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Research Article

Generalized Nonlinear Volterra-Fredholm Type Integral Inequality with Two Variables

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We establish a class of new nonlinear retarded Volterra-Fredholm type integral inequalities, with two variables, where known function w in integral functions in Q.-H. Ma and J. Pečarić, 2008 is changed into the functions w_1, w_2. By adopting novel analysis techniques, such as change of variable, amplification method, differential and integration, inverse function, and the dialectical relationship between constants and variables, the upper bounds of the embedded unknown functions are estimated. The derived results can be applied in the study of solutions of ordinary differential equations and integral equations.

1. Introduction

Gronwall-Bellman inequality [1, 2] is an important tool in the study of existence, uniqueness, boundedness, oscillation, stability, and other qualitative properties of solutions of differential equations and integral equation. There can be found a lot of its generalizations in various cases from the literature (e.g., [3–6]).

Gronwall-Bellman inequality [1, 2] can be stated as follows. If u and f are nonnegative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)\,u(s)\,ds, \quad t \in [a, b], \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c \exp\left(\int_a^t f(s)\,ds\right), \quad t \in [a, b]. \tag{2}$$

During the past few years, some investigators have established a lot of useful and interesting integral inequalities in order to achieve various goals; see [7–25] and the references cited therein.

In 2004, Pachpatte [8] has established the linear Volterra-Fredholm type integral inequality with retardation

$$\begin{aligned} u(t) \le k &+ \int_{\alpha(t_0)}^{\alpha(t)} a(t, s) \\ &\times \left[f(s)\,u(s) + \int_{\alpha(t_0)}^{s} c(s, \tau)\,u(\tau)\,d\tau\right] ds \\ &+ \int_{\alpha(t_0)}^{\alpha(T)} b(t, s)\,u(s)\,ds, \quad \forall t \in I. \end{aligned} \tag{3}$$

In 2005, Agarwal et al. [9] investigated the inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t, s)\,w_i(u(s))\,ds, \quad t_0 \le t < t_1. \tag{4}$$

In 2006, Cheung [10] studied the inequality

$$\begin{aligned} u^p(x, y) \le a &+ \frac{p}{p-q} \\ &\times \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(s, t)\,u^q(s, t)\,dt\,ds \end{aligned}$$

$$+\frac{p}{p-q}\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} g_2(s,t)\,u^q(s,t)\times\psi(u(s,t))\,dt\,ds. \tag{5}$$

In 2008, Ma and Pečarić [13] have discussed the following useful nonlinear Volterra-Fredholm type integral inequality with retardation

$$\begin{aligned}
u(x,y) &\le k+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s,t)\times\Big[f(s,t)\,w(u(s,t)) +\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} h_2(\tau,\xi)\,w\times(u(\tau,\xi))\,d\tau\,d\xi\Big]\\
&\quad+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s,t)\times\Big[f(s,t)\,w(u(s,t)) +\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} h_2(\tau,\xi)\,w\times(u(\tau,\xi))\,d\tau\,d\xi\Big].
\end{aligned} \tag{6}$$

In 2011, Abdeldaim and Yakout [20] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t)\le u_0+\int_{\alpha(t_0)}^{t} f(s)\,u(s)\times\Big[u(s)+\int_{\alpha(t_0)}^{s} h(\tau)\times\Big[u(\tau)+\int_{\alpha(t_0)}^{\tau} g(\xi)\times u(\xi)\,d\xi\Big]d\tau\Big]ds. \tag{7}$$

In this paper, on the basis of [13, 20], we discuss a new retarded nonlinear Volterra-Fredholm type integral inequality

$$\begin{aligned}
u(x,y) &\le k+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\,w(u(s_1,t_1))\times\Big[f_1(s_1,t_1)\,w_1(u(s_1,t_1)) +\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\quad\times\Big[f_2(s_2,t_2)\,w_2(u(s_2,t_2)) +\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\,w_3(u(s_3,t_3))\,dt_3ds_3\Big]dt_2ds_2\Big]dt_1ds_1\\
&\quad+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\,w(u(s_1,t_1))\times\Big[f_1(s_1,t_1)\,w_1(u(s_1,t_1)) +\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\quad\times\Big[f_2(s_2,t_2)\,w_2(u(s_2,t_2)) +\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\,w_3(u(s_3,t_3))\,dt_3ds_3\Big]dt_2ds_2\Big]dt_1ds_1,
\end{aligned} \tag{8}$$

where k is a constant, where the known function w in integral functions in [13] is replaced to the functions w_1, w_2. The upper bound estimation of the unknown function is given by integral inequality technique. Furthermore, we apply our result to retarded nonlinear Volterra-Fredholm type equations for estimation.

