Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 423063, 10 pages
http://dx.doi.org/10.1155/2014/423063

Research Article

Cusped and Smooth Solitons for the Generalized Camassa-Holm Equation on the Nonzero Constant Pedestal

Dong Li, Yongan Xie, and Shengqiang Tang

School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin, Guangxi 541004, China

Correspondence should be addressed to Yongan Xie; xyan@guet.edu.cn

Received 21 November 2013; Revised 23 December 2013; Accepted 25 December 2013; Published 21 January 2014

Academic Editor: Weiguo Rui

We investigate the traveling solitary wave solutions of the generalized Camassa-Holm equation $u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}$ on the nonzero constant pedestal $\lim_{\xi\to\pm\infty}u(\xi) = A$. Our procedure shows that the generalized Camassa-Holm equation with nonzero constant boundary has cusped and smooth soliton solutions. Mathematical analysis and numerical simulations are provided for these traveling soliton solutions of the generalized Camassa-Holm equation. Some exact explicit solutions are obtained. We show some graphs to explain our these solutions.

1. Introduction

In 1993, Camassa and Holm [1] derived a nonlinear wave equation (Camassa-Holm equation)

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{1}$$

and obtained the peakon wave solution of the form $u = ce^{-|x-ct|}$. Whereafter, (1) has been researched by many authors [2–7]. Because (1) possesses rich dynamics and complex properties, recently, many authors are interested in its generalized forms. In particular, Liu and Qian [8] suggested a generalized Camassa-Holm equation,

$$u_t + 2ku_x - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}, \tag{2}$$

and obtained the explicit expressions of the peakon solution of (2). Afterwards, Tian and Song [9] gave some physical significance of this equation and obtained some peakon solutions with special wave speeds. Kalisch [10] studied the stability of solitary wave solution of (2). He et al. [11] constructed some exact traveling wave solutions by using the integral bifurcation method. Liu and Liang [12] studied the explicit nonlinear wave solutions and their bifurcations of (2).

When $k = 0$, (2) transforms into the following equation:

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{3}$$

For (3), there are some related works. Shen and Xu [13] discussed the existence of smooth and nonsmooth traveling waves. Khuri [14] obtained a singular wave solution composed of triangle functions. Wazwaz [15, 16] acquired eleven exact traveling wave solutions composed of triangle functions or hyperbolic functions. Liu and Ouyang [17] obtained a peakon solution composed of hyperbolic functions. Liu and Guo [18] investigated the periodic blow-up solutions and their limit forms. Wang and Tang [19] obtained two exact solutions. Yomba [20, 21] gave two methods, the sub-ODE method and the generalized auxiliary equation method, to obtain the exact solution of (3). Liu and Pan [22] studied the coexistence of multifarious solutions.

In this paper, we use the Qiao and Zhang method [23] to investigate the traveling solitary wave solutions of (3) on the nonzero constant pedestal

$$\lim_{\xi\to\pm\infty} u(\xi) = A \neq 0. \tag{4}$$

Since Qiao and Zhang presented this method, many authors applied it to different nonlinear models and obtained a variety of new type soliton solutions. Zhang and Qiao [24] discussed the traveling wave solutions for the Degasperis-Procesi equation

$$m_t + m_xu + 3mu_x = 0, \quad m = u - u_{xx} \tag{5}$$

on the nonzero constant pedestal and found new cusped and peaked soliton solutions. Qiao [25] proposed a new completely integrable wave equation:

$$m_t + m_x\left(u^2 - u_x^2\right) + 2m^2 u_x = 0, \quad m = u - u_{xx}, \quad (6)$$

and obtained new cusped, one-peak, W/M-shape-peaks soliton solutions. Later, Chen et al. [26, 27] studied the osmosis $K(2,2)$ equation

$$u_t \pm \left(u^2\right)_x \pm \left(u^2\right)_{xxx} = 0 \quad (7)$$

under the inhomogeneous boundary condition and obtained smooth, peaked, cusped soliton solutions of the osmosis $K(2,2)$ equation by using the phase portrait analytical technique. Wei et al. [28] investigated the generalized KP-MEW(2,2) equation

$$\left(u_t + \left(u^2\right)_x + \left(u^2\right)_{xxt}\right)_x + u_{yy} = 0 \quad (8)$$

on the nonzero constant pedestal and acquired smooth, peaked, cusped, and loop soliton solutions. More works on single peak soliton are reported [29–32].

2. Some Preliminary Results

Substituting $u(x,t) = u(\xi)$ and $\xi = x - ct$ into (3), we have

$$-cu' + cu''' + 3u^2u' = 2u'u'' + uu''', \quad (9)$$

where "$\prime$" is the derivative with respect to ξ. Integrating (9) once, we yield

$$-cu + cu'' + u^3 = \frac{1}{2}\left(u'\right)^2 + uu'' + g_1, \quad (10)$$

where $g_1 \in R$ is an integration constant.

Further, we get

$$\left(u'\right)^2 = \frac{u^4 - 2cu^2 - 4g_1u - 4g_2}{2(u-c)}, \quad (11)$$

where $g_2 \in R$ is an integration constant.

Let us solve (11) with the following boundary condition:

$$\lim_{\xi\to\pm\infty} u(\xi) = A \neq 0, \quad (12)$$

where A is a constant. Equation (11) can be cast into the following ordinary differential equation:

$$\left(u'\right)^2 = \frac{(u-A)^2\left(u^2 + 2Au + 3A^2 - 2c\right)}{2(u-c)}. \quad (13)$$

When $c - A^2 \geq 0$, then (13) reduces to

$$\left(u'\right)^2 = \frac{(u-A)^2(u-B_1)(u-B_2)}{2(u-c)}, \quad (14)$$

where

$$B_1 = -A + \sqrt{2\left(c - A^2\right)}, \qquad B_2 = -A - \sqrt{2\left(c - A^2\right)}. \quad (15)$$

Obviously, $B_1 \geq B_2$.

