_n, x_n) + (1-\gamma_n)\alpha_n\beta_n\|p\|^2. \tag{25}$$

Then Lemma 10 ensures that $\lim_{n\to\infty}\varphi(\Pi_D x_n, x_n)$ exists.

Step 5. $\lim_{n\to\infty}\Pi_D x_n = v_0$, where v_0 is the same as that in Step 3.

From Lemma 5, we have $\varphi(v_0, \Pi_D x_n) \le \varphi(v_0, x_n) - \varphi(\Pi_D x_n, x_n)$. Thus

$$\begin{aligned}
\limsup_{n\to\infty} \varphi(v_0, \Pi_D x_n) &\le \lim_{n\to\infty}\varphi(v_0, x_n) \\
&\quad - \lim_{n\to\infty}\varphi(\Pi_D x_n, x_n) \\
&= h(v_0) - \lim_{n\to\infty}\varphi(\Pi_D x_n, x_n) \le 0
\end{aligned} \tag{26}$$

Therefore, Lemma 1 implies that $\Pi_D x_n \to v_0$, as $n \to \infty$.

Step 6. $x_n \rightharpoonup v_0$ where v_0 is the same as that in Step 3.

From Lemma 6, we know that,

$$\forall y \in D, \langle \Pi_D x_n - y, J\Pi_D x_n - Jx_n \rangle \le 0. \tag{27}$$

Since J is weakly sequentially continuous, then from Step 5, we have $J\Pi_D x_n \rightharpoonup Jv_0$, as $n \to \infty$.

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x_0$, as $j \to \infty$. From Step 2, $x_0 \in D$. And, $Jx_{n_j} \rightharpoonup Jx_0$, as $j \to \infty$. Substituting $\{x_n\}$ by $\{x_{n_j}\}$ in (27) and taking limits on both sides, we have

$$\forall y \in D, \qquad \langle v_0 - y, Jv_0 - Jx_0 \rangle \le 0. \tag{28}$$

Letting $y = x_0$ in (28), then $\langle v_0 - x_0, Jv_0 - Jx_0 \rangle \le 0$, which implies that $x_0 = v_0$, since J is strictly monotone.

Suppose there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup x_1$, as $l \to \infty$. Then $x_1 \in D$ and $Jx_{n_l} \rightharpoonup Jx_1$, as $l \rightarrow +\infty$. Repeating the above process, we know that $x_1 = v_0$. Therefore, all of the weakly convergent subsequences of $\{x_n\}$ converge weakly to the same element v_0, and then $x_n \rightharpoonup v_0$ which satisfies (13), as $n \rightarrow \infty$.

This completes the proof. □

If, in Theorem 11, the Banach space E reduces to the Hilbert space H, then we have the following theorem.

Theorem 12. *Let H be a Hilbert space and let D be the same as that in Theorem 11. Let $A_i, B_j \subset H \times H$ be m-accretive mappings, where $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, M$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the same as those in Theorem 11. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{r_{n,i}\}, \{s_{n,j}\} \subset (0, +\infty)$ satisfy some conditions presented in Theorem 11.*

Let $\{x_n\}$ be generated by the following scheme:

$$\begin{aligned} & x_1 \in H, \\ & u_n = (1 - \alpha_n) x_n, \\ & v_n = (1 - \beta_n) x_n + \beta_n U_n u_n, \\ & x_{n+1} = \gamma_n x_n + (1 - \gamma_n) W_n v_n, \quad n \geq 1, \end{aligned} \tag{C}$$

where $U_n = a_0 I + a_1 (I + r_{n,1} A_1)^{-1} + \cdots + a_N (I + r_{n,N} A_N)^{-1}$ and $W_n = b_0 I + b_1 (I + s_{n,1} B_1)^{-1} + b_2 (I + s_{n,2} B_2)^{-1} (I + s_{n,1} B_1)^{-1} + \cdots + b_M (I + s_{n,M} B_M)^{-1} (I + s_{n,M-1} B_{M-1})^{-1} \cdots (I + s_{n,1} B_1)^{-1}$. Then $\{x_n\}$ converges weakly to the unique element $p_0 \in D$, where $p_0 = \lim_{n \to \infty} P_D x_n$ and P_D is the metric projection from H onto D.