2. Main Result

Throughout this paper, $I_1 = [x_0, M]$ and $I_2 = [y_0, N]$ are the given subsets of real numbers $\mathbf{R}$, $\mathbf{R}_+ = [0, +\infty)$, $\Delta = I_1 \times I_2$, $C^1(M, S)$ denotes the class of continuously differentiable functions defined on set M with range in the set S, $C(M, S)$ denotes the class of continuous functions defined on set M with range in the set S, and $\alpha'(t)$ denotes the derived function of a function $\alpha'(t)$.

Theorem 1. *Suppose that $k > 0$ is a constant; functions $u(x, y), h_3(x, y), f_i(x, y), h_i(x, y) \in C(\Delta, \mathbf{R}_+)$, $(i = 1, 2)$, both $\alpha \in C^1(I_1, I_1)$ and $\beta \in C^1(I_2, I_2)$ are nondecreasing with $\alpha(x) \le x$ on I_1, $\beta(y) \le y$ on I_2, $w, w_1, w_2/w_1, w_3/w_1, w_3/w_2 \in C(\mathbf{R}_+, \mathbf{R}_+)$ are nondecreasing functions with $w_i(u) > 0$ $(i = 1, 2, 3)$ for $u > 0$,*

$$W_1(z) = \int_c^z \frac{ds}{w(s)\, w_1(s)}, \quad c > 0, \quad z \in (0, +\infty), \quad W_1(+\infty) = +\infty, \tag{9}$$

$$W_2(z) = \int_c^z \frac{w_1\left(W_1^{-1}(s)\right) ds}{w_2\left(W_1^{-1}(s)\right)}, \quad c > 0, \quad z \in (0, +\infty), \quad W_2(+\infty) = +\infty, \tag{10}$$

$$W_3(z) = \int_c^z \frac{w_2\left(W_1^{-1}\left(W_2^{-1}(s)\right)\right) ds}{w_3\left(W_1^{-1}\left(W_2^{-1}(s)\right)\right)}, \quad c > 0, \quad z \in (0, +\infty), \quad W_3(+\infty) = +\infty, \tag{11}$$

$$\begin{aligned} H(u) &= W_3\{W_2\{W_1(2u-k)\}\} - W_3 \\ &\quad \times \Bigg\{W_2\Bigg\{W_1(u) + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1, t_1) f_1(s_1, t_1)\, dt_1 ds_1\Bigg\} \\ &\quad + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1, t_1) \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \times f_2(s_2, t_2)\, dt_2 ds_2\right] dt_1 ds_1\Bigg\} \\ &\quad - \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1, t_1) \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \times \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3)\, dt_3 ds_3\right] dt_2 ds_2\right] dt_1 ds_1 \end{aligned} \tag{12}$$

is increasing, and $H(u) = 0$ has a solution c for $u > k$. If $u(x, y)$ satisfies (8), then

$$\begin{aligned} u(x, y) \le W_1^{-1}\Bigg\{W_2^{-1}\Bigg\{W_3^{-1}\Bigg\{W_3\Bigg\{W_2\Bigg\{W_1(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1, t_1) f_1(s_1, t_1)\, dt_1 ds_1\Bigg\} \\ + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1, t_1) \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) f_2(s_2, t_2)\, dt_2 ds_2\right] dt_1 ds_1\Bigg\} \\ + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1, t_1) \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \times \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3)\, dt_3 ds_3\right] dt_2 ds_2\right] dt_1 ds_1\Bigg\}\Bigg\}\Bigg\}, \\ \forall (x, y) \in \Delta, \end{aligned} \tag{13}$$

where W_i^{-1} $(i = 1, 2, 3)$ are inverse functions of W_i, respectively.

Proof. Let $z_1(x, y)$ denote the function on the right-hand side of (8), which is positive and nondecreasing in each of the variables $(x, y) \in \Delta$. From (8), we have

$$u(x,y) \le z_1(x,y), \quad \forall (x,y) \in \Delta, \tag{14}$$

$$\begin{aligned} z_1(x_0,y) = k + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} & h_1(s_1,t_1)\, w(u(s_1,t_1)) \\ & \times \Big[f_1(s_1,t_1)\, w_1(u(s_1,t_1)) \\ & + \int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \\ & \times \Big[f_2(s_2,t_2)\, w_2(u(s_2,t_2)) \\ & + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3) \\ & \times w_3(u(s_3,t_3))\, dt_3 ds_3 \Big] dt_2 ds_2 \Big] dt_1 ds_1. \end{aligned} \tag{15}$$