Remark 1. In the existing research on this method, the cases on $(u-A)^2(u-v_1)/(u-v_2)$ and $(u-A)^2(u-v_3)(u-v_4)/(u-v_5)^2$ have been discussed, but the case on $(u-A)^2(u-v_6)(u-v_7)/(u-v_8)$ (v_i $(i = 1,\ldots,8) \doteq$ constant) has not been discussed. So we consider it is very meaningful researching this new case on this method, and we can obtain some new soliton solutions from this case.

Definition 2. A wave function $u(\xi)$ is called smooth soliton solution, if $u(\xi)$ is smooth and $\lim_{\xi\uparrow\xi_0} u'(\xi) = -\lim_{\xi\downarrow\xi_0} u'(\xi) = 0$.

Definition 3. A wave function $u(\xi)$ is called cuspon solution, if $u(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi\uparrow\xi_0} u'(\xi) = -\lim_{\xi\downarrow\xi_0} u'(\xi) = +\infty$ (or $-\infty$).

Without loss of generality, we assume $\xi_0 = 0$.

3. The Parametric Conditions and Phase Portraits of Existence of Soliton Solutions of the Generalized Camassa-Holm Equation (3)

By virtue of the above analysis, we know that soliton solitons for the generalized Camassa-Holm Equation (3) must satisfy the following initial and boundary values problem:

$$\left(u'\right)^2 = \frac{(u-A)^2\left(u^2 + 2Au + 3A^2 - 2c\right)}{2(u-c)},$$
$$u(0) \in \{c, B_1, B_2\}, \quad (16)$$
$$\lim_{\xi\to\pm\infty} u(\xi) = A.$$

Lemma 4. *Suppose that one of the following five conditions holds:*

(i) $c < A^2$, $A \leq c$;

(ii) $c = A^2$, $A \leq c$;

(iii) $A^2 < c < 3A^2$, $A \leq c$;

(iv) $3A^2 = c$, $c < A$;

(v) $3A^2 < c$, $c < A$.

Then (3) *has trivial solution* $u(\xi) \equiv A$.

Proof. (i) If $c < A^2$ and $A \leq c$, then we have $u^2 + 2Au + 3A^2 - 2c > 0$. When $A < c$, (13) leads to $(u')^2 = (u-A)^2(u^2 + 2Au + 3A^2 - 2c)/2(u-c) \leq 0$. For $A = c$, (13) can be cast into $(u')^2 = (1/2)(u-A)(u^2 + 2Au + 3A^2 - 2c) \leq 0$.

(ii) When $c = A^2$ and $A \leq c$, then we have $u^2 + 2Au + 3A^2 - 2c = (u+A)^2 \geq 0$. If $A < c$, (14) changes into $(u')^2 = (u-A)^2(u+A)^2/2(u-c) \leq 0$. If $A = c$, (14) transforms into $(u')^2 = (1/2)(u-A)(u+A)^2 \leq 0$.

(iii) For $A^2 < c < 3A^2$ and $A \leq c$, then we obtain $u^2 + 2Au + 3A^2 - 2c = (u-B_1)(u-B_2) > 0$. If $A < c$, (14) leads to $(u')^2 = (u-A)^2(u-B_1)(u-B_2)/2(u-c) \leq 0$. If $A = c$, (14) changes into $(u')^2 = (1/2)(u-A)(u-B_1)(u-B_2) \leq 0$.

(iv) If $3A^2 = c$ and $c < A$, then we get $u^2 + 2Au + 3A^2 - 2c = (u-A)(u+3A) < 0$ and (14) can be cast into $(u')^2 = (u-A)^3(u+3A)/2(u-3A^2) \leq 0$.

(v) When $3A^2 < c$ and $c < A$, then we have $u^2 + 2Au + 3A^2 - 2c = (u - B_1)(u - B_2) < 0$ and (14) transforms into $(u')^2 = (u - A)^2(u - B_1)(u - B_2)/2(u - c) \leq 0$.

The fact that $(u')^2 \geq 0$ implies $u' = 0$ and $u(\xi) \equiv A$. □

Obviously, we get that the generalized Camassa-Holm Equation (3) with nonzero boundary condition has soliton solutions when A and c do not belong to the above five cases. Then we obtain the generalized Camassa-Holm Equation (3) with nonzero boundary condition having soliton solutions, when $c < A^2, c < A$; $c = A^2, c < A$; $A^2 < c < 3A^2, c < A$; and $3A^2 = c, A \leq c$; $3A^2 < c, A \leq c$.

For the cases on $A^2 < c < 3A^2, c < A, B_1 = c$; $3A^2 = c, A = c$, and $3A^2 < c, A \leq c, B_1 = c$, Liu and Qian [8] and Tian and Song [9] researched that the generalized Camassa-Holm Equation (3) has smooth soliton and peakon solutions as similar as follows:

$$u(\xi) = m\left(\frac{1 - e^{-n|\xi|}}{1 + e^{-n|\xi|}}\right)^2 + p,$$
$$u(\xi) = m\left(\frac{1 - e^{-n(|\xi|+s)}}{1 + e^{-n(|\xi|+s)}}\right)^2 + p, \tag{17}$$

where m, n, and p are constants, and $s > 0$ is an integration constant.

In fact, when $c < A^2, c < A$; $c = A^2, c < A$; $A^2 < c < 3A^2$, $c < A$; and $3A^2 = c, A < c$; $3A^2 < c, A \leq c$, the generalized Camassa-Holm Equation (3) has also other forms of the smooth soliton and cuspon. Because (11) is equivalent to the two-dimensional system

$$u' = y,$$
$$y' = \frac{u^3 - cu - (1/2)y^2 + g_1}{u - c}. \tag{18}$$

From (18), we can obtain the phase portraits of existence of soliton solutions of the generalized Camassa-Holm Equation (3) under the inhomogeneous boundary condition, when A and c belong to the above five cases (see Figure 1).

