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Research Article

The Solutions to Matrix Equation $AX = B$ with Some Constraints

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Let P be a given Hermitian matrix satisfying $P^2 = I$. Using the eigenvalue decomposition of P, we consider the least squares solutions to the matrix equation $AX = B$ with the constraints $PX = XP$ and $X^* = X$. A similar problem of this matrix equation with generalized constrained is also discussed.

1. Introduction

Throughout we denote the complex $m \times n$ matrix space by $\mathbb{C}^{m\times n}$. The symbols I, A^*, A^{-1}, and $\|A\|$ stand for the identity matrix with the appropriate size, the conjugate transpose, the inverse, and the Frobenius norm of $A \in \mathbb{C}^{m\times n}$, respectively.

It is a very active research topic to study solutions to various matrix equations [1–4]. There are many authors who have investigated the classical matrix equation

$$AX = B \tag{1}$$

with different constraints such as symmetric, reflexive, Hermitian-generalized Hamiltonian, and repositive definite [5–9]. By special matrix decompositions such as singular value decompositions (SVDs) and CS decompositions [10–12], Hu and his collaborators [13–15], Dai [16], and Don [17] have presented the existence conditions and detailed representations of constrained solutions for (1) with corresponding constraints, respectively. For instance, Peng and Hu [18] presented the eigenvectors-involved solutions to (1) with reflexive and antireflexive constraints; Wang and Yu [19] derived the bi(skew-)symmetric solutions and the bi(skew-)symmetric least squares solutions with the minimum norm to this matrix equation; Qiu and Wang [20] proposed an eigenvectors-free method to (1) with $PX = XP$ and $X^* = sX$ constraints, where P is a Hermitian involutory matrix and $s = \pm 1$.

Inspired by the work mentioned above, we focus on the matrix equation (1) with $PX = XP$ and $X^* = X$ constraints, which can be described as follows: find X such that

$$\left\{\|AX - B\|^2 = \min, PX = XP, X^* = X\right\}. \tag{2}$$

Moreover, we also discuss the least squares solutions of (1) with $PX = XGPG^*$ and $X^* = X$ constraints, where G is a given unitary matrix of order n.

In Section 2, we present the least squares solutions to the matrix equation (1) with the constraints $PX = XP$ and $X^* = X$. In Section 3, we derive the least squares solutions to the matrix equation (1) with the constraints $PX = XGPG^*$ and $X^* = X$. In Section 4, we give an algorithm and a numerical example to illustrate our results.

2. Least Squares Solutions to the Matrix Equation (1) with the Constraints $PX = XP$ and $X^* = X$

It is required to transform the constrained problem to unconstrained one. To this end, let

$$P = U \operatorname{diag}\left(I_k, -I_{n-k}\right) U^* \tag{3}$$

be the eigenvalue decomposition of the Hermitian matrix P with unitary matrix U. Obviously, $PX = XP$ holds if and only if

$$\operatorname{diag}\left(I_k, -I_{n-k}\right) \overline{X} = \overline{X} \operatorname{diag}\left(I_k, -I_{n-k}\right), \tag{4}$$

where $\overline{X} = U^* X U$. Partitioning

$$\overline{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad X_{11} \in \mathbb{C}^{k\times k},\ X_{22} \in \mathbb{C}^{(n-k)\times(n-k)}, \quad (5)$$

(4) is equivalent to

$$X_{12} = -X_{12}, \qquad X_{21} = -X_{21}. \quad (6)$$

Therefore,

$$X = U \operatorname{diag}(X_{11}, X_{22}) U^*, \quad X_{11} \in \mathbb{C}^{k\times k}, \quad X_{22} \in \mathbb{C}^{(n-k)\times(n-k)}. \quad (7)$$

The constraint $X^* = X$ is equivalent to

$$X = U \operatorname{diag}(X_1, X_2) U^*, \quad X_i^* = X_i,\ i = 1, 2, \quad (8)$$

with $X_1 \in \mathbb{C}^{k\times k}$, $X_2 \in \mathbb{C}^{(n-k)\times(n-k)}$.