Differentiating $z_1(x, y)$ with respect to x, using (14), we have

$$\begin{aligned} & \frac{\partial}{\partial x} z_1(x,y) \\ & = \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x),t_1)\, w(u(\alpha(x),t_1)) \\ & \quad \times \Big[f_1(\alpha(x),t_1)\, w_1(u(\alpha(x),t_1)) \\ & \quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \\ & \quad \times \Big[f_2(s_2,t_2)\, w_2(u(s_2,t_2)) \\ & \quad + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3) \\ & \quad \times w_3(u(s_3,t_3))\, dt_3 ds_3 \Big] dt_2 ds_2 \Big] dt_1 \\ & \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x),t_1)\, w(z_1(\alpha(x),t_1)) \\ & \quad \times \Big[f_1(\alpha(x),t_1)\, w_1(z_1(\alpha(x),t_1)) \\ & \quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \\ & \quad \times \Big[f_2(s_2,t_2)\, w_2(z_1(s_2,t_2)) \\ & \quad + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, dt_1 \\ & \quad \times w_3(z_1(s_3,t_3))\, dt_3 ds_3 \Big] dt_2 ds_2 \Big] \end{aligned} \tag{16}$$

by the monotonicity of w, w_1, w_2, w_3, and z_1 and the property of α, β. From (16), we have

$$\begin{aligned} & \frac{(\partial/\partial x)\, z_1(x,y)}{w(z_1(x,y))\, w_1(z_1(x,y))} \\ & \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x),t_1) \\ & \quad \times \Big[f_1(\alpha(x),t_1) \\ & \quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \\ & \quad \times \Big[f_2(s_2,t_2) \frac{w_2(z_1(s_2,t_2))}{w_1(z_1(s_2,t_2))} \end{aligned}$$

$$+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3) \times\frac{w_3(z_1(s_3,t_3))}{w_1(z_1(s_3,t_3))}dt_3ds_3\Bigg]dt_2ds_2\Bigg]dt_1. \tag{17}$$

Integrating both sides of the above inequality from x_0 to x, we obtain

$$\begin{aligned}
&W_1(z_1(x,y))\\
&\leq W_1(z_1(x_0,y))\\
&\quad+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\\
&\qquad\quad+\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\qquad\qquad\times\Bigg[f_2(s_2,t_2)\frac{w_2(z_1(s_2,t_2))}{w_1(z_1(s_2,t_2))}\\
&\qquad\qquad\quad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\\
&\qquad\qquad\quad\times\frac{w_3(z_1(s_3,t_3))}{w_1(z_1(s_3,t_3))}dt_3ds_3\Bigg]dt_2ds_2\Bigg]dt_1ds_1\\
&\leq W_1(z_1(x_0,y))\\
&\quad+\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} h_1(s_1,t_1)f_1(s_1,t_1)\,dt_1ds_1\\
&\quad+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\\
&\qquad\times\Bigg[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\qquad\quad\times\Bigg[f_2(s_2,t_2)\frac{w_2(z_1(s_2,t_2))}{w_1(z_1(s_2,t))}\\
&\qquad\qquad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\\
&\qquad\qquad\times\frac{w_3(z_1(s_3,t_3))}{w_1(z_1(s_3,t_3))}dt_3ds_3\Bigg]dt_2ds_2\Bigg]dt_1ds_1,
\end{aligned} \tag{18}$$

for all $(x,y)\in[x_0,X]\times[y_0,Y]$, $X\in I_1$, $Y\in I_2$, and X,Y are chosen arbitrarily, where W_1 is defined by (9).

Let $z_2(x,y)$ denote the function on the right-hand side of (18), which is positive and nondecreasing in each of the variables $(x,y)\in[x_0,X]\times[y_0,Y]$. From (18), we have

$$z_1(x,y)\leq W_1^{-1}(z_2(x,y)),\quad \forall (x,y)\in[x_0,X]\times[y_0,Y], \tag{19}$$

$$z_2(x_0,y)=W_1(z_1(x_0,y)) +\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} h_1(s_1,t_1)f_1(s_1,t_1)\,dt_1ds_1. \tag{20}$$

