Research Article

On the Global Dissipative and Multipeakon Dissipative Behavior of the Two-Component Camassa-Holm System

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The global dissipative and multipeakon dissipative behavior of the two-component Camassa-Holm shallow water system after wave breaking was studied in this paper. The underlying approach is based on a skillfully defined characteristic and a set of newly introduced variables which transform the original system into a Lagrangian semilinear system. It is the transformation, together with the associated properties, that allows for the continuity of the solution beyond collision time to be established, leading to a uniquely global dissipative solution, which constructs a semigroup, and the multipeakon dissipative solution.

1. Introduction

In view of the wide applications in fluid dynamics, nonlinear optics, biochemistry, microbiology, physics, and many other fields, the study of the dynamic behavior of shallow water wave represents an important subject of research [1–5]. The Camassa-Holm (CH) equation [1] has been widely used to model the unidirectional propagation of shallow water waves over a flat bottom. The nonstandard properties of the CH equation set it apart from the classical soliton equations such as KdV, the first two remarkable of which are that it has peaked solitons [1, 6] and is able to model wave breaking [1, 2]. The presence of breaking waves means that the solution remains bounded while its slope becomes unbounded in finite time [2, 5]. After wave breaking, the solutions of the CH equation become uniquely as either global conservative [7–9] or global dissipative solutions [10–12].

In this paper, we focus on the two-component Camassa–Holm shallow water system [13–15]:

$$m_t + u m_x + 2 u_x m + \sigma \rho \rho_x = 0, \quad t > 0, \ x \in R,$$
$$\rho_t + (u \rho)_x = 0, \quad t > 0, \ x \in R, \tag{1}$$

with $m = \sigma_1 u - u_{xx}$, $\sigma = 1$, $\sigma_1 = 1$ (or in the "short wave" limit, $\sigma_1 = 0$). Here $u(x, t)$ represents the horizontal

velocity of the fluid and $\rho(x, t)$ is in connection with the free-surface elevation from equilibrium (or scalar density). This system appears originally in [16] and then derived by Constantin and Ivanov [13] in the context of shallow water theory. System (1) is an extension of the Camassa-Holm (CH) equation by combing its integrability property with compressibility or free-surface elevation dynamics in its shallow water interpretation [6, 17]; analogously it is formally integrable [13–15] in the sense that it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ:

$$\Psi_{xx} = \left(-\sigma \zeta^2 \rho^2 + \zeta m + \frac{\sigma_1}{4} \right) \Psi,$$
$$\Psi_t = \left(\frac{1}{2} \zeta - u \right) \Psi_x + \frac{1}{2} \cdot u_x \Psi. \tag{2}$$

The Cauchy problem for the two-component Camassa-Holm (CH2) system has been studied extensively [18–22]. It has been shown that the CH2 system is locally well-posed with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s > 3/2$ [18]. The system also has global strong solutions which blow up in finite time [19, 21, 22] and a global weak solution [23]. However, the problem about continuation of the solutions beyond wave breaking, although interesting and important, has not been

explicitly addressed yet to the best of our knowledge. In our recent work [20], we studied the continuation beyond wave breaking by applying an approach based on a novel transformation that transforms system (1) into an equivalent semilinear system of original differential equation. Such treatment makes it possible to investigate the continuity of the solution beyond collision time, leading to the multipeakon conservative solution and a global conservative solution where the energy is conserved for almost all times.

It should be noted that both conservation and dissipation are important features associated with the system. The global conservation property of the CH2 system has been obtained in [20], and the dissipative behavior of the modified coupled two-component Camassa-Holm system, different from the CH2 system, has also been established in [24, 25]; however, there is no effort made in the literature on the study of the global dissipative and the multipeakon dissipative solutions of such system studied in this paper to the best of the authors' knowledge. In this work, we explore a new approach to establish a global and stable dissipative solution of the CH2 system, which allows for the construction of the global dissipative multipeakon solution of this system. Our study is also motivated by the early work [10, 11] in the study of the global dissipative solution of the CH equation. The main difference is that here we deal with a coupled system and consider explicitly the mutual effect between two components, which literally makes the analysis more challenging as compared to the single one considered in [10, 11]. The key to circumvent the difficulty is to use the skillfully defined characteristic and a new set of variables, which allows for the establishment of the global dissipative solutions of system (1). Furthermore, it is useful to understand whether or not system (1) has the multipeakon solution as with the CH equation [12], an important aspect related to the solutions near wave breaking. And this work develops such multipeakon solution and confirms that the semigroup of the global dissipative solution preserves the multipeakon structure.

The remainder of this paper is organized as follows. Section 2 presents the transformation from the original system to a Lagrangian semilinear system. The global solutions of the equivalent semilinear system are obtained in Section 3, which are transformed into the global dissipative solutions of the original system in Section 4. We establish the multipeakon dissipative solutions for the original system in Section 5. To demonstrate the feature of the solution, two numerical examples are considered in Section 6. Finally the paper is closed in Section 7.