Partition $U = (U_1, U_2)$ and denote

$$A_1 = AU_1, \qquad A_2 = AU_2, \qquad B_1 = BU_1, \qquad B_2 = BU_2; \quad (9)$$

then assume that the singular value decomposition of A_1 and A_2 is as follows:

$$A_1 = M_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} N_1^*, \qquad A_2 = M_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} N_2^*, \quad (10)$$

where M_1, M_2, N_1, and N_2 are unitary matrices, $\Sigma_1 = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$, $\alpha_i > 0$ $(i = 1, \ldots, r)$, $r = \operatorname{rank}(A_1)$, $\Sigma_2 = \operatorname{diag}(\beta_1, \ldots, \beta_l)$, $\beta_j > 0$ $(j = 1, \cdots, l)$, and $l = \operatorname{rank}(A_2)$.

Theorem 1. *Given $A, B \in \mathbb{C}^{m\times n}$. Then the least squares solutions to the matrix equation* (1) *with the constraints $PX = XP$ and $X^* = X$ can be expressed as*

$$X = U \operatorname{diag}\begin{pmatrix} N_1 \begin{pmatrix} \dfrac{\Sigma_1^{-1} B_{11} + B_{11}^* \Sigma_1^{-1}}{2} & \Sigma_1^{-1} B_{12} \\ B_{12}^* \Sigma_1^{-1} & X_{14} \end{pmatrix} N_1^* & 0 \\ 0 & N_2 \begin{pmatrix} \dfrac{\Sigma_2^{-1} B_{21} + B_{21}^* \Sigma_2^{-1}}{2} & \Sigma_2^{-1} B_{22} \\ B_{22}^* \Sigma_2^{-1} & X_{24} \end{pmatrix} N_2^* \end{pmatrix} U^*, \quad (11)$$

where $X_{14} = X_{14}^*$ and $X_{24} = X_{24}^*$ are arbitrary matrix.

Proof. According to (8) and the unitary invariance of Frobenius norm

$$\begin{aligned} \|AX - B\| &= \|AU \operatorname{diag}(X_1, X_2) U^* - B\| \\ &= \|AU \operatorname{diag}(X_1, X_2) - BU\|. \end{aligned} \quad (12)$$

By (9), the least squares problem is equivalent to

$$\|AX - B\| = \|(A_1 X_1 - B_1, A_2 X_2 - B_2)\|. \quad (13)$$

We get

$$\|AX - B\|^2 = \|A_1 X_1 - B_1\|^2 + \|A_2 X_2 - B_2\|^2. \quad (14)$$

According to (10), the least squares problem is equivalent to

$$\begin{aligned} \|AX - B\|^2 &= \left\| M_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} N_1^* X_1 - B_1 \right\|^2 + \left\| M_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} N_2^* X_2 - B_2 \right\|^2 \\ &= \left\| \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} N_1^* X_1 N_1 - M_1^* B_1 N_1 \right\|^2 + \left\| \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} N_2^* X_2 N_2 - M_2^* B_2 N_2 \right\|^2. \end{aligned} \quad (15)$$

Assume that

$$N_1^* X_1 N_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{bmatrix}, \qquad N_2^* X_2 N_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{23} & X_{24} \end{bmatrix},$$
$$M_1^* B_1 N_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{13} & B_{14} \end{bmatrix}, \qquad M_2^* B_2 N_2 = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}. \quad (16)$$

Then we have

$$\begin{aligned} \|AX - B\|^2 &= \left\| \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} \\ B_{13} & B_{14} \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{21} & X_{22} \\ X_{23} & X_{24} \end{bmatrix} - \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix} \right\|^2 \\ &= \|\Sigma_1 X_{11} - B_{11}\|^2 + \|\Sigma_2 X_{21} - B_{21}\|^2 \\ &\quad + \|\Sigma_1 X_{12} - B_{12}\|^2 + \|\Sigma_2 X_{22} - B_{22}\|^2 \\ &\quad + \|B_{13}\|^2 + \|B_{14}\|^2 + \|B_{23}\|^2 + \|B_{24}\|^2. \end{aligned} \quad (17)$$

Hence

$$\|AX - B\|^2 = \min \tag{18}$$

is solvable if and only if there exist $X_{11}, X_{12}, X_{21}, X_{22}$ such that

$$\begin{aligned} \|\Sigma_1 X_{11} - B_{11}\|^2 &= \min, \qquad \|\Sigma_1 X_{12} - B_{12}\|^2 = \min, \\ \|\Sigma_2 X_{21} - B_{21}\|^2 &= \min, \qquad \|\Sigma_2 X_{22} - B_{22}\|^2 = \min. \end{aligned} \tag{19}$$