The phase portraits of (3) are shown in Figure 1 under different parametric conditions.

(1-1) $c < A^2, c < A$; (1-2) $c = A^2, c < A$; (1-3) $A^2 < c < 3A^2, c < A, c < B_1$; (1-4) $A^2 < c < 3A^2, c < A, B_1 < c$; (1-5) $c = 3A^2, 0 < A < c$; (1-6) $c = 3A^2, A < -1 < c$; (1-7) $c = 3A^2, -1 < A < 0 < c$; (1-8) $3A^2 < c, A = c$; (1-9) $3A^2 < c, A < c, c < B_1$; (1-10) $3A^2 < c, A < c, B_1 < c$.

4. Cusped and Smooth Solitons for the Generalized Camassa-Holm Equation (3)

In this section, by using the phase portrait analytical technique, which has been developed by Li and Dai [33], we get cusped and smooth soliton solutions of the generalized Camassa-Holm Equation (3) under the inhomogeneous boundary condition.

Case 1 ($c < A^2, c < A$). By the standard phase portrait analysis (see Figure 1(1-1)), we have $u(0) = c < A$. From (13), we yield

$$u' = -\frac{(u - A)\sqrt{u^2 + 2Au + 3A^2 - 2c}}{\sqrt{2(u - c)}}\operatorname{sign}(\xi). \tag{19}$$

Taking the integration of both sides of (19), we can obtain the implicit cuspon solution $u(\xi)$ defined by

$$-\frac{\sqrt{2W}}{c - W - A}\left[-S_1(u) + \frac{1}{1 - \alpha}\left(S_2(u) - \alpha\frac{\operatorname{sn}(u)}{\operatorname{dn}(u)}\right)\right] = |\xi| + K_1, \tag{20}$$

where $K_1 = 0$ is an integration constant,

$$S_1(u) = F\left[\arccos\left(\frac{W + c - u}{W - c + u}\right), k\right],$$
$$S_2(u) = \prod\left[\arccos\left(\frac{W + c - u}{W - c + u}\right), \frac{\alpha^2}{\alpha^2 - 1}, k\right],$$
$$W = \sqrt{(A + c)^2 + 2(A^2 - c)},$$
$$k = \sqrt{\frac{W - A - c}{2W}}, \qquad \alpha = \frac{c - W - A}{c + W - A}, \tag{21}$$
$$\operatorname{sn}(u) = \sqrt{1 - \left(\frac{W + c - u}{W - c + u}\right)^2},$$
$$\operatorname{dn}(u) = \sqrt{1 - k^2\operatorname{sn}^2(u)}.$$

Remark 5. $F(\phi, k)$ is the elliptic integral of first kind, and $\Pi(\phi, \tau, k)$ is the elliptic integral of third kind [34].

The profile of cusped soliton solution is shown in Figure 2(2-1).

Case 2 ($c = A^2, c < A$). Equation (14) can be cast into

$$(u')^2 = \frac{(u - A)^2(u + A)^2}{2(u - A^2)}. \tag{22}$$

By the standard phase portrait analysis (see Figure 1(1-2)), we have $u(0) = c < A$. From (22), we get

$$u' = -\frac{(u - A)(u + A)}{\sqrt{2(u - A^2)}}\operatorname{sign}(\xi). \tag{23}$$

Let $h(u) = -1/(u - A)(u + A)$; then $h(c) = -1/(c - A)(c + A)$, and

$$\int\sqrt{2(u - A^2)}h(u)\,du = \int\operatorname{sign}(\xi)\,d\xi. \tag{24}$$

Inserting $h(u) = h(c) + O(u)$ into (24) and using the initial condition $u(0) = c$, we obtain

$$\frac{1}{3}\left[2\left(u - A^2\right)\right]^{3/2}h(c)(1 + O(1)) = |\xi|. \tag{25}$$

