Research Article
The Banach-Saks Properties in Orlicz-Lorentz Spaces

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The Banach-Saks index of an Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ for both function and sequence case, is computed with respect to its Matuszewska-Orlicz indices of φ. It is also shown that an Orlicz-Lorentz function space has weak Banach-Saks (resp., Banach-Saks) property if and only if it is separable (resp., reflexive).

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. A bounded sequence $\{x_n\} \subset X$ is called a *weak Banach-Saks sequence* whenever there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that its Cesàro means converge in norm to zero; that is,

$$\frac{1}{n}\sum_{i=1}^{n} y_i \longrightarrow 0. \tag{1}$$

The space X is said to satisfy the *weak Banach-Saks property*, in short $X \in (wBS)$, if every weakly null sequence $\{x_n\}$ in X is a weak Banach-Saks sequence. Recall that a Banach space X satisfies the *Banach-Saks property* (BS) if for every bounded sequence $\{x_n\}$ in X, there is a subsequence $\{y_j\}$ such that its Cesàro means converge; that is, the sequence $\{(1/m)\sum_{j=1}^{m} y_j\}$ is convergent in norm. It is well known that a Banach space has the (BS)-property if and only if it is reflexive and it has the (wBS)-property [1].

Schreier constructed a separable Banach space without the weak Banach-Saks property [2] (which is called a Schreier space) and later it was shown by Baernstein [3] that there is a reflexive separable Banach space without the weak Banach-Saks property. Hence the space ℓ_∞ does not have the weak Banach-Saks property, since every separable space is embeddable in ℓ_∞.

Following Johnson [4], given $1 \leq p \leq \infty$, a bounded sequence $\{x_n\} \subset X$ is called a BS_p *sequence* if there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that

$$\sup_{m \in \mathbb{N}} m^{-1/p} \left\| \sum_{i=1}^{m} y_i \right\| < \infty. \tag{2}$$

Here $m^{1/\infty} = 1$ for all $m \in \mathbb{N}$. We then say that X has the p-*Banach-Saks* property, shortly $X \in (BS_p)$, if each weakly null sequence contains a BS_p subsequence. It is clear that every Banach space has a (BS_1) property. The *Banach-Saks index* is the number $\gamma(X) = \sup\{p : X \in (BS_p)\}$. The set $\Gamma(X) = \{p : X \in (BS_p)\}$ is always an interval $[1, \gamma(X))$ or $[1, \gamma(X)]$. The Bessaga-Pełczyński method [5, Proposition 1.a.12] shows that $\Gamma(\ell_p) = [1, p]$ and the Schur property of ℓ_1 shows that $\Gamma(\ell_1) = [1, \infty]$. The Banach-Saks index of Nakano sequence space was computed in [6].

The stronger property (S_p) was introduced by Knaust and Odell in [7]. It is said that X has property (S_p), $1 < p \leq \infty$, if every weakly null sequence in X has a subsequence $\{x_k\}$ so that there is a constant $C > 0$ such that, for all $m \in \mathbb{N}$ and all real sequences $a = (a_n) \in \ell_p$,

$$\left\| \sum_{j=1}^{m} a_j x_j \right\| \leq C \|a\|_p, \tag{3}$$

where $\|a\|_p$ is the ℓ_p norm.

It is clear that $(S_p) \Rightarrow (BS_p) \Rightarrow (wBS)$ for all $1 < p \leq \infty$. The Elton c_0-theorem [8, Theorem III 3.5] states that $(BS_\infty) \Leftrightarrow (S_\infty)$ and a Banach space X has (BS_∞) if and only if every normalized weakly null sequence contains a subsequence which is equivalent to the unit vector basis of c_0, which is equivalent to the case in which every subspace of X has the Dunford-Pettis property (for details see [9]). In general, the two properties (BS_p) and (S_p) are not equivalent if $1 < p < \infty$ [10]; however, Rakov [11, Theorem 3] showed that if $1 < q < p < \infty$, then (BS_p) implies (S_q). Various weak Banach-Saks properties and their relations were shown in [6] for Musielak-Orlicz sequence spaces.

In this paper we compute the Banach-Saks index of Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ and show that Orlicz-Lorentz function space $\Lambda_{\varphi,w}(I)$ has the weak Banach-Saks property (resp., Banach-Saks property) if and only if it is separable (resp., reflexive). We start with several preliminary results about Orlicz-Lorentz spaces.

2. Preliminaries

Let $I = \mathbb{N}$ or $I = (0, a)$ with $0 < a \leq \infty$, equipped with the counting measure if $I = \mathbb{N}$ and the Lebesgue measure if $I = (0, a)$. If $A \subset I$ is a measurable set, we denote by $|A|$ its measure. By $L^0(I)$ we denote the collection of all real valued measurable functions on I. In the case when $I = \mathbb{N}$ the elements are sequences $x = (x(n))$, and in the other cases they are real valued Lebesgue measurable functions x.