Remark 13. Compared to the work in [10], we may find that Theorem 11 is not a simple extension from the case of m-accretive mappings to maximal monotone operators. In (A), different A_i and B_j have different coefficients while in (12), different A_i have the same coefficients.

3. Weak Convergence Theorems Based on the Generalized Resolvent

Theorem 14. *Let E be a real smooth and uniformly convex Banach space. Let $A_i, B_j \subset E^* \times E$ be maximal monotone operators, where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Suppose that both $J : E \rightarrow E^*$ and $J^{-1} : E^* \rightarrow E$ are weakly sequentially continuous and $\widetilde{\widetilde{D}} := (\bigcap_{i=1}^{N} A_i^{-1} 0) \bigcap (\bigcap_{j=1}^{M} B_j^{-1} 0) \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative scheme (B), where $S_n := a_0 I + a_1 R_{r_{n,1}}^{A_1} + a_2 R_{r_{n,2}}^{A_2} + \cdots + a_N R_{r_{n,N}}^{A_N}$, and $R_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i J)^{-1}$, for $i = 1, 2, \ldots, N$, $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$, $\sum_{k=0}^{N} a_k = 1$. $W_n := b_0 I + b_1 R_{s_{n,1}}^{B_1} + b_2 R_{s_{n,2}}^{B_2} R_{s_{n,1}}^{B_1} + \cdots + b_M R_{s_{n,M}}^{B_M} R_{s_{n,M-1}}^{B_{M-1}} \cdots R_{s_{n,1}}^{B_1}$, where $R_{s,j}^{B_j} = (I + s_{n,j} B_j J)^{-1}$, for $j = 1, 2, \ldots, M$, $0 < b_k < 1$, for $k = 0, 1, 2, \ldots, M$, $\sum_{k=0}^{M} b_k = 1$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{r_{n,i}\}$, $\{s_{n,j}\} \subset (0, +\infty)$ satisfy the following conditions:*

(i) *$\alpha_n \rightarrow 0$, as $n \rightarrow \infty$;*

(ii) *$\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \gamma_n) < +\infty$;*

(iii) *$\liminf_{n \to +\infty} \beta_n > 0 \limsup_{n \to +\infty} \gamma_n < 1$;*

(iv) *$\liminf_{n \to \infty} r_{n,i} > 0$ and $\liminf_{n \to \infty} s_{n,j} > 0$, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$.*

Then $\{x_n\}$ converges weakly to the unique element $v_0 \in \widetilde{D}$, where

$$\widetilde{D} := \left[\bigcap_{i=1}^{N} (A_i J)^{-1} 0 \right] \bigcap \left[\bigcap_{j=1}^{M} (B_j J)^{-1} 0 \right]. \tag{29}$$

Proof. We will split the proof into four steps.

Step 1. $\widetilde{D} \neq \emptyset$.

Since $\widetilde{\widetilde{D}} \neq \emptyset$, then we may choose $p \in \widetilde{\widetilde{D}}$, which implies that $p \in A_i^{-1} 0$ and $p \in B_j^{-1} 0$, for $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, M$. Thus $0 \in A_i p = A_i J J^{-1} p$ and $0 \in B_j p = B_j J J^{-1} p$, for $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, M$. And then $J^{-1} p \in (A_i J)^{-1} 0$ and $J^{-1} p \in (B_j J)^{-1} 0$, for $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, M$. Therefore, $J^{-1} p \in \widetilde{D}$ which implies that $\widetilde{D} \neq \emptyset$.