Differentiating $z_2(x,y)$ with respect to x, by the monotonicity of z_2, W_1^{-1}, w_2/w_1, and w_3/w_1, the property of α, β, and (19), we have

$$\begin{aligned}
&\frac{\partial}{\partial x}z_2(x,y)\\
&=\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x),t_1)\\
&\quad\times\Bigg[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\qquad\times\Bigg[f_2(s_2,t_2)\frac{w_2(z_1(s_2,t_2))}{w_1(z_1(s_2,t_2))}\\
&\qquad\quad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\\
&\qquad\qquad\times\frac{w_3(z_1(s_3,t_3))}{w_1(z_1(s_3,t_3))}dt_3ds_3\Bigg]dt_2ds_2\Bigg]dt_1\\
&\leq\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x),t_1)\\
&\quad\times\Bigg[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\\
&\qquad\times\Bigg[f_2(s_2,t_2)\frac{w_2(W_1^{-1}(z_2(s_2,t_2)))}{w_1(W_1^{-1}(z_2(s_2,t_1)))}\\
&\qquad\quad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\\
&\qquad\quad\times\frac{w_3(W_1^{-1}(z_2(s_3,t_3)))}{w_1(W_1^{-1}(z_2(s_3,t_3)))}dt_3ds_3\Bigg]dt_2ds_2\Bigg]dt_1
\end{aligned} \tag{21}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. From (21), we have

$$\begin{aligned}
&\frac{w_1\left(W_1^{-1}\left(z_2(x,y)\right)\right)(\partial/\partial x)\, z_2(x,y)}{w_2\left(W_1^{-1}\left(z_2(x,y)\right)\right)} \\
&\le \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x), t_1) \\
&\quad\times\left[\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\right. \\
&\quad\times\left[f_2(s_2,t_2)\right. \\
&\quad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3) \\
&\quad\left.\left.\times\frac{w_3\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}{w_2\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}dt_3\,ds_3\right]dt_2\,ds_2\right]dt_1,
\end{aligned}\tag{22}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. From (22), we have

$$\begin{aligned}
W_2(z_2(x,y)) &\le W_2(z_2(x_0,y)) \\
&\quad+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \\
&\quad\times\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\right. \\
&\quad\times\left[f_2(s_2,t_2)\right. \\
&\quad+\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3) \\
&\quad\left.\left.\times\frac{w_3\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}{w_2\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}dt_3\,ds_3\right]dt_2\,ds_2\right]dt_1\,ds_1 \\
&\le W_2(z_2(x_0,y)) \\
&\quad+\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} h_1(s_1,t_1)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\, f_2(s_2,t_2)\, dt_2\,ds_2\right]dt_1\,ds_1 \\
&\quad+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \\
&\quad\times\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\right. \\
&\quad\left.\times\left[\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\,\frac{w_3\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}{w_2\left(W_1^{-1}\left(z_2(s_3,t_3)\right)\right)}dt_3\,ds_3\right]dt_2\,ds_2\right]dt_1\,ds_1,
\end{aligned}\tag{23}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$, where W_2 is defined by (10). Let $z_3(x, y)$ denote the function on the right-hand side of (23), which is positive and nondecreasing in each of the variables $(x, y) \in [x_0, X] \times [y_0, Y]$. Then (23) is equivalent to

$$z_2(x, y) \le W_2^{-1}(z_3(x, y)), \quad \forall (x, y) \in [x_0, X] \times [y_0, Y], \tag{24}$$

$$\begin{aligned} z_3(x_0, y) &= W_2(z_2(x_0, y)) \\ &+ \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h_1(s_1, t_1) \\ &\quad \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \right. \\ &\quad \left. \times f_2(s_2, t_2)\, dt_2 ds_2 \right] dt_1 ds_1. \end{aligned} \tag{25}$$

Differentiating $z_3(x, y)$ with respect to x, using (24), we have

$$\begin{aligned} \frac{\partial}{\partial x} z_3(x, y) &= \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x), t_1) \\ &\quad \times \left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \right. \\ &\quad \left. \times \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3) \frac{w_3\left(W_1^{-1}(z_2(s_3, t_3))\right)}{w_2(W_1^{-1}(z_2(s_3, t_3)))} dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 \\ &\le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x), t_1) \\ &\quad \times \left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \right. \\ &\quad \left. \times \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3) \frac{w_3\left(W_1^{-1}\left(W_2^{-1}(z_3(s_3, t_3))\right)\right)}{w_2(W_1^{-1}(W_2^{-1}(z_3(s_3, t_3))))} dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1, \end{aligned} \tag{26}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. From (26), using the monotonicity of z_3, W_1^{-1}, W_2^{-1}, and w_3/w_2 and the property of α, β, we have