For a measurable function x, the decreasing rearrangement of x will be denoted by x^*, defined as $x^*(t) = \inf\{s > 0 : d_{|x|}(s) \leq t\}, t \geq 0$, where $d_x(s) = |\{t : x(t) \geq s\}|, s \geq 0$. For two measurable functions x and y, the relation $x \prec\prec y$ means that, for all $t \geq 0$, we have

$$\int_0^t x^*(s)\, ds \leq \int_0^t y^*(s)\, ds. \tag{4}$$

For a sequence $x = (x(n))$, x^* will denote the decreasing rearrangement of x. That is, letting $\tilde{x}(t) = \sum_{n=1}^\infty x(n)\chi_{[n-1,1)}$, we have $x^*(n) = \tilde{x}^*(n-1)$ for all $n \in \mathbb{N}$. For sequences x and y, $x \prec\prec y$ means that for all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^n x^*(k) \leq \sum_{k=1}^n y^*(k). \tag{5}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an *Orlicz function*; that is, φ is convex, strictly increasing, and $\varphi(0) = 0$. Let the weight function $w \in L^0(I)$ be nonnegative and nonincreasing. For $I = (0, a)$, $0 < a \leq \infty$, the *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I) = \Lambda_{\varphi,w}(0, a)$ is the collection of all functions $f \in L^0(0, a)$ such that

$$\|f\| = \|f\|_{\varphi,w} = \inf\left\{\epsilon > 0 : I\left(\frac{f}{\epsilon}\right) \leq 1\right\} < \infty, \tag{6}$$

where $I(f) = I_{\varphi,w}(f) = \int_I \varphi(f^*)w$. Analogously in case when $I = \mathbb{N}$, the *Orlicz-Lorentz sequence space* $d(\varphi, w) = \Lambda_{\varphi,w}(\mathbb{N})$ contains all real sequences $x = (x(n))$ such that

$$\|x\| = \|x\|_{\varphi,w} = \inf\left\{\epsilon > 0 : I\left(\frac{x}{\epsilon}\right) \leq 1\right\} < \infty, \tag{7}$$

where $I(x) = I_{\varphi,w}(x) = \sum_{n=1}^\infty \varphi(x^*(n))w(n)$. It is well known that the space $(\Lambda_{\varphi,w}(I), \|\cdot\|)$ is a rearrangement invariant (r.i.) Banach function space satisfying the Fatou property [12–15]. The space $\Lambda_{\varphi,w}^0(0, a)$ (resp., $d^0(\varphi, w)$) is the order continuous part of $\Lambda_{\varphi,w}(0, a)$ (resp., $d(\varphi, w)$). In the sequence case $d^0(\varphi, w)$ is the closure of the linear span of the unit vectors (e_i) in $d(\varphi, w)$. If $1 \leq p < \infty$ and $\varphi(u) = u^p$, then $d(\varphi, w) = d(p, w)$ and $\Lambda_{\varphi,w}(0, a) = \Lambda_{p,w}(0, a)$ [16]. If $w \equiv 1$, then $\Lambda_{\varphi,w}(0, a) = L_\varphi(0, a)$ is an Orlicz function space, and $d(\varphi, w) = \ell_\varphi$ is an Orlicz sequence space. In this case also $\Lambda_{\varphi,w}^0(0, a) = L_\varphi^0(0, a)$ and $d^0(\varphi, w) = \ell_\varphi^0$ are subspaces of order continuous elements in Orlicz spaces.

We may assume that $\varphi(1) = 1$. In fact, the space $\Lambda_{\psi,w}(I)$, where $\psi(t) = \varphi(\alpha t)$ with $\varphi(\alpha) = 1$, is isometric to $\Lambda_{\varphi,w}(I)$; we have $\|f\|_{\psi,w} = \alpha\|f\|_{\varphi,w}$.

Definition 1. Given an Orlicz function φ, one defines the growth conditions connected to its *lower* and *upper Matuszewska-Orlicz indices* in three different categories, at zero, at infinity, and on $\mathbb{R}_+$.

We say that, for $1 \leq p < \infty$, $\varphi \in \Delta_0^{*p}$, $\varphi \in \Delta_\infty^{*p}$, and $\varphi \in \Delta^{*p}$, whenever

$$\sup_{\substack{0<a\leq1\\0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty, \qquad \sup_{\substack{0<a\leq1\\t>1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty,$$

$$\sup_{\substack{0<a\leq1\\t>0}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty, \tag{8}$$

respectively. Analogously we say that, given $1 \leq q < \infty$, $\varphi \in \Delta_0^q$, $\varphi \in \Delta_\infty^q$, and $\varphi \in \Delta^q$, whenever

$$\inf_{\substack{0<a\leq1\\0<t<1}} \frac{\varphi(at)}{\varphi(t)a^q} > 0, \qquad \inf_{\substack{0<a\leq1\\t>1}} \frac{\varphi(at)}{\varphi(t)a^q} > 0,$$

$$\inf_{\substack{0<a\leq1\\t>0}} \frac{\varphi(at)}{\varphi(t)a^q} > 0, \tag{9}$$

respectively.