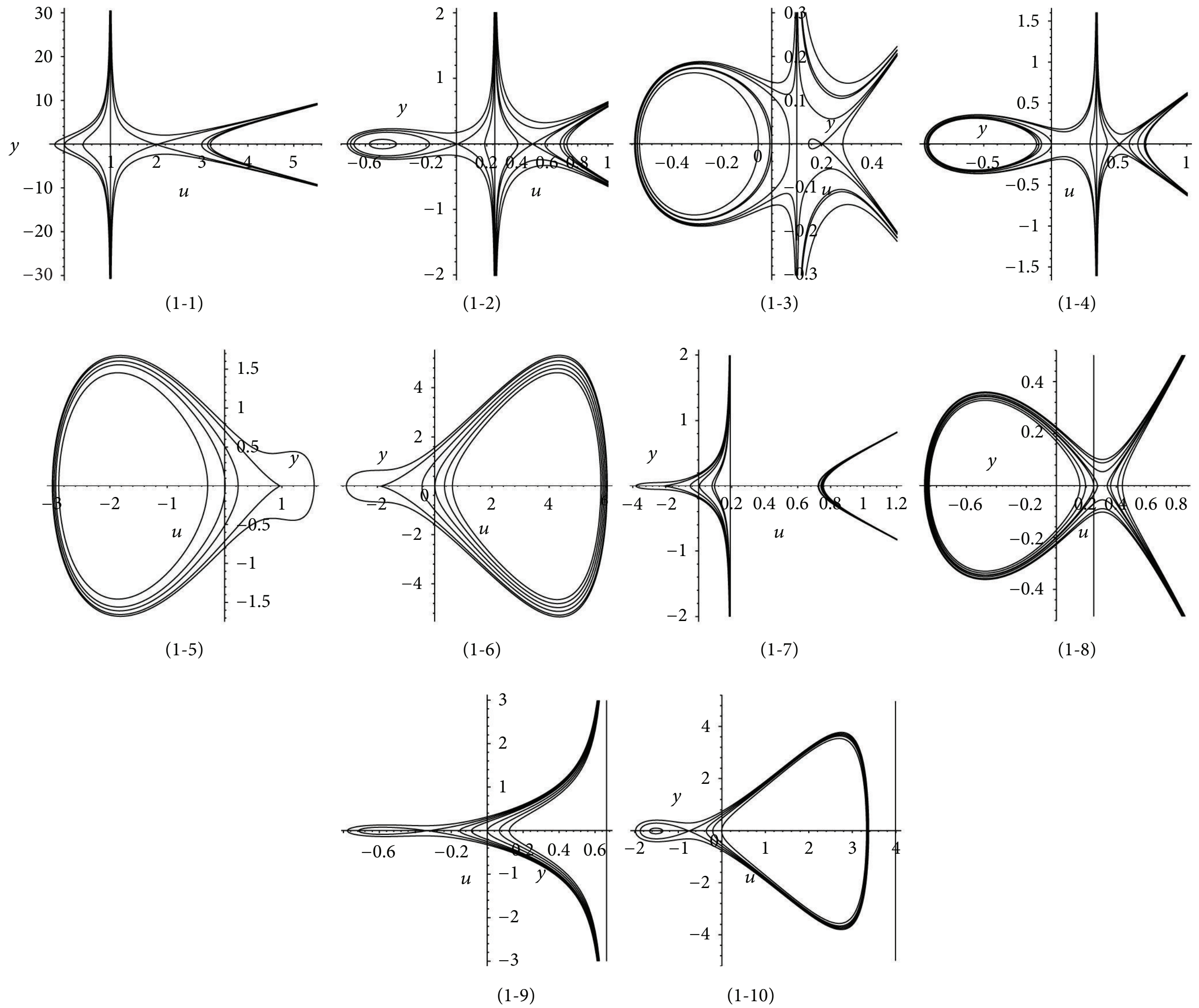

Figure 1: Phase portraits of (3) under the inhomogeneous boundary condition.

Thus,

$$u = \frac{1}{2}|\xi|^{2/3}\left(\frac{3}{h(c)}\right)^{2/3}(1+O(1))^{-2/3}+A^2, \quad \xi \longrightarrow 0, \tag{26}$$

which implies $u = O(|\xi|^{2/3})$. Therefore, we have

$$\begin{aligned} u &= \frac{1}{2}\left(\frac{3}{h(c)}\right)^{2/3}|\xi|^{2/3}+O(|\xi|)+A^2, \quad \xi \longrightarrow 0, \\ u' &= \frac{1}{3}\left(\frac{3}{h(c)}\right)^{2/3}|\xi|^{-1/3}+O(1), \quad \xi \longrightarrow 0. \end{aligned} \tag{27}$$

So we can get the implicit cuspon solution $u(\xi)$ defined by

$$\frac{A-1}{\sqrt{2(A-A^2)}}I_1(u)-\frac{\sqrt{2}(A+1)}{\sqrt{A+A^2}}I_2(u) = |\xi|+K_2, \tag{28}$$

where

$$\begin{aligned} I_1(u) &= \ln\left|\frac{\sqrt{u-A^2}-\sqrt{A-A^2}}{\sqrt{u-A^2}+\sqrt{A-A^2}}\right|, \\ I_2(u) &= \arctan\left(\sqrt{\frac{u-A^2}{A+A^2}}\right). \end{aligned} \tag{29}$$

Remark 6. The proof of other cuspons is similar to the above proof.

Because $u(0)=c$, the constant K_2 is defined by

$$K_{2_c} = \frac{A-1}{\sqrt{2(A-A^2)}}I_1(c)-\frac{\sqrt{2}(A+1)}{\sqrt{A+A^2}}I_2(c) = 0. \tag{30}$$

The profile of cusped soliton solution is shown in Figure 2 (2-2).

Case 3 ($A^2 < c < 3A^2$, $c < A$). In this case, we discuss two conditions: (1) $B_1 > c$; (2) $B_1 < c$.

(1) When $B_1 > c$, by the standard phase portrait analysis (see Figure 1(1-3)), we have $c < u(0) = B_1 < A$. From (14), we have

$$u' = -\frac{(u-A)\sqrt{(u-B_1)(u-B_2)}}{\sqrt{2(u-c)}}\operatorname{sign}(\xi). \tag{31}$$

As same as the above, we can obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$-\frac{2\sqrt{2}(B_1-c)}{(B_1-A)\sqrt{B_1-B_2}}V(u) = |\xi| + K_3, \tag{32}$$

where

$$V(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_1}{u-c}}\right), \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right). \tag{33}$$

For $u(0) = B_1$, the constant K_3 is defined by $K_{3_{B_1}} = 0$. For this smooth soliton solution, we get an exact explicit form [35]

$$\begin{aligned} u(\xi) &= \left(B_1 - c\cdot\sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{B_1-B_2}|\xi|}{2\sqrt{2}(B_1-c)}, \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right)\right)\right) \\ &\quad\times\left(1-\sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{B_1-B_2}|\xi|}{2\sqrt{2}(B_1-c)}, \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right)\right)\right)^{-1}. \end{aligned} \tag{34}$$

The profile of smooth soliton solution is shown in Figure 2(2-3).

(2) When $B_1 < c$, by the standard phase portrait analysis (see Figure 1(1-4)), we have $B_1 < u(0) = c < A$. Taking the integration of both sides of (31), we can yield the implicit cuspon solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}(c-B_1)}{(A-B_1)\sqrt{c-B_2}}(O_1(u) - O_2(u)) = |\xi| + K_4, \tag{35}$$

where

$$\begin{aligned} O_1(u) &= \prod\left(\arcsin\left(\sqrt{\frac{u-c}{u-B_1}}\right), \frac{A-B_1}{A-c}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right), \\ O_2(u) &= F\left(\arcsin\left(\sqrt{\frac{u-c}{u-B_1}}\right), \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \end{aligned} \tag{36}$$

By view of $u(0) = c$, the constant K_4 is defined by $K_{4_c} = 0$. The profile of cusped soliton solution is shown in Figure 2(2-4).