2. The Original System and the Equivalent Lagrangian System

For simplicity, we consider here the associated evolution for positive times (of course, one would get similar results for negative times just by changing the initial condition u_0 into $-u_0$). Let us introduce an operator $\Lambda = (1 - \partial_x^2)^{-1}$, which can be expressed by its associated Green's function $G = e^{-|x|}/2$

such as $\Lambda f(x) = G * f(x) = 1/2 \cdot \int_R e^{-|x-x'|} f(x') dx'$ for all $f \in L^2(R)$. If we define P as

$$
\begin{aligned}
P(t, x) &= G * \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) \\
&= \frac{1}{2} \cdot \int_R e^{-|x-x'|} \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) (t, x') dx',
\end{aligned}
\tag{3}
$$

where $v = \rho - 1$, then (1) can be rewritten as

$$
\begin{aligned}
u_t + uu_x + P_x &= 0, \quad t > 0, \ x \in R, \\
v_t + uv_x + vu_x + u_x &= 0, \quad t > 0, \ x \in R.
\end{aligned}
\tag{4}
$$

Moreover, for regular solutions, we have the total energy

$$
E(t) = \int_R u^2 + u_x^2 + v^2 dx
\tag{5}
$$

is constant in time. Thus (4) possesses the H^1-norm conservation law defined as

$$
\|z\|_{H^1} = \|u\|_{H^1} + \|v\|_{L^2} = \left(\int_R \left[u^2 + u_x^2 + v^2 \right] dx \right)^{1/2},
\tag{6}
$$

where $z = (u, v)$. Since $z = (u, v) \in H^1 \times W$ with $W = L^2 \cap L^\infty$, Young's inequality ensures $P \in H^1$.

We reformulate system (4) into a Lagrangian equivalent semilinear system as follows. Let $z(t, x) = (u, v)(t, x)$ denote the solution of system (4). For given initial data $y(0, \xi)$, we define the corresponding characteristic $y(t, \xi)$ as the solution of

$$
y_t(t, \xi) = u(t, y(t, \xi)),
\tag{7}
$$

where the variable ξ is identified with a "particle." The change in Lagrangian energy distribution along the particle path is given by

$$
h(t, \xi) = \left(u^2 + u_x^2 + v^2 \right) \circ yy_\xi.
\tag{8}
$$

It is not hard to check that

$$
\left(u^2 + u_x^2 + v^2 \right)_t + \left(u \left(u^2 + u_x^2 + v^2 \right) \right)_x = \left(u^3 - 2uP \right)_x.
\tag{9}
$$

Then it follows from (7) and (9) that

$$
\frac{dh}{dt} = \left(u^3 - 2uP \right)_x \circ yy_\xi = \left(3U^2 - 2P \right) U_\xi - 2UP_x y_\xi.
\tag{10}
$$

Throughout the following, we use the notation

$$
\begin{aligned}
U(t, \xi) &= u(t, y(t, \xi)), \qquad V(t, \xi) = v(t, y(t, \xi)), \\
N(t, \xi) &= u_x(t, y(t, \xi)).
\end{aligned}
\tag{11}
$$

In the following, we drop the variable t for simplicity. After the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, we obtain the following expressions for P_x and P; namely,

$$P_x(\xi) = -\frac{1}{4} \cdot \int_R \operatorname{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|}$$
$$\times [h + (U^2 + 2V) y_\xi](\xi') d\xi', \quad (12)$$

$$P(\xi) = \frac{1}{4} \cdot \int_R e^{-|y(\xi) - y(\xi')|} [h + (U^2 + 2V) y_\xi](\xi') d\xi',$$

where we have taken that y is an increasing function for any fixed time t for granted (the validity will be proved later) and have used the fact that $h = (u^2 + u_x^2 + v^2) \circ y y_\xi$. From the definition of the characteristic, it follows that

$$U_t(t, \xi) = u_t(t, y) + u_x(t, y) y_t(t, \xi) = -P_x \circ y(t, \xi),$$

$$V_t(t, \xi) = v_t(t, y) + v_x(t, y) y_t(t, \xi)$$
$$= -[(v+1) u_x] \circ y(t, \xi),$$
$$\quad (13)$$
$$N_t(t, \xi) = u_{xt}(t, y) + u_{xx}(t, y) y_t(t, \xi)$$
$$= \left(u^2 - \frac{u_x^2}{2} + \frac{v^2}{2} + v - P\right) \circ y(t, \xi).$$