The Matuszewska-Orlicz indices, lower α_φ^0, α_φ^∞, and α_φ and upper β_φ^0, β_φ^∞, and β_φ, are then defined as follows:

$$\alpha_\varphi^0 = \sup\left\{p : \varphi \in \Delta_0^{*p}\right\}, \qquad \alpha_\varphi^\infty = \sup\left\{p : \varphi \in \Delta_\infty^{*p}\right\},$$

$$\alpha_\varphi = \sup\left\{p : \varphi \in \Delta^{*p}\right\},$$

$$\beta_\varphi^0 = \inf\left\{q : \varphi \in \Delta_0^q\right\}, \qquad \beta_\varphi^\infty = \inf\left\{q : \varphi \in \Delta_\infty^q\right\},$$

$$\beta_\varphi = \inf\left\{q : \varphi \in \Delta^q\right\}. \tag{10}$$

It is clear that $\alpha_\varphi = \min(\alpha_\varphi^0, \alpha_\varphi^\infty)$ and $\beta_\varphi = \max(\beta_\varphi^0, \beta_\varphi^\infty)$. We say that φ satisfies *conditions* Δ_2^0, Δ_2^∞, and Δ_2, whenever

$$\sup_{0<t<1} \frac{\varphi(2t)}{\varphi(t)} < \infty, \qquad \sup_{t>1} \frac{\varphi(2t)}{\varphi(t)} < \infty, \qquad \sup_{t>0} \frac{\varphi(2t)}{\varphi(t)} < \infty, \tag{11}$$

respectively. Let φ_1 and φ_2 be Orlicz functions. We call them *equivalent at zero*, *equivalent at infinity*, or just *equivalent*, if there is $D > 0$ such that $\varphi_1(D^{-1}t) \leq \varphi_2(t) \leq \varphi_1(Dt)$, for all $0 < t < 1$, $t > 1$, or for all $t > 0$, respectively. It is well known and easy to prove that Matuszewska-Orlicz indices of Orlicz functions are preserved by the corresponding equivalence relations (e.g., see [17]); that is, if φ_1 and φ_2 are equivalent (resp., equivalent at 0 and equivalent at ∞), then $\alpha_{\varphi_1} = \alpha_{\varphi_2}$ (resp., $\alpha_{\varphi_1}^0 = \alpha_{\varphi_2}^0$ and $\alpha_{\varphi_1}^\infty = \alpha_{\varphi_2}^\infty$). Similar equalities hold for upper indices. Recall also that any upper index of φ is finite if and only if φ satisfies the corresponding condition Δ_2 [17, 18].

We also have that if the Orlicz functions φ_1 and φ_2 are equivalent (resp., equivalent at infinity and equivalent at zero), then $\Lambda_{\varphi_1,w}(0,\infty) = \Lambda_{\varphi_2,w}(0,\infty)$ (resp., $\Lambda_{\varphi_1,w}(0,a) = \Lambda_{\varphi_2,w}(0,a)$, $a < \infty$; $d(\varphi_1,w) = d(\varphi_2,w)$) with equivalent norms.

The space $\Lambda_{\varphi,w}(0,\infty)$ (resp., $\Lambda_{\varphi,w}(0,a)$ $(a < \infty)$; $d(\varphi,w)$) is separable if and only if $\varphi \in \Delta_2$ and $W(\infty) = \infty$ (resp., $\varphi \in \Delta_2^\infty$; $\varphi \in \Delta_2^0$ and $W(\infty) = \infty$), if $a = \infty$ (resp., $a < \infty$; $I = \mathbb{N}$). The separability is also equivalent to the case in which the space $\Lambda_{\varphi,w}(0,\infty)$ does not contain the isomorphic copy of ℓ_∞ [12, 15, 19].

Lemma 2. *Let $1 \leq p, q < \infty$;*

(1) *([20, Lemma 6], [21, Lemma 3], and [22]) An Orlicz function $\varphi \in \Delta^{*p}$ (resp., $\varphi \in \Delta_\infty^{*p}$; $\varphi \in \Delta_0^{*p}$) if and only if there exists an Orlicz function ψ equivalent (resp., equivalent at ∞; equivalent at 0) to φ such that $\psi(t^{1/p})$ is convex;*

(2) *([20, Lemma 5], [21, Lemma 2], and [22]) An Orlicz function $\varphi \in \Delta^q$ (resp., $\varphi \in \Delta_\infty^q$; $\varphi \in \Delta_0^q$) if and only if there exists an Orlicz function ψ equivalent (resp., equivalent at ∞ and equivalent at 0) to φ such that $\psi(t^{1/q})$ is concave.*

We will use the following functions:

$$W(t) = \int_0^t w(s)\,ds \quad \text{for } t \in (0,a), \text{ or}$$
$$W(m) = \sum_{i=1}^m w(i) \quad \text{for } m \in \mathbb{N}. \tag{12}$$

Recall that the weight w is *regular* whenever there exists $C > 0$ such that $W(s) \leq Csw(s)$ for all $s \in I$. Since always $sw(s) \leq W(s)$, the weight w is regular if the functions $W(s)$ and $sw(s)$ are equivalent on I. Notice that for any regular weight w on $I = (0,\infty)$ or $I = \mathbb{N}$ it holds that $W(\infty) = \infty$.