Case 4 ($3A^2 = c$, $A < c$). (1) When $0 < A < c$, by the standard phase portrait analysis (see Figure 1(1-5)), we have $u(0) = B_2 < 0 < B_1 = A < c$. Equation (14) transforms into

$$(u')^2 = \frac{(u-A)^2(u-A)(u+3A)}{2(u-3A^2)}. \tag{37}$$

From (37), we have

$$u' = -(u-A)\sqrt{\frac{(u-A)(u+3A)}{2(u-3A^2)}}\operatorname{sign}(\xi). \tag{38}$$

Taking the integration of both sides of (38), we can obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{\sqrt{6}}{2}\sqrt{\frac{A(A+1)}{A}}\left(\frac{3A-1}{3A+3}P_1(u) + \frac{4}{3A+3}P_2(u)\right) = |\xi| + K_5, \tag{39}$$

where

$$\begin{aligned} P_1(u) &= \prod\left(\arcsin\left(\sqrt{\frac{3A+u}{4A}}\right), 1, \frac{2}{\sqrt{3+3A}}\right), \\ P_2(u) &= F\left(\arcsin\left(\sqrt{\frac{3A+u}{4A}}\right), \frac{2}{\sqrt{3+3A}}\right). \end{aligned} \tag{40}$$

For $u(0) = B_2$, we obtain $K_{5_{B_2}} = 0$. The profile of smooth soliton solution is shown in Figure 2(2-5).

(2) When $A < 0$, by virtue of (37), we have

$$u' = (u-A)\sqrt{\frac{(u-A)(u+3A)}{2(u-3A^2)}}\operatorname{sign}(\xi). \tag{41}$$

In this case, we discuss two conditions: (i) $A < -1$; (ii) $-1 < A < 0$.

(i) When $A < -1$, by the standard phase portrait analysis (see Figure 1(1-6)), we have $A = B_2 \le u \le B_1 < c$. Taking the integration of (41) on the interval $[A, B_1]$, thus, we obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{-3\sqrt{2}(1+A)}{2\sqrt{3A^2-A}}H(u) = |\xi| + Q_1, \tag{42}$$

where

$$H(u) = \prod\left(\arcsin\left(\sqrt{\frac{(3A-1)(3A+u)}{4(3A^2-u)}}\right), 1, \sqrt{\frac{-4}{3A-1}}\right). \tag{43}$$

Because $u(0) = B_1$, we obtain $Q_{1_{B_1}} = 0$. For this smooth soliton solution, we get an exact explicit form

$$u(\xi) = \left(12A^2\sin^2\left(\Pi^{-1}\left(\frac{2\sqrt{3A^2-A}|\xi|}{-3\sqrt{2}(1+A)}, 1, \sqrt{\frac{-4}{3A-1}}\right)\right) - 3A(3A-1)\right) \times \left((3A-1) + 4\sin^2\left(\Pi^{-1}\left(\frac{2\sqrt{3A^2-A}|\xi|}{-3\sqrt{2}(1+A)}, 1, \sqrt{\frac{-4}{3A-1}}\right)\right)\right)^{-1}. \quad (44)$$

The profile of smooth soliton solution is shown in Figure 2(2-6).

(ii) When $-1 < A < 0$, by the standard phase portrait analysis (see Figure 1(1-7)), we have $A = B_2 \le u < c < B_1$. Integrating (41) on the interval $[A, c)$, we obtain the implicit cuspon solution $u(\xi)$ defined by

$$\frac{3\sqrt{2}(A+1)}{4\sqrt{-A}}\left[R_1(u) - R_2(u)\right] = |\xi| + Q_2, \quad (45)$$

where

$$R_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{4(3A^2-u)}{(3A-1)(3A+u)}}\right), 1, \sqrt{\frac{3A-1}{-4}}\right),$$
$$R_2(u) = F\left(\arcsin\left(\sqrt{\frac{4(3A^2-u)}{(3A-1)(3A+u)}}\right), \sqrt{\frac{3A-1}{-4}}\right). \quad (46)$$

From $u(0) = c$, we obtain $Q_{2_c} = 0$. The profile of cusped soliton solution is shown in Figure 2(2-7).

Case 5 ($3A^2 < c$, $A \le c$). (1) When $A = c$, (14) can be cast into

$$(u')^2 = \frac{1}{2}(u-A)(u-B_1)(u-B_2). \quad (47)$$

Because $3A^2 < c$, we have $B_2 < A < B_1$. By the standard phase portrait analysis (see Figure 1(1-8)), we get $B_2 \le u \le A = c < B_1$. By view of (47), we obtain

$$u' = \frac{\sqrt{2}}{2}\sqrt{(u-A)(u-B_1)(u-B_2)}\,\mathrm{sign}(\xi). \quad (48)$$

As same as the above, we can get the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}}{\sqrt{B_1-B_2}}F\left(\arcsin\left(\sqrt{\frac{u-B_2}{A-B_2}}\right), \sqrt{\frac{A-B_2}{B_1-B_2}}\right) = |\xi| + K_6. \quad (49)$$

For $u(0) = B_2$, the constant K_6 is defined by $K_{6_{B_2}} = 0$. For this smooth soliton solution, we can give an exact explicit form

$$u(\xi) = (A-B_2) \times \sin^2\left(F^{-1}\left(\frac{\sqrt{(B_1-B_2)}|\xi|}{2\sqrt{2}}, \sqrt{\frac{A-B_2}{B_1-B_2}}\right)\right) + B_2. \quad (50)$$

The profile of smooth soliton solution is shown in Figure 2(2-8).

(2) When $A < c$, we discuss three cases: (i) $B_1 > c$; (ii) $B_1 < c$; (iii) $B_1 = c$.

(i) When $B_1 > c$, by the standard phase portrait analysis (see Figure 1(1-9)), we have $u(0) = B_2 \le u < A$ or $A \le u < u(0) = c$.