Step 2. $\{x_n\}$ is bounded.

For $\forall p \in \widetilde{D}$, noticing the definition of the Lyapunov functional and by using Lemma 9 repeatedly, we have

$$\begin{aligned} & \varphi(x_{n+1}, p) \\ & \quad \leq \gamma_n \varphi(x_n, p) + (1 - \gamma_n) \varphi(T_n v_n, p) \\ & \quad \leq \gamma_n \varphi(x_n, p) + (1 - \gamma_n) \\ & \qquad \times \left[b_0 \varphi(v_n, p) + b_1 \varphi\left(R_{s_{n,1}}^{B_1} v_n, p\right) + b_2 \varphi\left(R_{s_{n,2}}^{B_2} R_{s_{n,1}}^{B_1} v_n, p\right) \right. \\ & \qquad \left. + \cdots + b_M \varphi\left(R_{s_{n,M}}^{B_M} R_{s_{n,M-1}}^{B_{M-1}} \cdots R_{s_{n,1}}^{B_1} v_n, p\right) \right] \\ & \quad \leq \gamma_n \varphi(x_n, p) + (1 - \gamma_n) \varphi(v_n, p) \\ & \quad \leq \gamma_n \varphi(x_n, p) + (1 - \gamma_n) \\ & \qquad \times \left[(1 - \beta_n) \varphi(x_n, p) + \beta_n \varphi(S_n u_n, p) \right] \\ & \quad \leq \left[1 - \beta_n (1 - \gamma_n) \right] \varphi(x_n, p) \\ & \qquad + (1 - \gamma_n) \beta_n \left[a_0 \varphi(u_n, p) + \sum_{i=1}^{N} a_i \varphi\left(R_{r_{n,i}}^{A_i} u_n, p\right) \right] \\ & \quad \leq \left[1 - \beta_n (1 - \gamma_n) \right] \varphi(x_n, p) + (1 - \gamma_n) \beta_n \varphi(u_n, p) \\ & \quad \leq \left[1 - \alpha_n \beta_n (1 - \gamma_n) \right] \varphi(x_n, p) + (1 - \gamma_n) \alpha_n \beta_n \|p\|^2. \end{aligned} \tag{30}$$

Lemma 10 ensures that $\lim_{n \to \infty} \varphi(x_n, p)$ exists, which ensures that $\{x_n\}$ is bounded.

Step 3. $\omega(x_n) \subset \widetilde{D}$, where $\omega(x_n)$ is the set of weak limit points of all of the weakly convergent subsequences of $\{x_n\}$.

Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. So there exists a subsequence of $\{x_n\}$; for simplicity, we still denote it by $\{x_n\}$ such that $x_n \rightharpoonup x$, $n \rightarrow \infty$.

Using Lemma 9 again, we have for $\forall p \in \widetilde{D}$

$$\begin{aligned}
&\varphi(x_{n+1}, p) \\
&\quad \leq \gamma_n \varphi(x_n, p) + (1-\gamma_n)\varphi(T_n v_n, p) \\
&\quad \leq \gamma_n \varphi(x_n, p) + (1-\gamma_n) \\
&\qquad \times \left[(b_0 + b_1 + \cdots + b_{M-1})\varphi(v_n, p)\right. \\
&\qquad\qquad \left. + b_M \varphi\left(R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n, p\right)\right] \\
&\quad \leq \gamma_n \varphi(x_n, p) + (1-\gamma_n)(1-b_M)\varphi(v_n, p) \\
&\qquad + b_M (1-\gamma_n)\varphi\left(R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n, p\right) \\
&\qquad - b_M (1-\gamma_n) \\
&\qquad \times \varphi\left(R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n, R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n\right) \\
&\quad \leq \gamma_n \varphi(x_n, p) + (1-\gamma_n) \\
&\qquad \times \left[(1-\beta_n)\varphi(x_n, p) + \beta_n \varphi(S_n u_n, p)\right] - b_M (1-\gamma_n) \\
&\qquad \times \varphi\left(R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n, R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n\right) \\
&\quad \leq \left[1 - \alpha_n \beta_n (1-\gamma_n)\right]\varphi(p, x_n) \\
&\qquad + (1-\gamma_n)\alpha_n \beta_n \|p\|^2 - b_M (1-\gamma_n) \\
&\qquad \times \varphi\left(R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n, R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n\right).
\end{aligned} \tag{31}$$