Let us introduce another variable $\varsigma(t, \xi)$, such as $\varsigma(t, \xi) = y(t, \xi) - \xi$ (it will turn out that $\varsigma \in L^\infty(R)$). With these new variables, we now derive an equivalent system of (4):

$$\varsigma_t = U, \qquad U_t = -P_x, \qquad V_t = -(V+1) N,$$
$$\quad (14)$$
$$N_t = U^2 - \frac{N^2}{2} + \frac{V^2}{2} + V - P,$$

where P and P_x are given by (12). Differentiating the first two equations in (14) with respect to ξ yields

$$\varsigma_{\xi t} = U_\xi,$$

$$U_{\xi t} = \frac{h}{2} + \left(\frac{U^2}{2} + V - P\right) y_\xi, \quad (15)$$

$$h_t = (3U^2 - 2P) U_\xi - 2UP_x y_\xi,$$

which is semilinear with respect to the variables y_ξ, U_ξ, and h. Dissipative solutions differ from conservative solutions when particles collide, that is, where $y_\xi(t, \xi) = 0$ for ξ in an interval of positive length. If we solve (14) and (15), we obtain the conservative solution. However, to obtain the dissipative solution, we impose that when particles collide, they lose their energy; that is, if $y_\xi(\tau, \xi) = 0$ for some τ, then we set $h(\tau, \xi) = 0$. One can show that $y_\xi(\tau, \xi) = 0$ implies $U_\xi(\tau, \xi) = 0$ so that the system (15) implies that $y_\xi(t, \xi) = U_\xi(t, \xi) = h(t, \xi) = 0$ for $t \geq \tau$. Thus, we can define $\tau(\xi)$ to be the first time when $y_\xi(t, \xi)$ vanishes; namely,

$$\tau(\xi) = \sup\{t \in R^+ \mid y_\xi(t', \xi) > 0 \ \forall \ 0 \leq t' < t\}. \quad (16)$$

However, from this definition $\tau(\xi)$ in (16), it is not clear whether τ is measurable or not. We now replace this definition by the following one. Let

$$A_t = \bigcup_{n \geq 1} \bigcap_{t_i \leq t} \left\{\xi \in R \mid y_\xi(t, \xi) > \frac{1}{n}\right\}, \quad (17)$$

where $\{t_i\}$ is a dense countable subset of $[0, T]$ and the sets A_t are measurable for all t, with $A_{t'} \subset A_t$ for $t < t'$. Let us set $\tau^n(\xi) = \sum_{i=0}^{2^n} 2^{-n} iT \chi_{i,n}(\xi)$ with $\chi_{i,n}$ being the indicator function of the set $A_{2^{-n}iT} / A_{2^{-n}(i+1)T}$. It is not hard to know that $\tau^n(\xi)$ is by construction measurable and increasing with respect to n, which is also bounded by T. Thus, we define

$$\tau(\xi) = \lim_{n \to \infty} \tau^n(\xi), \quad (18)$$

and τ is a measurable function. One can check that this definition coincided with (16). Hence, the expressions for P_x and P in (12) become

$$P_x(\xi) = -\frac{1}{4} \cdot \int_{t < \tau(\xi)} \operatorname{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|}$$
$$\times [h + (U^2 + 2V) y_\xi](\xi') d\xi',$$

$$P(\xi) = \frac{1}{4} \cdot \int_{t < \tau(\xi)} e^{-|y(\xi) - y(\xi')|} [h + (U^2 + 2V) y_\xi](\xi') d\xi',$$
$$\quad (19)$$

and the modified system to be solved here reads

$$\varsigma_t = U, \qquad U_t = -P_x, \qquad V_t = -(V+1) N,$$
$$\quad (20)$$
$$N_t = U^2 - \frac{N^2}{2} + \frac{V^2}{2} + V - P,$$

$$\varsigma_{\xi t} = \chi_{\{t < \tau(\xi)\}} U_\xi,$$

$$U_{\xi t} = \chi_{\{t < \tau(\xi)\}} \left(\frac{h}{2} + \left(\frac{U^2}{2} + V - P\right) y_\xi\right), \quad (21)$$

$$h_t = \chi_{\{t < \tau(\xi)\}} \left((3U^2 - 2P) U_\xi - 2UP_x y_\xi\right),$$

where χ_B is the indicator function of the set B, which can be regarded as an O.D.E. in the Banach space $E = L^\infty \cap W \cap W \cap W \cap W \cap W \cap L^1$.

3. Global Solutions of the Equivalent System

To obtain the global existence of solutions, we start from a contraction argument that offers the local existence of solutions, which is proved in the Banach space E. Note that global solutions of (20) may not exist for all initial data in E. However, they exist when the initial data $X_0 = (\varsigma_0, U_0, V_0, N_0, h_0)$ belongs to the set Γ which is defined as follows.