$$\begin{aligned} \frac{w_2\left(W_1^{-1}\left(W_2^{-1}(z_3(x, y))\right)\right)(\partial/\partial x) z_3(x, y)}{w_3(W_1^{-1}(W_2^{-1}(z_3(x, y))))} &\le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} h_1(\alpha(x), t_1) \\ &\quad \times \left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t_1} h_2(s_2, t_2) \right. \\ &\quad \left. \times \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3, t_3)\, dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1, \end{aligned} \tag{27}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. Integrating both sides of the above inequality from x_0 to x, we obtain

$$W_3(z_3(x,y)) \le W_3(z_3(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1, \quad \forall (x,y) \in [x_0, X] \times [y_0, Y], \tag{28}$$

where W_3 is defined by (11).

From (19), (24), and (28), we have

$$\begin{aligned} z_1(x,y) &\le W_1^{-1}(z_2(x,y)) \\ &\le W_1^{-1}\left(W_2^{-1}(z_3(x,y))\right) \\ &\le W_1^{-1}\left\{W_2^{-1}\left\{W_3^{-1}\left\{W_3(z_3(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \right.\right.\right. \\ &\qquad \left.\left.\left. \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1 \right\}\right\}\right\}, \\ &\qquad \forall (x,y) \in [x_0, X] \times [y_0, Y]. \end{aligned} \tag{29}$$

Substituting (20) and (25) into (29), we have

$$\begin{aligned} z_1(x,y) &\le W_1^{-1}(z_2(x,y)) \le W_1^{-1}\left(W_2^{-1}(z_3(x,y))\right) \\ &\le W_1^{-1}\left\{W_2^{-1}\left\{W_3^{-1}\left\{W_3\left\{W_2\left\{W_1(z_1(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h_1(s_1,t_1) f_1(s_1,t_1)\, dt_1 ds_1 \right\}\right.\right.\right.\right. \\ &\qquad \left. + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h_1(s_1,t_1) \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) f_2(s_2,t_2)\, dt_2 ds_2 \right] dt_1 ds_1 \right\} \\ &\qquad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \\ &\qquad \left.\left.\left. \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1 \right\}\right\}\right\}, \\ &\qquad \forall (x,y) \in [x_0, X] \times [y_0, Y]. \end{aligned} \tag{30}$$

Since X, Y are chosen arbitrarily, we have

$$\begin{aligned}
z_1(x,y) \le{}& W_1^{-1} \\
&\times \Bigg\{ W_2^{-1} \Bigg\{ W_3^{-1} \Bigg\{ W_3 \Bigg\{ W_2 \Bigg\{ W_1(z_1(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) f_1(s_1,t_1)\, dt_1 ds_1 \Bigg\} \\
&\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) f_2(s_2,t_2)\, dt_2 ds_2 \right] dt_1 ds_1 \Bigg\} \\
&\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \\
&\quad \times \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \left[\int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, dt_3 ds_3 \right] dt_2 ds_2 \right] dt_1 ds_1 \Bigg\} \Bigg\} \Bigg\}, \\
& \forall (x,y) \in \Delta.
\end{aligned} \tag{31}$$

By the definition of z_1 and (15), we have

$$\begin{aligned}
&2z_1(x_0,y) - k \\
&= k + 2\int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\, w(u(s_1,t_1)) \\
&\quad \times \Bigg[f_1(s_1,t_1)\, w_1(u(s_1,t_1)) \\
&\quad + \int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) \\
&\quad \times \left[f_2(s_2,t_2)\, w_2(u(s_2,t_2)) + \int_{\alpha(x_0)}^{s_2} \int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\, w_3(u(s_3,t_3))\, dt_3 ds_3 \right] dt_2 ds_2 \Bigg] dt_1 ds_1 \\
&= z_1(M,N).
\end{aligned} \tag{32}$$

From (31) and (32), we have

$$\begin{aligned}
2z_1(x_0,y) - k \le{}& W_1^{-1} \\
&\times \Bigg\{ W_2^{-1} \Bigg\{ W_3^{-1} \Bigg\{ W_3 \Bigg\{ W_2 \Bigg\{ W_1(z_1(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1) f_1(s_1,t_1)\, dt_1 ds_1 \Bigg\} \\
&\quad + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1) \left[\int_{\alpha(x_0)}^{s_1} \int_{\beta(y_0)}^{t_1} h_2(s_2,t_2) f_2(s_2,t_2)\, dt_2 ds_2 \right] dt_1 ds_1 \Bigg\}
\end{aligned}$$