A Banach lattice is said to be *p-convex* (resp., *p-concave*) for some $1 \leq p < \infty$ if there is a constant $K > 0$ such that

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \leq K \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$
$$\left(\text{resp.,} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq K \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \right) \tag{13}$$

for every choice of vectors $x_1, \ldots, x_n$ in X. A Banach lattice is said to satisfy *upper p-estimate* (resp., *lower p-estimate*) if the definition of *p-convexity* (resp., *p-concavity*) holds true for any choice of disjointly supported elements $x_1, \ldots, x_n$ in X. It is known that, given $1 \leq p < \infty$, if X is p-convex (resp., p-concave), then X is r-convex (resp., r-concave) for $1 \leq r \leq p$ (resp., $r \geq p$) [23, 24].

Theorem 3 (see [25]). *The space $\Lambda_{\varphi,w}(0,\infty)$ (resp., $\Lambda_{\varphi,w}(0,a)$ $(a < \infty)$; $d(\varphi,w)$) has finite concavity if and only if φ satisfies the Δ_2 (resp., Δ_2^∞; Δ_2^0) condition and the weight w is regular.*

Recall that a Banach lattice has the finite concavity if and only if it is of finite cotype [26].

Theorem 4 (see [19]). *Let $1 \leq p \leq \infty$;*

(1) *[19, Theorem 7.18] For $I = (0,\infty)$, the following conditions are equivalent:*

    (i) *ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $\Lambda_{\varphi,w}^0(0,\infty)$,*

    (ii) *either $p \in [\alpha_\varphi^0, \beta_\varphi^0] \cup [\alpha_\varphi^\infty, \beta_\varphi^\infty] = [\alpha_\varphi, \beta_\varphi]$ or, for some $c > 0$, $\int_0^\infty \varphi(ct^{-1/p})w(t)dt < \infty$;*

(2) *[19, Theorem 6.13] For $I = (0,a)$, $a < \infty$, the following assertions are equivalent:*

    (i) *ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $\Lambda_{\varphi,w}^0(0,a)$,*

    (ii) *$p \in [\alpha_\varphi^\infty, \beta_\varphi^\infty]$,*

    (iii) *ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $L_\varphi^0(0,a)$;*

(3) *[19, Theorem 7.8] For $I = \mathbb{N}$ the following statements are equivalent (where c_0 is meant in place of ℓ^p if $p = \infty$):*

    (i) *ℓ^p is order isomorphic to a sublattice of $d_0(w,\varphi)$,*

    (ii) *ℓ^p is isomorphic to a subspace of $d_0(w,\varphi)$,*

    (iii) *$p \in [\alpha_\varphi^0, \beta_\varphi^0]$,*

    (iv) *ℓ^p is isomorphic to a subspace of h_φ.*

3. Main Results

Theorem 5. *If $1 < p < \infty$ and $\varphi \in \Delta^{*p}$ (resp., $\varphi \in \Delta_\infty^{*p}$ and $\varphi \in \Delta_0^{*p}$), then $\Lambda_{\varphi,w}(0,\infty)$ (resp., $\Lambda_{\varphi,w}(0,a)$, $a < \infty$; $d(\varphi,w)$) is p-convex. If $\Lambda_{\varphi,w}(0,\infty)$ (resp., $\Lambda_{\varphi,w}(0,a)$, $a < \infty$; $d(\varphi,w)$) is p-convex for some $1 < p < \infty$, then $p \leq \alpha_\varphi$ (resp., $p \leq \alpha_\varphi^\infty$; $p \leq \alpha_\varphi^0$).*

Proof. (1) By Lemma 2 we assume that $\psi(t) = \varphi(t^{1/p})$ is convex. Then we have the equality $\|f\|_{\Lambda_{\psi,w}^{(p)}(I)} = \|f\|_{\varphi,w}$, where $\Lambda_{\psi,w}^{(p)}(I)$ is a p-convexification of $\Lambda_{\psi,w}(I)$. Indeed,

$$
\begin{aligned}
\|f\|_{\Lambda_{\psi,w}^{(p)}(I)} &= \left\||f|^p\right\|_{\psi,w}^{1/p} \\
&= \inf\left\{\epsilon^{1/p} > 0 : \int_I \psi\left(\frac{\left(|f|^p\right)^*}{\epsilon}\right) \le 1\right\} \\
&= \inf\left\{\epsilon^{1/p} > 0 : \int_I \varphi\left(\frac{f^*}{\epsilon^{1/p}}\right) \le 1\right\} = \|f\|_{\varphi,w}.
\end{aligned}
\tag{14}
$$

Since ψ is a convex Orlicz function, the space $\Lambda_{\psi,w}(I)$ is a Banach space, so its p-convexification $\Lambda_{\psi,w}^{(p)}(I) = \Lambda_{\varphi,w}(I)$ is p-convex [26].

We only show the second part in the case of $I = (0, \infty)$. If $p > \alpha_\varphi = \min(\alpha_\varphi^0, \alpha_\varphi^\infty)$, then, by Theorem 4(1), there is $\alpha_\varphi \le r < p$ such that ℓ^r is an order copy in $\Lambda_{\varphi,w}(0, \infty)$, and thus the space cannot be p-convex. □

Example 6. In general, the parallel statement on concavity in Theorem 5 does not hold. As shown by Pisier in [26, Example 1.f.19], the Lorentz space $d(p, w)$, where $w_n = n^{1/p} - (n-1)^{1/p}$ and $1 < p < 2$, is not 2-concave even though the function $\varphi(t) = t^p$ satisfies condition Δ_2 and w is regular.