For $u(0) = B_2$, from (14), we have

$$u' = (A-u)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi). \quad (51)$$

Taking the integration of (51) on the interval $[B_2, A]$, thus, we obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}}{\sqrt{B_1-B_2}}\left(\frac{c-A}{A-B_2}G_1(u) + G_2(u)\right) = |\xi| + W_1, \quad (52)$$

where

$$G_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_2}{c-B_2}}\right), \frac{c-B_2}{A-B_2}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right),$$
$$G_2(u) = F\left(\arcsin\left(\sqrt{\frac{u-B_2}{c-B_2}}\right), \sqrt{\frac{c-B_2}{B_1-B_2}}\right). \quad (53)$$

The constant W_1 is defined by $W_{1_{B_2}} = 0$. The profile of smooth soliton solution is shown in Figure 2(2-9).

For $u(0) = c$, by view of (14), we obtain

$$u' = (u-A)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi). \quad (54)$$

Taking the integration of both sides of (54), thus, we can yield the implicit solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}(B_1-c)}{(B_1-A)\sqrt{B_1-B_2}}\left[E_1(u) - E_2(u)\right] = |\xi| + W_2, \quad (55)$$

where

$$E_1(u) = \prod\left[\arcsin\left(\sqrt{\frac{(B_1-B_2)(c-u)}{(c-B_2)(B_1-u)}}\right), \frac{(c-B_2)(B_1-A)}{(B_1-B_2)(c-A)}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right],$$

$$E_2(u) = F\left[\arcsin\left(\sqrt{\frac{(B_1-B_2)(c-u)}{(c-B_2)(B_1-u)}}\right), \sqrt{\frac{c-B_2}{B_1-B_2}}\right]. \quad (56)$$

For $u(0)=c$, W_2 is defined by $W_{2_c}=0$. The profile of cusped soliton solution is shown in Figure 2(2-10).

(ii) When $B_1 < c$, by the standard phase portrait analysis (see Figure 1(1-10)), we have $B_2 < A < B_1 < c$.

For $u(0)=B_2$, from (14), we get

$$u' = (A-u)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\operatorname{sign}(\xi); \quad (57)$$

we yield

$$\Theta_1(u) \equiv \int \theta_1(u)\,du = |\xi| + M_1, \quad (58)$$

where

$$\theta_1(u) = -\frac{1}{(u-A)}\sqrt{\frac{2(u-c)}{(u-B_1)(u-B_2)}}, \quad (59)$$

and M_1 is an integration constant. Taking the integration of $\theta_1(u)$ on the interval $[B_2, A]$, thus, we obtain the implicit solution $u(\xi)$ defined by

$$\begin{aligned}\Theta_1(u) &= \frac{2\sqrt{2(c-B_2)}}{B_1-B_2} \times \left[\left(\frac{B_1-B_2}{A-B_2} - \frac{B_1-B_2}{c-B_2}\right)T_1(u) + \frac{B_1-B_2}{c-B_2}T_2(u)\right] \\ &= |\xi| + M_1,\end{aligned} \quad (60)$$

where

$$T_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_2}{B_1-B_2}}\right), \frac{B_1-B_2}{A-B_2}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right),$$

$$T_2(u) = F\left(\arcsin\left(\sqrt{\frac{u-B_2}{B_1-B_2}}\right), \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \quad (61)$$

The constant M_1 is defined by $M_{1_{B_2}}=0$. Because $\theta_1(u) > 0$, we know that the $\Theta_1(u)$ is strictly increasing on the interval $[B_2, A]$;

$$\Theta_1(u) = \Theta_{1_{[B_2,A]}}(u) \quad (62)$$

has the inverse denoted by $u(\xi) = \Theta_1^{-1}(|\xi|)$. The profile of smooth soliton solution is shown in Figure 2(2-11).

For $u(0)=B_1$, by view of (14), we obtain

$$u' = (u-A)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\operatorname{sign}(\xi),$$
$$\Theta_2(u) \equiv \int \theta_2(u)\,du = |\xi| + M_2, \quad (63)$$

where

$$\theta_2(u) = \frac{1}{(u-A)}\sqrt{\frac{2(u-c)}{(u-B_1)(u-B_2)}}, \quad (64)$$

and M_2 is an integration constant. Taking the integration of $\theta_2(u)$ on the interval $[A, B_1]$, thus, we obtain the implicit solution $u(\xi)$ defined by

$$\Theta_2(u) = -\frac{2\sqrt{2}(c-B_1)}{(A-B_1)\sqrt{c-B_2}}N(u) = |\xi| + M_2, \quad (65)$$

where

$$N(u) = \prod\left(\arcsin\left(\sqrt{\frac{(c-B_2)(B_1-u)}{(B_1-B_2)(c-u)}}\right), \frac{(B_1-B_2)(A-c)}{(c-B_2)(A-B_1)}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \quad (66)$$

When $u(0)=B_1$, the constant M_2 is defined by $M_{2_{B_1}}=0$. From $\theta_2(u) > 0$, we know that the $\Theta_2(u)$ is strictly increasing on the interval $[A, B_1]$;

$$\Theta_2(u) = \Theta_{2_{[A,B_1]}}(u) \quad (67)$$

has the inverse denoted by $u(\xi) = \Theta_2^{-1}(|\xi|)$.