$$+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1) \times\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\left[\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\,dt_3ds_3\right]dt_2ds_2\right]dt_1ds_1\Bigg\}\Bigg\}\Bigg\} \tag{33}$$

or

$$\begin{aligned} &W_3\{W_2\{W_1(2z_1(x_0,y)-k)\}\} \\ &\le W_3\left\{W_2\left\{W_1(z_1(x_0,y))+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\,f_1(s_1,t_1)\,dt_1ds_1\right\}\right. \\ &\quad\left.+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,f_2(s_2,t_2)\,dt_2ds_2\right]dt_1ds_1\right\} \\ &\quad+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\left[\int_{\alpha(x_0)}^{s_2}\int_{\beta(y_0)}^{t_2} h_3(s_3,t_3)\,dt_3ds_3\right]dt_2ds_2\right]dt_1ds_1. \end{aligned} \tag{34}$$

By the definition of H, the assumption of Theorem 1, and (34), we observe that

$$H(z_1(x_0,y)) \le 0 = H(c). \tag{35}$$

Since H is increasing, from (14), (31), and (35), we have the desired estimation (13).

We consider a special case of Theorem 1. If $u(x,y)$ satisfies nonlinear Volterra-Fredholm type integral inequality with retardation,

$$\begin{aligned} u(x,y) \le{}& k \\ &+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\,w(u(s_1,t_1)) \\ &\quad\times\left[f_1(s_1,t_1)\,w_1(u(s_1,t_1))+\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,w_2(u(s_2,t_2))\,dt_2ds_2\right]dt_1ds_1 \\ &+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1)\,w(u(s_1,t_1)) \\ &\quad\times\left[f_1(s_1,t_1)\,w_1(u(s_1,t_1))+\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,w_2(u(s_2,t_2))\,dt_2ds_2\right]dt_1ds_1, \\ &\forall (x,y)\in\Delta. \end{aligned} \tag{36}$$

□

Corollary 2. *Let functions $u(x,y)$, $f_1(x,y)$, $h_1(x,y)$, $h_2(x,y)$, α, β, w, w_1, w_2/w_1, W_1, and W_2 and constant k be as in Theorem 1. Suppose that*

$$\begin{aligned}
H(u) &= W_2\{W_1(2u-k)\} \\
&\quad - W_2\left\{W_1(u) + \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1) f_1(s_1,t_1)\,dt_1 ds_1\right\} \\
&\quad - \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} h_1(s_1,t_1) \\
&\qquad \times \left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,dt_2 ds_2\right] dt_1 ds_1
\end{aligned} \tag{37}$$

is increasing and $H(u)=0$ has a solution c for $u>k$. If $u(x,y)$ satisfies (36), *then*

$$\begin{aligned}
u(x,y) &\le W_1^{-1}\left\{W_2^{-1}\left\{W_2\left\{W_1(c) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\right.\right.\right. \\
&\qquad \left.\times f_1(s_1,t_1)\,dt_1 ds_1\right\} \\
&\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) \\
&\qquad \left.\left.\times \left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,dt_2 ds_2\right] dt_1 ds_1\right\}\right\}.
\end{aligned} \tag{38}$$

for all $(x,y)\in\Delta$, where W_i^{-1} $(i=1,2)$ are inverse functions of W_i, respectively.

Remark 3. In Corollary 2, when $w_1(x,y)\equiv w_2(x,y)$ on Δ, $W_2(u)=u-c$, and $W_2^{-1}(u)=u+c$, (38) is equivalent to

$$\begin{aligned}
u(x,y) &\le W_1^{-1} \\
&\quad \times \left\{W_1(c) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1) f_1(s_1,t_1)\,dt_1 ds_1\right. \\
&\qquad \left. + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,dt_2 ds_2\right] dt_1 ds_1\right\} \\
&= W_1^{-1}\left\{W_1(c) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} h_1(s_1,t_1)\left[f_1(s_1,t_1) + \int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} h_2(s_2,t_2)\,dt_2 ds_2\right] dt_1 ds_1\right\}, \\
&\qquad \forall (x,y)\in\Delta.
\end{aligned} \tag{39}$$

Corollary 2 reduces to Theorem 3.1$'$ in [13].

3. Application

In this section, we apply our result in Theorem 1 to study the retarded Volterra-Fredholm integral equations with two variables, which demonstrates that our result is useful to investigate the qualitative properties of solutions of some retarded Volterra-Fredholm integral equations with two variables.