Proposition 7. *Let* $1 < p \le 2$. *If* $\varphi \in \Delta^{*p}$ *and* $\varphi \in \Delta_2$ *if* $a = \infty$ *(resp.,* $\varphi \in \Delta_\infty^{*p}$ *and* $\varphi \in \Delta_2^\infty$ *in the case when* $a < \infty$*) and* w *is regular, then* $\Lambda_{\varphi,w}(0, a)$, $0 < a \le \infty$, *has type* p.

Proof. By Theorem 5 the space is p-convex, and thus in view of Theorem 3 and by applying the well-known fact [26, Proposition 1.f.3], the space has type p. □

Rakov [27, Theorem 1] showed that if a Banach space has type $1 < p \le 2$, then it has p-Banach-Saks property. Hence we get the following.

Theorem 8. *If a Banach space* X *has type* p *for some* $1 < p \le 2$, *then* $X \in (BS_p)$. *Consequently* $p \le \gamma(X)$.

Notice that for every separable r.i. space E on $[0, 1]$, it is shown in [28] that $\gamma(E) \le 2$. Because every r.i. function space X on $[0, \infty)$ contains the r.i. space $\widetilde{X} = \{f\chi_{[0,1]} : f \in X\}$ isometrically, we get the following.

Proposition 9 (see [29]). *If* X *is a separable r.i. space on* $(0, a)$, $0 < a \le \infty$, *then* $\gamma(X) \le 2$.

Recall that a Banach space X is said to be p-smooth ($1 \le p \le 2$) if its modulus of smoothness $\rho_X(\epsilon)$ is majorized by the function ϵ^p. That is, there is a constant $A > 0$ such that, for all $\epsilon > 0$,

$$
\rho_X(\epsilon) := \inf_{\{\|x\|=\|y\|=1\}} \frac{\|x+\epsilon y\| + \|x-\epsilon y\| - 2}{2} \le A\epsilon^p. \tag{15}
$$

Rakov [27, Theorem 1] showed that if X is a p-smooth space, then it has the (S_p) property. It is well known [26] that a Banach lattice is isomorphic to a p-smooth Banach space for some $1 < p \le 2$ if and only if it has the type p and satisfies a lower q-estimate for some $1 \le q < \infty$. Notice also that the finite concavity is equivalent to the lower q-estimate for some $q < \infty$ [26]. Then we obtain immediately the following corollary as a consequence of definition of lower indices and by Propositions 7 and 9 and Theorem 8.

Corollary 10. *Let* $1 < p \le 2$. *If* $\varphi \in \Delta^{*p}$ *and* $\varphi \in \Delta_2$ *if* $a = \infty$ *(resp.,* $\varphi \in \Delta_\infty^{*p}$ *and* $\varphi \in \Delta_2^\infty$ *in the case when* $a < \infty$*) and* w *is regular, then* $\Lambda_{\varphi,w}(0, a)$, $0 < a \le \infty$, *has the* S_p-*property.*

Consequently, if $\varphi \in \Delta_2$ *when* $a = \infty$ *(resp.,* $\varphi \in \Delta_2^\infty$ *in the case when* $a < \infty$*) and* w *is regular, then*

$$
\begin{aligned}
\min\left(2, \alpha_\varphi\right) &\le \gamma\left(\Lambda_{\varphi,w}(0, \infty)\right), \\
\min\left(2, \alpha_\varphi^\infty\right) &\le \gamma\left(\Lambda_{\varphi,w}(0, a)\right), \\
a &< \infty.
\end{aligned}
\tag{16}
$$

The next corollary is a result of the facts that $\gamma(\ell^\infty) = 1$ and that if φ does not satisfy appropriate condition Δ_2, then $\Lambda_{\varphi,w}(I)$ contains an isomorphic copy of ℓ^∞.

Corollary 11. *If* $\varphi \notin \Delta_2$ *when* $a = \infty$, $\varphi \notin \Delta_2^\infty$ *in the case when* $a < \infty$, *and* $\varphi \notin \Delta_2^0$ *when* $I = \mathbb{N}$, *then*

$$
\begin{aligned}
\gamma\left(\Lambda_{\varphi,w}(0, \infty)\right) &= 1, \qquad \gamma\left(\Lambda_{\varphi,w}(0, a)\right) = 1, \\
a &< \infty, \quad \gamma\left(d\left(\varphi, w\right)\right) = 1.
\end{aligned}
\tag{17}
$$

Corollary 12. *An Orlicz-Lorentz space* $\Lambda_{\varphi,w}^0(0, \infty)$ *(resp.,* $\Lambda_{\varphi,w}^0(0, a)$, $a < \infty$; $d^0(w, \varphi)$*) does not contain an isomorphic copy of* ℓ^1 *if and only if* $\alpha_\varphi > 1$ *(resp.,* $\alpha_\varphi^\infty > 1$; $\alpha_\varphi^0 > 1$*).*

Proof. This hypothesis for sequence spaces is an immediate consequence of Theorem 4(3).