It follows from (19) that

$$\begin{aligned} X_{11} &= \frac{\Sigma_1^{-1} B_{11} + B_{11}^* \Sigma_1^{-1}}{2}, \qquad X_{12} = \Sigma_1^{-1} B_{12}, \\ X_{21} &= \frac{\Sigma_2^{-1} B_{21} + B_{21}^* \Sigma_2^{-1}}{2}, \qquad X_{22} = \Sigma_2^{-1} B_{22}. \end{aligned} \tag{20}$$

Substituting (20) into (16) and then into (8), we can get that the form of X is (11). □

3. Least Squares Solutions to the Matrix Equation (1) with the Constraints $PX = XGPG^*$ and $X^* = X$

In this section, we generalize the constraints $PX = XP$ to $PX = XGPG^*$, where G is a given unitary matrix of order n. Obviously, the constraint is equal to

$$PXG = XGP. \tag{21}$$

Notice that (1) can be equivalently rewritten in

$$AXG = BG. \tag{22}$$

Denoting by $Y = XG$ and setting $C = BG$, the equation becomes

$$AY = C, \tag{23}$$

with the constraints $PY = YP$ and $Y^* = Y$.

Therefore, the least squares solutions to matrix equation (1) with the constraints $PX = XGPG^*$ and $X^* = X$ can be solved similar to Theorem 1.

Theorem 2. *Given $A, B \in \mathbb{C}^{m\times n}$. Then the least squares solutions to the matrix equation* (1) *with the constraints $PX = XGPG^*$ and $X^* = X$ can be expressed as*

$$X = U \operatorname{diag}\begin{pmatrix} N_1 \begin{pmatrix} \dfrac{\Sigma_1^{-1} C_{11} + C_{11}^* \Sigma_1^{-1}}{2} & \Sigma_1^{-1} C_{12} \\ C_{12}^* \Sigma_1^{-1} & Y_{14} \end{pmatrix} N_1^* & 0 \\ 0 & N_2 \begin{pmatrix} \dfrac{\Sigma_2^{-1} C_{21} + C_{21}^* \Sigma_2^{-1}}{2} & \Sigma_2^{-1} C_{22} \\ C_{22}^* \Sigma_2^{-1} & Y_{24} \end{pmatrix} N_2^* \end{pmatrix} U^* G^*, \tag{24}$$

where $Y_{14} = Y_{14}^*$ and $Y_{24} = Y_{24}^*$ are arbitrary matrix.

4. An Algorithm and Numerical Examples

Based on the main results of this paper, we in this section propose an algorithm for finding the least squares solutions to the matrix equation $AX = B$ with the constraints $PX = XP$ and $X^* = X$. All the tests are performed by MATLAB 6.5 which has a machine precision of around 10^{-16}.

Algorithm 3. (1) Input $A, B \in \mathbb{C}^{m\times n}, P \in \mathbb{C}^{n\times n}$ and compute $U \in \mathbb{C}^{n\times n}$, $I_k \in \mathbb{C}^{k\times k}$, $-I_{n-k} \in \mathbb{C}^{(n-k)\times(n-k)}$ by the eigenvalue decomposition to P.

(2) Compute A_1, A_2, B_1, B_2 according to (9).

(3) Compute $N_1, N_2, M_1, M_2, \Sigma_1, \Sigma_2$ by the singular value decomposition of A_1, A_2.

(4) Compute $B_{11}, B_{12}, B_{21}, B_{22}$ according to (16).

(5) Compute X by Theorem 1.

Example 4. Suppose

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.2i & 0 & 0 \\ 0 & 0 & 0 & 0.8i \end{bmatrix}, \\ B &= \begin{bmatrix} -3 & -0.8i & -1-3i & -1 \\ -1-i & -1 & 9i & -7 \\ -2 & -2 & 2i & -2 \end{bmatrix}, \\ P &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{25}$$

Applying Algorithm 3, we obtain the following:

$$U = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1.2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.8i & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.8 & 3.i \\ -i & -1+i \\ -2i & 2i \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 & -1-3i \\ -7 & 9i \\ -2 & 2i \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 1 & 0 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$\Sigma_1 = [1.2], \qquad \Sigma_2 = [0.8], \qquad B_{11} = [i],$$

$$B_{12} = [-1-i], \qquad B_{21} = [2i], \qquad B_{22} = [-2],$$

$$X = \begin{bmatrix} 3 & -0.83+0.83i & 0 & 0 \\ -0.83-0.83i & 0 & 0 & 0 \\ 0 & 0 & -2 & 2.5 \\ 0 & 0 & 2.5 & 0 \end{bmatrix}. \tag{26}$$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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