Then (31) implies that

$$\begin{aligned}
&b_M (1-\gamma_n)\varphi\left(R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n, R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n\right) \\
&\quad \leq \left[1 - \alpha_n \beta_n (1-\gamma_n)\right]\varphi(x_n, p) \\
&\qquad + (1-\gamma_n)\alpha_n \beta_n \|p\|^2 - \varphi(x_{n+1}, p).
\end{aligned} \tag{32}$$

Similar to the discussion of (17) in Step 2 in Theorem 11, we have

$$R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n - R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n \longrightarrow 0, \tag{33}$$

as $n \to \infty$.

Then, similar to the discussions of (19) and (20), we have

$$\begin{gathered}
R^{B_{M-1}}_{s_{n,M-1}} R^{B_{M-2}}_{s_{n,M-2}} \cdots R^{B_1}_{s_{n,1}} v_n - R^{B_{M-2}}_{s_{n,M-2}} R^{B_{M-3}}_{s_{n,M-3}} \cdots R^{B_1}_{s_{n,1}} v_n \longrightarrow 0, \\
R^{B_{M-2}}_{s_{n,M-2}} R^{B_{M-3}}_{s_{n,M-3}} \cdots R^{B_1}_{s_{n,1}} v_n - R^{B_{M-3}}_{s_{n,M-3}} R^{B_{M-4}}_{s_{n,M-4}} \cdots R^{B_1}_{s_{n,1}} v_n \longrightarrow 0, \\
\vdots \\
R^{B_1}_{s_{n,1}} v_n - v_n \longrightarrow 0.
\end{gathered} \tag{34}$$

On the other hand, noticing (30) and using Lemma 9, we have

$$\begin{aligned}
&\varphi(x_{n+1}, p) \\
&\quad \leq \left[1 - \beta_n (1-\gamma_n)\right]\varphi(x_n, p) \\
&\qquad + (1-\gamma_n)\beta_n \left[a_0 \varphi(u_n, p) + \sum_{i=1}^{N} a_i \varphi\left(R^{A_i}_{r_{n,i}} u_n, p\right)\right] \\
&\quad \leq \left[1 - \beta_n (1-\gamma_n)\right]\varphi(x_n, p) + (1-\gamma_n) \\
&\qquad \times \beta_n \left\{a_0 \varphi(u_n, p) + \sum_{i=1}^{N} a_i \left[\varphi(u_n, p) - \varphi\left(u_n, R^{A_i}_{r_{n,i}} u_n\right)\right]\right\} \\
&\quad \leq \left[1 - \alpha_n \beta_n (1-\gamma_n)\right]\varphi(p, x_n) + (1-\gamma_n)\alpha_n \beta_n \|p\|^2 \\
&\qquad - (1-\gamma_n)\beta_n \sum_{i=1}^{N} a_i \varphi\left(u_n, R^{A_i}_{r_{n,i}} u_n\right),
\end{aligned} \tag{35}$$

which implies that

$$R^{A_i}_{r_{n,i}} u_n - u_n \longrightarrow 0, \tag{36}$$

as $n \to \infty$, $i = 1, 2, \ldots, N$.