Let now $a < \infty$. From [19, Corollary 3.5(2), (3)], the space $\Lambda_{\varphi,w}^0(0, a)$ contains an isomorphic copy of ℓ^1 if and only if either ℓ^1 is an order isomorphic copy, that is, in view of Theorem 4(2), $\alpha_\varphi^\infty = 1$, or the sequence of functions $\min(t^{-1}, n)$ is bounded in $L_\varphi(0, a)$. However $\alpha_\varphi^\infty > 1$ implies that there exists $p > 1$ such that $\varphi(\beta u) \ge K\beta^p\varphi(u)$ for all $u, \beta \ge 1$ and some $K > 0$. Hence for any constant $c > 0$ and large enough $n \in \mathbb{N}$,

$$
\begin{aligned}
\int_0^a \varphi\left(c\min\left(t^{-1}, n\right)\right) dt &\ge \int_0^{1/n} \varphi(cn) \, dt \\
&\ge \int_0^{1/n} Kc^p n^p \varphi(1) \, dt \\
&= K\varphi(c) n^{p-1} \longrightarrow \infty.
\end{aligned}
\tag{18}
$$

Thus whenever $\alpha_\varphi^\infty > 1$ the sequence $\min(t^{-1}, n)$ is not bounded, and the proof is done in case of finite interval.

Assume that $a = \infty$. Again applying Corollary 3.5(2) in [19], we get dichotomy that ℓ^1 is an isomorphic copy in $\Lambda^0_{\varphi,w}(0,\infty)$ if and only if either ℓ^1 is an order isomorphic copy in $\Lambda^0_{\varphi,w}(0,\infty)$ or the sequence $\min(t^{-1}\chi_{(0,1)}(t), n)$ is bounded in $L_\varphi(0,\infty)$. Assuming now that $\alpha_\varphi > 1$, we can show similarly as above that the second alternative cannot occur. Recall then that, by Theorem 4(1), ℓ^1 is an order isomorphic copy in the space if and only if either $\alpha_\varphi = 1$ or, for some $c > 0$, $\int_0^\infty \varphi(ct^{-1})w(t)dt < \infty$. We will exclude again the second possibility. In fact if $\alpha_\varphi > 1$, then for some $p > 1$ and $K > 0$ and for all $\beta \geq 1$ and $u \geq 0$, $\varphi(\beta u) \geq K\beta^p\varphi(u)$. Therefore for every $c > 0$ and some $t_0 < 1$ such that $w(t_0) > 0$, we have

$$
\int_0^\infty \varphi\left(ct^{-1}\right)w(t)\,dt \geq \int_0^{t_0} \varphi\left(ct^{-1}\right)w(t_0)\,dt
$$
$$
\geq K\varphi(c)w(t_0)\int_0^{t_0} t^{-p}dt = \infty,
$$
(19)

and this completes the proof. □

Corollary 13. *Assume that $\Lambda_{\varphi,w}(0,a)$, $0 < a \leq \infty$, does not contain an isomorphic copy of ℓ^1, equivalently that $\alpha_\varphi > 1$ in case $a = \infty$ and $\alpha_\varphi^\infty > 1$ in case $a < \infty$. Then*

$$
\gamma\left(\Lambda_{\varphi,w}(0,\infty)\right) \leq \min\left(2, \alpha_\varphi\right),
$$
$$
\gamma\left(\Lambda_{\varphi,w}(0,a)\right) \leq \min\left(2, \alpha_\varphi^\infty\right),
$$
(20)
$$
a < \infty.
$$

Proof. By Proposition 9 the Banach-Saks index of any r.i. space over the interval $(0,a)$ cannot be bigger than 2. On the other hand, recall that $\Gamma(\ell_p) = [1,p]$ for $1 < p < \infty$ and $\Gamma(c_0) = \Gamma(\ell_1) = [1,\infty]$. The space ℓ_∞ does not have wBS property, so $\Gamma(\ell_\infty) = 1$. Hence Theorem 4 shows that

$$
\gamma\left(\Lambda_{\varphi,w}(0,a)\right)
$$
$$
\leq A
$$
$$
:= \inf\left\{p > 1 : \ell^p \text{ is an isomorphic copy in } \Lambda_{\varphi,w}(0,a)\right\},
$$
(21)

where $A \leq \alpha_\varphi$ when $a = \infty$ and $A \leq \alpha_\varphi^\infty$ if $a < \infty$. □

Finally by Corollaries 10 and 11 we get the result on Banach-Saks index in the Orlicz-Lorentz spaces over the interval $(0,a)$, $a \leq \infty$.

Theorem 14. *Assume that $\Lambda_{\varphi,w}(0,a)$, $0 < a \leq \infty$, does not contain an isomorphic copy of ℓ^1 and equivalently that $\alpha_\varphi > 1$ in case $a = \infty$ and $\alpha_\varphi^\infty > 1$ in case $a < \infty$. Assume also that*

$\varphi \in \Delta_2$ *when $a = \infty$, $\varphi \in \Delta_2^\infty$ when $a < \infty$, and w is regular. Then*

$$
\min\left(2, \alpha_\varphi\right) = \gamma\left(\Lambda_{\varphi,w}(0,\infty)\right),
$$
$$
\min\left(2, \alpha_\varphi^\infty\right) = \gamma\left(\Lambda_{\varphi,w}(0,a)\right),
$$
(22)
$$
a < \infty.
$$

Using the Bessaga-Pełczynski method, we get the following.