For this smooth soliton solution, we give an exact explicit form

$$u(\xi) = \left(c(B_1-B_2) \times \sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{c-B_2}\,|\xi|}{2\sqrt{2}(B_1-c)}, \frac{(B_1-B_2)(A-c)}{(c-B_2)(A-B_1)}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right)\right) - B_1(c-B_2)\right)$$

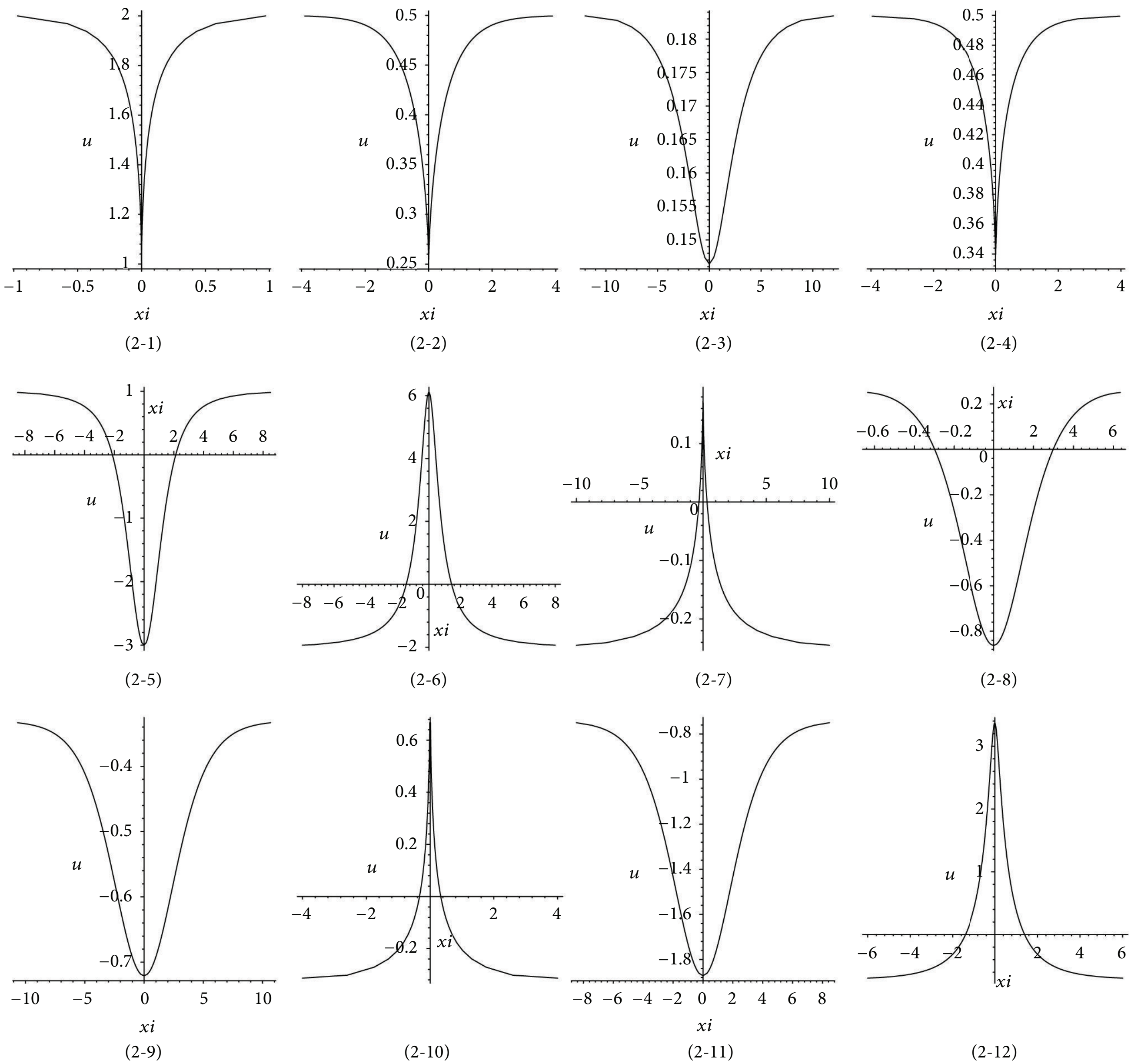

Figure 2: Profiles of soliton solution.

$$\times\Bigg(\left(B_1 - B_2\right) \times \sin^2\left(\Pi^{-1}\left(\frac{\left(A - B_1\right)\sqrt{c - B_2}\,|\xi|}{2\sqrt{2}\left(B_1 - c\right)}, \frac{\left(B_1 - B_2\right)\left(A - c\right)}{\left(c - B_2\right)\left(A - B_1\right)}, \sqrt{\frac{B_1 - B_2}{c - B_2}}\right)\right) - c + B_2\Bigg)^{-1}. \tag{68}$$

The profile of smooth soliton solution is shown in Figure 2 (2-12).

The profile of soliton solution of (3) is shown in Figure 2 under special values of c and A.

(2-1) $c = 1$, $A = 2$; (2-2) $c = 1/4$, $A = 1/2$; (2-3) $c = 1/10$, $A = 1/5$; (2-4) $c = 1/3$, $A = 1/2$; (2-5) $c = 3$, $A = 1$; (2-6) $c = 12$, $A = -2$; (2-7) $c = 3/16$, $A = -1/4$; (2-8) $c = A = 1/4$; (2-9, 10) $c = 2/3$, $A = -1/3$; (2-11, 12) $c = 4$, $A = -3/4$.

5. Conclusion

In this paper, we research the soliton solutions of the generalized Camassa-Holm Equation (3) under inhomogeneous boundary condition. The parametric conditions and phase portraits of existence of the cuspon and smooth soliton solutions are given. We obtain cuspon and smooth soliton solutions of the generalized Camassa-Holm Equation (3). Some exact explicit solutions are obtained. We show some graphs to explain our these solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11361017 and 11161013), the Natural Science Foundation of Guangxi (2012GXNSFAA053003), and the Innovation Project of GUET Graduate Education (XJYC2012021 and XJYC2012022).

References

[1] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[2] T. Qian and M. Tang, "Peakons and periodic cusp waves in a generalized Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 12, no. 7, pp. 1347–1360, 2001.

[3] Z.-R. Liu, R.-Q. Wang, and Z.-J. Jing, "Peaked wave solutions of Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 19, no. 1, pp. 77–92, 2004.

[4] A. Constantin and W. A. Strauss, "Stability of peakons," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 603–610, 2000.