We consider the retarded Volterra-Fredholm integral equation of the form

$$\begin{aligned}
v(x,y) &= g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} F_1\left\{s_1,t_1,v(s_1-d(s_1),t_1-e(t_1)),\right. \\
&\qquad \left.\times \int_{x_0}^{s_1}\int_{y_0}^{t_1} F_2[s_2,t_2,v(s_2-d(s_2),t_2-e(t_2))]\,dt_2 ds_2\right\} dt_1 ds_1 \\
&\quad + \int_{x_0}^{M}\int_{y_0}^{N} F_1\left\{s_1,t_1,v(s_1-d(s_1),t_1-e(t_1)),\int_{x_0}^{s_1}\int_{y_0}^{t_1} F_2[s_2,t_2,v(s_2-d(s_2),t_2-e(t_2))]\,dt_2 ds_2\right\} dt_1 ds_1
\end{aligned} \tag{40}$$

for all $(x, y) \in [x_0, M] \times [y_0, N]$, where $g, v \in C(\Delta, \mathbf{R})$, $x - d(x) \in C^1(I_1, I_1)$, and $y - e(y) \in C^1(I_2, I_2)$ are strictly increasing with $d(x_0) = 0$, $e(y_0) = 0$, $d(x) \geq 0$, $e(y) \geq 0$, $d'(x) < 1$, $e'(y) < 1$, and $F_1, F_2 \in C(\Delta \times \mathbf{R}, \mathbf{R})$. Let $\alpha(x) = x - d(x)$, $\beta(y) = y - e(y)$; then $\alpha(x), \beta(y)$ satisfy the condition in Theorem 1 and are invertible functions.

The following corollary gives the bound on the solution of (40).

Corollary 4. *Let* $k = \max_{(x,y)\in\Delta}\{|g(x, y)|\}$. *Suppose that* F_1, F_2 *in* (40) *satisfy the conditions*

$$\left|F_1(s,t,x,y)\right| \leq h_1(s,t)\, w(|x|)\left[f_1(s,t)\, w_1(|x|) + |y|\right],$$
$$\left|F_2(s,t,x)\right| \leq h_2(s,t)\, w_2(|x|), \tag{41}$$

where $f_1(s,t)$, $h_1(s,t)$, $h_2(s,t)$, $w(s)$, $w_1(s)$, *and* $w_2(s)$ *are as in Theorem 1. Assume that the function*

$$\begin{aligned} H^*(u) = {} & W_2\{W_1(2u-k)\} \\ & - W_2\left\{W_1(u) + \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} Eh_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right) f_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right) dt_1 ds_1\right\} \\ & - \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} Eh_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} Eh_2\left(\alpha^{-1}(s_2), \beta^{-1}(t_2)\right) dt_2 ds_2\right] dt_1 ds_1 \end{aligned} \tag{42}$$

is increasing and $H^*(u) = 0$ *has a solution c for* $u > k$. *If* $v(x, y)$ *is a solution of* (40) *on* Δ, *then*

$$\begin{aligned} |v(x,y)| \leq {} & W_1^{-1} \\ & \times \left\{W_2^{-1}\left\{W_2\left\{W_1(c) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} Eh_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right) f_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right) dt_1 ds_1\right\}\right.\right. \\ & \left.\left. + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} Eh_1\left(\alpha^{-1}(s_1), \beta^{-1}(t_1)\right)\left[\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} Eh_2\left(\alpha^{-1}(s_2), \beta^{-1}(t_2)\right) dt_2 ds_2\right] dt_1 ds_1\right\}\right\}, \end{aligned} \tag{43}$$

where $E = \max_{x\in I_1}(1/\alpha'(\alpha^{-1}(x)))\max_{y\in I_2}(1/\beta'(\beta^{-1}(y))) < \infty) < \infty$ *and* W_1, W_2, W_1^{-1}, *and* W_2^{-1} *are as in Theorem 1.*

Proof. Using the conditions (41), we have

$$\begin{aligned} |v(x,y)| \leq {} & |g(x,y)| \\ & + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s_1,t_1)\, w\left(\left|v\left(s_1 - d(s_1), t_1 - e(t_1)\right)\right|\right) \\ & \times\left[f_1(s_1,t_1)\, w_1\left(\left|v\left(s_1 - d(s_1), t_1 - e(t_1)\right)\right|\right)\right. \\ & \left. + \int_{x_0}^{s_1}\int_{y_0}^{t_1} h_2(s_2,t_2)\, w_2\left(\left|v\left(s_2 - d(s_2), t_2 - e(t_2)\right)\right|\right) dt_2 ds_2\right] dt_1 ds_1 \end{aligned}$$