Theorem 15. *Let $1 < p < \infty$. If an order continuous sequence space X is p-convex, then X has the (S_p) property. Consequently $p \leq \gamma(X)$.*

Proof. Since X is order continuous, the standard unit vectors $\{e_n\}$ form a basis of X. Suppose that $\{x_n\}$ is a normalized weakly null sequence in X. By the Bessaga-Pełczyński method [5, Proposition 1.a.12], we may assume that $x_n = y_n + z_n$ for all n, where $\{y_n\}$ is a block basic sequence of $\{e_n\}$ and $\|z_n\| \leq 1/2^n$ for all n. Then for real numbers $a_1, \ldots, a_n$ with $\sum_{j=1}^n |a_j|^p \leq 1$, the p-convexity of X and disjointness of the supports of y_n's show that

$$
\left\|\sum_{j=1}^n a_j x_j\right\| \leq \left\|\sum_{j=1}^n a_j y_j\right\| + \sum_{j=1}^n |a_j|\,\|z_j\|
$$
$$
\leq \left\|\left(\sum_{j=1}^n |a_j y_j|^p\right)^{1/p}\right\| + \left(\sum_{j=1}^n |a_j|^p\right)^{1/p}\left(\sum_{j=1}^n \frac{1}{2^{jq}}\right)^{1/q}
$$
$$
\leq 2C\left(\sum_{j=1}^n |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |a_j|^p\right)^{1/p}\left(\sum_{j=1}^n \frac{1}{2^{jq}}\right)^{1/q}
$$
$$
\leq (2C+1)\left(\sum_{j=1}^n |a_j|^p\right)^{1/p},
$$
(23)

where C is the constant from the p-convexity of X and q is the conjugate of p satisfying $1/p + 1/q = 1$. The proof is done. □

As a consequence of the above Theorems 15 and 5 about p-convexity and Theorem 4 about ℓ^p-copies, we get the Banach-Saks index in sequence spaces as follows.

Theorem 16. *Assume that $d(\varphi, w)$ does not contain an isomorphic copy of ℓ^1 and equivalently that $\alpha_\varphi^0 > 1$. Assume also that $\varphi \in \Delta_2^0$ and $W(\infty) = \infty$. Then*

$$
\gamma\left(d\left(\varphi, w\right)\right) = \alpha_\varphi^0.
$$
(24)

Proof. By Theorems 5 and 15, $\alpha_\varphi^0 \leq \gamma(d(\varphi,w))$ and

$$
\gamma\left(d\left(\varphi, w\right)\right) \leq \inf\left\{p > 1 : \ell^p \subset d(\varphi, w)\right\}.
$$
(25)

Since $\gamma(\ell_p) = p$ $(1 < p < \infty)$, Theorem 4 completes the proof. □

We finish with a result on the weak Banach-Saks property.

Theorem 17. *A separable Orlicz-Lorentz space* $\Lambda_{\varphi,w}(0,a)$, $a \le \infty$, *has the weak Banach-Saks property.*

Proof. Let $\{x_n\} \subset \Lambda_{\varphi,w}(0,a)$ be weakly null and assume without loss of generality that $\|x_n\| = 1$, $n \in \mathbb{N}$. Then by [29, Proposition 3.2] there exists a subsequence $\{x'_n\}$ splitted into three subsequences

$$x'_n = y_n + z_n + d_n, \tag{26}$$

such that $\{y_n\}, \{z_n\}, \{d_n\} \subset \Lambda_{\varphi,w}(0,a)$ are bounded and

(a) $y_1^* = y_n^*$, $\|y_n\| \le 1$, and $y_n \to 0$ weakly;

(b) $z_n z_m = 0$ for $n \ne m$, $\|z_n\| \le 2$, $z_n \to 0$ in measure, and $z_n \to 0$ weakly;

(c) $\|d_n\| \to 0$.

We need to show that there is a subsequence of $\{x'_n\}$ whose Cesàro means approach zero in norm. It is obvious that we can do it for the sequence $\{d_n\}$ satisfying (c). We will show this now for the sequence $\{y_n\}$. By [29, Proposition 4.3] there exists a subsequence $\{y_{n(k)}\} \subset \{y_n\}$, $x \in L_1(0,a) + L_\infty(0,a)$, such that

$$\frac{1}{N}\sum_{k=1}^{N} y_{m(k)} \longrightarrow x \tag{27}$$

in $L_1(0,a) + L_\infty(0,a)$ for all further subsequences $\{y_{m(k)}\} \subset \{y_{n(k)}\}$ (we can assume that $y_{m(1)} = y_1$). Hence

$$\frac{1}{N}\sum_{k=1}^{N} y_{m(k)} \longrightarrow x \quad \text{in measure on } (0,a). \tag{28}$$