Since $u_n = (1-\alpha_n)x_n$ and $u_n \rightharpoonup x$, then $u_n \rightharpoonup x$, as $n \to \infty$. Now, (36) implies that $R^{A_i}_{r_{n,i}} u_n \rightharpoonup x$. Let $z_i^* \in A_i z_i$, then

$$\left\langle z_i^* - \frac{u_n - R^{A_i}_{r_{n,i}} u_n}{r_{n,i}}, z_i - J R^{A_i}_{r_{n,i}} u_n \right\rangle \geq 0. \tag{37}$$

Since J is weakly sequentially continuous, then, letting $n \to \infty$, (37) ensures that

$$\left\langle z_i^*, z_i - Jx \right\rangle \geq 0. \tag{38}$$

Since A_i is maximal monotone, then $Jx \in A_i^{-1}0$, which implies that $x \in \bigcap_{i=1}^{N} (A_i J)^{-1} 0$.

From $R^{A_i}_{r_{n,i}} u_n \rightharpoonup x$, we have $S_n u_n \rightharpoonup x$, as $n \to \infty$. Since $v_n = (1-\beta_n)x_n + \beta_n S_n u_n$, then $v_n \rightharpoonup x$, as $n \to \infty$. Using (34), we have $R^{B_1}_{s_{n,1}} v_n \rightharpoonup x, R^{B_2}_{s_{n,2}} R^{B_1}_{s_{n,1}} v_n \rightharpoonup x, \ldots, R^{B_M}_{s_{n,M}} R^{B_{M-1}}_{s_{n,M-1}} \cdots R^{B_1}_{s_{n,1}} v_n \rightharpoonup x$, as $n \to \infty$. By induction, similar to the proof of $x \in (A_i J)^{-1} 0$, we know that $x \in \bigcap_{j=1}^{M} (B_j J)^{-1} 0$.

Therefore, $x \in \widetilde{D}$, and then $\omega(x_n) \subset \widetilde{D}$.

Step 4. $x_n \rightharpoonup v_0$, as $n \to \infty$, where v_0 is the unique element in $\widetilde{D}$.

From Steps 2 and 3, we know that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_0 \in \widetilde{D}$, as $i \to \infty$.

If there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup v_1 \in \widetilde{D}$, as $j \to \infty$, then from Step 1, we know that

$$\begin{aligned}
&\lim_{n\to\infty} [\varphi(x_n, v_0) - \varphi(x_n, v_1)] \\
&= \lim_{i\to\infty} \left[\varphi\left(x_{n_i}, v_0\right) - \varphi\left(x_{n_i}, v_1\right)\right] \\
&= \lim_{i\to\infty} \left[\|v_0\|^2 - \|v_1\|^2 + 2\left\langle x_{n_i}, Jv_1 - Jv_0\right\rangle\right] \\
&= \|v_0\|^2 - \|v_1\|^2 + 2\langle v_0, Jv_1 - Jv_0\rangle .
\end{aligned} \tag{39}$$

Similarly,

$$\begin{aligned}
&\lim_{n\to\infty} [\varphi(x_n, v_0) - \varphi(x_n, v_1)] \\
&= \lim_{j\to\infty} \left[\varphi\left(x_{n_j}, v_0\right) - \varphi\left(x_{n_j}, v_1\right)\right] \\
&= \lim_{i\to\infty} \left[\|v_0\|^2 - \|v_1\|^2 + 2\left\langle x_{n_j}, Jv_1 - Jv_0\right\rangle\right] \\
&= \|v_0\|^2 - \|v_1\|^2 + 2\langle v_1, Jv_1 - Jv_0\rangle .
\end{aligned} \tag{40}$$

From (39) and (40), we have $\langle v_1 - v_0, Jv_1 - Jv_0\rangle = 0$, which implies that $v_0 = v_1$.

This completes the proof. □

Remark 15. If, in Theorem 14, the Banach space E reduces to the Hilbert space H, then the result of Theorem 12 is still true. That is, Theorems 14 and 11 are the same in the frame of Hilbert spaces.

Remark 16. Next, we will present an example to show that the assumptions that $D \neq \emptyset$ in Theorem 11 and $\widetilde{D} \neq \emptyset$ in Theorem 14 are meaningful.