[5] A. Constantin and L. Molinet, "Orbital stability of solitary waves for a shallow water equation," *Physica D*, vol. 157, no. 1-2, pp. 75–89, 2001.

[6] A. Constantin, V. Gerdjikov, and R. Ivanov, "Inverse scattering transform for the Cammassa-Holm equation," *Inverse Problems*, vol. 22, no. 6, pp. 2197–2207, 2006.

[7] S. Abbasbandy and E. J. Parkes, "Solitary smooth hump solutions of the Camassa-Holm equation by means of the homotopy analysis method," *Chaos, Solitons & Fractals*, vol. 36, no. 3, pp. 581–591, 2008.

[8] Z. Liu and T. Qian, "Peakons and their bifurcation in a generalized Camassa-Holm equation," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 11, no. 3, pp. 781–792, 2001.

[9] L. Tian and X. Song, "New peaked solitary wave solutions of the generalized Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 19, no. 3, pp. 621–637, 2004.

[10] H. Kalisch, "Stability of solitary waves for a nonlinearly dispersive equation," *Discrete and Continuous Dynamical Systems. Series A*, vol. 10, no. 3, pp. 709–717, 2004.

[11] B. He, W. Rui, C. Chen, and S. Li, "Exact travelling wave solutions of a generalized Camassa-Holm equation using the integral bifurcation method," *Applied Mathematics and Computation*, vol. 206, no. 1, pp. 141–149, 2008.

[12] Z. Liu and Y. Liang, "The explicit nonlinear wave solutions and their bifurcations of the generalized Camassa-Holm equation," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 21, no. 11, pp. 3119–3136, 2011.

[13] J. Shen and W. Xu, "Bifurcations of smooth and non-smooth travelling wave solutions in the generalized Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 26, no. 4, pp. 1149–1162, 2005.

[14] S. A. Khuri, "New ansätz for obtaining wave solutions of the generalized Camassa-Holm equation," *Chaos, Solitons and Fractals*, vol. 25, no. 3, pp. 705–710, 2005.

[15] A. Wazwaz, "Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations," *Physics Letters A*, vol. 352, no. 6, pp. 500–504, 2006.

[16] A. Wazwaz, "New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 130–141, 2007.

[17] Z. Liu and Z. Ouyang, "A note on solitary waves for modified forms of Camassa-Holm and Degasperis-Procesi equations," *Physics Letters A*, vol. 366, no. 4-5, pp. 377–381, 2007.

[18] Z. Liu and B. Guo, "Periodic blow-up solutions and their limit forms for the generalized Camassa-Holm equation," *Progress in Natural Science*, vol. 18, no. 3, pp. 259–266, 2008.

[19] Q. Wang and M. Tang, "New exact solutions for two nonlinear equations," *Physics Letters A*, vol. 372, no. 17, pp. 2995–3000, 2008.

[20] E. Yomba, "The sub-ODE method for finding exact travelling wave solutions of generalized nonlinear Camassa-Holm, and generalized nonlinear Schrödinger equations," *Physics Letters A*, vol. 372, no. 3, pp. 215–222, 2008.

[21] E. Yomba, "A generalized auxiliary equation method and its application to nonlinear Klein-Gordon and generalized nonlinear Camassa-Holm equations," *Physics Letters A*, vol. 372, no. 7, pp. 1048–1060, 2008.

[22] Z. Liu and J. Pan, "Coexistence of multifarious explicit nonlinear wave solutions for modified forms of Camassa-Holm and Degaperis-Procesi equations," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 19, no. 7, pp. 2267–2282, 2009.

[23] Z. Qiao and G. Zhang, "On peaked and smooth solitons for the Camassa-Holm equation," *Europhysics Letters*, vol. 73, no. 5, pp. 657–663, 2006.

[24] G. Zhang and Z. Qiao, "Cuspons and smooth solitons of the Degasperis-Procesi equation under inhomogeneous boundary condition," *Mathematical Physics, Analysis and Geometry*, vol. 10, no. 3, pp. 205–225, 2007.

[25] Z. Qiao, "A new integrable equation with cuspons and W/M-shape-peaks solitons," *Journal of Mathematical Physics*, vol. 47, Article ID 112701, 9 pages, 2006.

[26] A. Chen and J. Li, "Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous boundary condition," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 758–766, 2010.

[27] L. Zhang, A. Chen, and J. Tang, "Special exact soliton solutions for the $K(2,2)$ equation with non-zero constant pedestal," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4448–4457, 2011.

[28] M. Wei, S. Tang, H. Fu, and G. Chen, "Single peak solitary wave solutions for the generalized KP-MEW$(2,2)$ equation under boundary condition," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 8979–8990, 2013.

[29] Z. Qiao, "The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic

submanifold," *Communications in Mathematical Physics*, vol. 239, no. 1-2, pp. 309–341, 2003.

[30] Z. Qiao and E. Fan, "Negative-order Korteweg-de Vries equations," *Physical Review E*, vol. 86, no. 1, Article ID 016601, 20 pages, 2012.

[31] Z. Qiao, "New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W-shape peak solitons," *Journal of Mathematical Physics*, vol. 48, Article ID 082701, 20 pages, 2007.

[32] J. Li and Z. Qiao, "Bifurcations and exact traveling wave solutions for a generalized Camassa-Holm equation," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 23, Article ID 1350057, 2013.

[33] J. Li and H. Dai, *On the Study of Singular Nonlinear Traveling Wave Equations: Dynamical System Approach*, Science Press, Beijing, China, 2007.

[34] B. Carlson, "Computing elliptic integrals by duplication," *Numerische Mathematik*, vol. 33, no. 1, pp. 1–16, 1979.

[35] J. Li and Z. Liu, "Smooth and non-smooth traveling waves in a nonlinearly dispersive equation," *Applied Mathematical Modelling*, vol. 25, no. 1, pp. 41–56, 2000.