(Recall that if $x_n \to 0$ in any r.i. space X, then $x_n \to 0$ in measure.) Since $y_{n(k)} \to 0$ weakly in $\Lambda_{\varphi,w}(0,a)$, we have, for any subsequence $(m(k))$ of $(n(k))$,

$$\frac{1}{N}\sum_{k=1}^{N} y_{m(k)} \longrightarrow 0 \quad \text{weakly in } L_1(0,a) + L_\infty(0,a). \tag{29}$$

Applying now [29, Lemma 4.4] we get that

$$\frac{1}{N}\sum_{k=1}^{N} y_{m(k)} \longrightarrow 0 \quad \text{in } L_1(0,a) + L_\infty(0,a) \tag{30}$$

for all subsequences $\{y_{m(k)}\} \subset \{y_{n(k)}\}$. Let now $\{y_{m(k)}\}$ be an arbitrary subsequence of $\{y_{n(k)}\}$. Set

$$a_N = \frac{1}{N}\sum_{k=1}^{N} y_{m(k)}. \tag{31}$$

By $y_{m(k)}^* = y_1^*$ and by subadditivity of the operator $t \mapsto \int_0^t f^*$, $f \in L^0(I)$, we get, for $t \ge 0$,

$$\int_0^t \left(\frac{1}{N}\sum_{k=1}^{N} y_{m(k)}\right)^* \le \frac{1}{N}\sum_{k=1}^{N}\int_0^t y_{m(k)}^* = \int_0^t y_1^*; \tag{32}$$

that is,

$$a_N \prec\prec y_1. \tag{33}$$

Since $a_N \to 0$ in measure and $a_N \prec\prec y_1$ for every $N \in \mathbb{N}$, by [29, Proposition 4.2],

$$\frac{1}{N}\sum_{k=1}^{N} y_{m(k)} \longrightarrow 0 \tag{34}$$

in $\Lambda_{\varphi,w}(0,a)$, which finishes the case (a).

In the case (b), by [19, Theorem 5.5], there exists a subsequence $\{z_{n(k)}\} \subset \{z_n\}$ such that any further subsequence $\{z_{m(k)}\}$ of $\{z_{n(k)}\}$ is equivalent to the unit vector basis in some Orlicz sequence space ℓ_ψ^0. Thus $z_{n(k)} \to 0$ weakly in ℓ_ψ^0, and by wBS-property of ℓ_ψ^0 (see, e.g., [27]), there is a subsequence $\{z_k'\} \subset \{z_{n(k)}\}$ such that $(1/n)\sum_{k=1}^{N} z_k' \to 0$ in ℓ_ψ^0 and thus in $\Lambda_{\varphi,w}(0,a)$. This completes case (b) and proves the theorem. $\square$

A Banach space X is said to have the *(resp., weakly) alternative signs Banach-Saks property* (ABS) (resp., (wABS)) if, for every bounded (resp., weakly null) sequence $(x_n)_{n=1}^\infty$, there is a subsequence (y_n) and a sequence (ϵ_n) of signs such that the Cesàro averages are norm convergent to zero. That is,

$$\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{i=1}^{n} \epsilon_i y_i\right\| = 0. \tag{35}$$

It is shown by Rosenthal [30, 31] that if X has the (ABS), then X has the weak Banach-Saks property. In case that X does not contain ℓ_1, then (ABS) is equivalent to the weak Banach-Saks property [31]. It is clear that ℓ_1 does not have the (ABS) and has the weak Banach-Saks property. Hence a Banach space X has the (ABS) if and only if X has the weak Banach-Saks property and does not contain ℓ_1. The notion of (wABS) was introduced in [32] and it was shown that (wABS) is equivalent to (wBS).

Corollary 18. *Let* $I = (0,a)$ *for some* $0 < a \le \infty$. *Then the following are equivalent:*

(1) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *has the Banach-Saks property (BS);*

(2) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *is reflexive;*

(3) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *does not contain isomorphic copies of* c_0 *or* ℓ_1;

(4) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *is separable and does not contain an isomorphic copy of* ℓ_1;

(5) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *has the weak Banach-Saks property (wBS) and does not contain an isomorphic copy of* ℓ_1;

(6) *Orlicz-Lorentz space* $\Lambda_{\varphi,w}(I)$ *has the alternative signs Banach-Saks property (ABS).*

Proof. Recall that a Banach lattice is reflexive if and only if it does not contain isomorphic copies of c_0 or ℓ_1

[26, Theorem 1.c.5]. Hence we get (2) ⇔ (3). Recall also that a Banach lattice X is a KB space if and only if X does not contain an isomorphic copy of c_0 [33, Theorem 14.12]. That is, if X is a Köthe function space, then it is equivalent to the case in which X is order continuous and satisfies the Fatou property. Since an Orlicz-Lorentz space satisfies the Fatou property and the order continuity is equivalent to the separability of the Orlicz-Lorentz space, (3) ⇔ (4) follows.

Since a nonseparable Orlicz-Lorentz space contains ℓ_∞ and it does not have the weak Banach-Saks property, Theorem 17 shows that (4) ⇔ (5). The preceding argument of the corollary shows that (5) ⇔ (6). Since (6) implies both (5) and (2), (6) implies (1). The proof is done. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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