Consider the following p_i-*Laplacian* Dirichlet boundary value problem:

$$\begin{aligned}
&-\operatorname{div}\left(|\nabla u|^{p_i-2}\nabla u\right) = f(x), \quad \text{a.e. } x \in \Omega; \\
&\gamma u = w, \quad \text{a.e. } x \in \Gamma,
\end{aligned} \tag{D}$$

where Ω is a bounded conical domain of the Euclidean space R^N $(N \geq 1)$ with its boundary $\Gamma \in C^1$, $2N/(N+1) < p_i < +\infty$, $f \in W^{1,p_i}(\Omega)$, and $w \in W^{1-1/p_i,p_i}(\Gamma)$ are given functions, where $W^{1-1/p_i,p_i}(\Gamma)$ is the trace space of $W^{1,p_i}(\Omega)$, $\gamma : W^{1,p_i}(\Omega) \to W^{1-1/p_i,p_i}(\Gamma)$ is the trace operator, $i = 1, 2, \ldots, M$.

Similar to the discussion in [11], we have the following results.

Proposition 17 (see [11]). *Define the mapping* $C_{p_i} : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ *by*

$$\left(v, C_{p_i}u\right) = \int_\Omega \left\langle |\nabla u|^{p_i-2}\nabla u, \nabla v\right\rangle dx - \int_\Omega f v \, dx, \tag{41}$$

for $\forall u, v \in W^{1,p_i}(\Omega)$. *Define the mapping* $B_{p_i} : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ *by*

$$\left(v, B_{p_i}u\right) = \int_\Omega \left\langle |\nabla u|^{p_i-2}\nabla u, \nabla v\right\rangle dx, \tag{42}$$

for $\forall u, v \in W^{1,p_i}(\Omega)$.

Then both B_{p_i} *and* C_{p_i} *are maximal monotone, for each* $i = 1, 2, \ldots, M$.

Proposition 18 (see [11]). *For* $f \in W^{1,p_i}(\Omega)$ *and* $w \in W^{1-1/p_i,p_i}(\Gamma)$, *nonlinear boundary problem* (D) *has a unique solution* $u(x) \in W^{1,p_i}(\Omega)$, *for each* $i = 1, 2, \ldots, M$. *Moreover,* $u(x) \in C_{p_i}^{-1}(0)$ *if and only if* $u(x)$ *is the solution of* (D), *which ensures that* $C_{p_i}^{-1}(0) \neq \emptyset$, *for each* $i = 1, 2, \ldots, M$.

We can easily get the following result.

Proposition 19. $B_{p_i}^{-1}(0) = \{u(x) \in W^{1,p_i}(\Omega) : u(x) \equiv$ *Constant*$\}$, *for each* $i = 1, 2, \ldots, M$, *which ensures that* $\bigcap_{i=1}^M B_{p_i}^{-1}(0) \neq \emptyset$. *And,* $\bigcap_{i=1}^M C_{p_i}^{-1}(0) \neq \emptyset$, *which means the following nonlinear* $(p_1, p_2, \ldots, p_M)$*-Laplacian elliptic systems* (E) *with Dirichlet boundary share the same solution:*

$$\begin{aligned}
&-\operatorname{div}\left(|\nabla u|^{p_1-2}\nabla u\right) = f(x), \quad \text{a.e. } x \in \Omega; \\
&-\operatorname{div}\left(|\nabla u|^{p_2-2}\nabla u\right) = f(x), \quad \text{a.e. } x \in \Omega; \\
&\qquad \vdots \\
&-\operatorname{div}\left(|\nabla u|^{p_M-2}\nabla u\right) = f(x), \quad \text{a.e. } x \in \Omega; \\
&\gamma u = w, \quad \text{a.e. } x \in \Gamma.
\end{aligned} \tag{E}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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