$$
\begin{aligned}
&+\int_{x_0}^{M}\int_{y_0}^{N} h_1(s_1,t_1)\,w\left(\left|v\left(s_1-d(s_1),t_1-e(t_1)\right)\right|\right)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\,w_1\left(\left|v\left(s_1-d(s_1),t_1-e(t_1)\right)\right|\right)\\
&\qquad\quad+\int_{x_0}^{s_1}\int_{y_0}^{t_1} h_2(s_2,t_2)\,w_2\left(\left|v\left(s_2-d(s_2),t_2-e(t_2)\right)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1\\
&\le k+\int_{x_0}^{x}\int_{y_0}^{y} h_1(s_1,t_1)\,w\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\,w_1\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\quad+\int_{x_0}^{s_1}\int_{y_0}^{t_1} h_2(s_2,t_2)\,w_2\left(\left|v\left(\alpha(s_2),\beta(t_2)\right)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1\\
&+\int_{x_0}^{M}\int_{y_0}^{N} h_1(s_1,t_1)\,w\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\,w_1\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\quad+\int_{x_0}^{s_1}\int_{y_0}^{t_1} h_2(s_2,t_2)\,w_2\left(\left|v\left(\alpha(s_2),\beta(t_2)\right)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1,
\end{aligned}
\tag{44}
$$

for all $(x,y)\in\Delta$. From (44), using change of variables, we obtain

$$
\begin{aligned}
&\le k+\int_{x_0}^{x}\int_{y_0}^{y} h_1(s_1,t_1)\,w\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\,w_1\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\quad+\int_{\alpha(x_0)}^{\alpha(s_1)}\int_{\beta(y_0)}^{\beta(t_1)}\frac{1}{\alpha'\left(\alpha^{-1}(s_2)\right)}\\
&\qquad\quad\times\frac{1}{\beta'\left(\beta^{-1}(t_2)\right)}h_2\\
&\qquad\quad\times\left(\alpha^{-1}(s_2),\beta^{-1}(t_2)\right)w_2\\
&\qquad\quad\times\left(\left|v(s_2,t_2)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1\\
&+\int_{x_0}^{M}\int_{y_0}^{N} h_1(s_1,t_1)\,w\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\times\Bigg[f_1(s_1,t_1)\,w_1\left(\left|v\left(\alpha(s_1),\beta(t_1)\right)\right|\right)\\
&\qquad\quad+\int_{\alpha(x_0)}^{\alpha(s_1)}\int_{\beta(y_0)}^{\beta(t_1)}\frac{1}{\alpha'\left(\alpha^{-1}(s_2)\right)}\frac{1}{\beta'\left(\beta^{-1}(t_2)\right)}h_2\\
&\qquad\quad\times\left(\alpha^{-1}(s_2),\beta^{-1}(t_2)\right)w_2\\
&\qquad\quad\times\left(\left|v(s_2,t_2)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1\\
&\le k+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} Eh_1\left(\alpha^{-1}(s_1),\beta^{-1}(t_1)\right)w\left(\left|v(s_1,t_1)\right|\right)\\
&\qquad\times\Bigg[f_1\left(\alpha^{-1}(s_1),\beta^{-1}(t_1)\right)w_1\left(\left|v(s_1,t_1)\right|\right)\\
&\qquad\quad+\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} Eh_2\left(\alpha^{-1}(s_2),\beta^{-1}(t_2)\right)w_2\\
&\qquad\quad\times\left(\left|v(s_2,t_2)\right|\right)dt_2\,ds_2\Bigg]dt_1\,ds_1\\
&+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} Eh_1\left(\alpha^{-1}(s_1),\beta^{-1}(t_1)\right)w\left(\left|v(s_1,t_1)\right|\right)\\
&\qquad\times\Bigg[f_1\left(\alpha^{-1}(s_1),\beta^{-1}(t_1)\right)w_1\left(\left|v(s_1,t_1)\right|\right)\\
&\qquad\quad+\int_{\alpha(x_0)}^{s_1}\int_{\beta(y_0)}^{t_1} Eh_2\left(\alpha^{-1}(s_2),\beta^{-1}(t_2)\right)w_2
\end{aligned}
$$

$$\times \left(\left|v\left(s_2,t_2\right)\right|\right) dt_2 ds_2$$
$$\left. \times \frac{1}{\alpha'\left(\alpha^{-1}\left(s_2\right)\right)} \right] dt_1 ds_1, \tag{45}$$

for all $(x, y) \in \Delta$. Applying the result of Theorem 1 to the inequality (45), we obtain the desired estimation (39). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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