Definition 1. The set Γ is composed of all (ς, U, V, N, h) such that

$$X = (\varsigma, U, V, N, \varsigma_\xi, U_\xi, h) \in E, \tag{22a}$$

$$y_\xi \geq 0, \qquad h \geq 0 \text{ almost everywhere}, \tag{22b}$$

$$y_\xi h = y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 \text{ almost everywhere}, \tag{22c}$$

$$\frac{1}{(y_\xi + h)} \in L^\infty(R), \tag{22d}$$

$$\lim_{\xi \to -\infty} \varsigma(\xi) = 0, \tag{22e}$$

$$g\left(y, U, V, y_\xi, U_\xi, h\right) - 1 \in W, \tag{22f}$$

with $y(\xi) = \varsigma(\xi) + \xi$, and $g(\mathbf{x})$ is given by

$$g(\mathbf{x}) = \begin{cases} |x_5| + 2\left(1 + x_2^2 + x_3^2\right)x_4, & \text{if } x \in \Omega, \\ x_4 + x_6, & \text{otherwise}, \end{cases} \tag{23}$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in R^6$, where Ω is the following subset of R^6:

$$\Omega = \left\{ \mathbf{x} \in R^6 \mid |x_5| + 2\left(1 + x_2^2 + x_3^2\right)x_4 \leq x_4 + x_6 \right. \\ \left. \text{and } x_5 \leq 0 \right\}. \tag{24}$$

We write system (21) in a compact form that

$$Y_t(t, \xi) = \chi_{\{t < \tau(\xi)\}}(\xi) F(X) Y(t, \xi), \tag{25}$$

where $Y = (\varsigma_\xi, U_\xi, h)$. For $t > \tau$, we set $Y(t, \xi) = Y(\tau(\xi), \xi)$. Note that, in this definition, we do not reset the energy density h to zero after collision but keep the value it reached just before the collision. Let us define B_M (the subset of E) as follows:

$$B_M = \left\{ X = (\varsigma, U, V, N, \varsigma_\xi, U_\xi, h) \in E \mid \|X\|_E \right. \\ \left. + \|g(X) - 1\|_W + \left\| \frac{1}{(y_\xi + h)} \right\|_{L_T^\infty L_R^\infty} \leq M \right\}. \tag{26}$$

Theorem 2. *Given initial data $X_0 = (\varsigma_0, U_0, V_0, N_0, h_0) \in \Gamma$, there exists a time $T > 0$ such that the system (20) and (21) admit a unique solution in $C^1([0, T], B_M)$.*

Proof. To obtain the local existence of solutions, it suffices to show that $F(X)$ given by (25) is a contraction, and, therefore, there exists a unique fixed point $(\varsigma, U, V, N, \varsigma_\xi, U_\xi, h) \in C([0, T], B_M)$ which is solution to (20) and (21).

Our main task is to prove the estimates for P and P_x defined by (19). Let us write $P_x(\xi)$ as $P_x(\xi) = P_x^1(\xi) + P_x^2(\xi)$ with

$$P_x^1(\xi) = -\frac{e^{-\varsigma(\xi)}}{4} \cdot \left(f_1 * \left[\chi_{\{t < \tau(\xi)\}} r_1\right]\right)(t, \xi), \tag{27}$$

$$P_x^2(\xi) = \frac{e^{\varsigma(\xi)}}{4} \cdot \left(f_2 * \left[\chi_{\{t < \tau(\xi)\}} r_2\right]\right)(t, \xi),$$

where

$$f_1(\xi) = \chi_{\{\xi > 0\}}(\xi) e^{-\xi}, \qquad r_1 = r_1^1 + r_1^2,$$

$$r_1^1 = e^\varsigma\left(U^2 + 2V\right)\left(1 + \varsigma_\xi\right), \qquad r_1^2 = e^\varsigma h,$$

$$f_2(\xi) = \chi_{\{\xi < 0\}}(\xi) e^{-\xi}, \qquad r_2 = r_2^1 + r_2^2, \tag{28}$$

$$r_2^1 = e^{-\varsigma}\left(U^2 + 2V\right)\left(1 + \varsigma_\xi\right), \qquad r_2^2 = e^{-\varsigma} h.$$

Young's inequalities imply that

$$\left\|P_x^1(t, \cdot)\right\|_W$$

$$\leq \frac{1}{4} \cdot \left\|e^{-\varsigma}\right\|_{L_R^\infty}$$

$$\times \left(\left\|f_1 * \left[\chi_{\{t < \tau(\xi)\}} r_1^1\right]\right\|_W + \left\|f_1 * \left[\chi_{\{t < \tau(\xi)\}} r_1^2\right]\right\|_W\right)$$

$$\leq C(M)\left(\left(\|f_1\|_{L_R^1} + \|f_1\|_{L_R^2}\right)\|r_1^1\|_{L_R^2}\right.$$

$$\left. + \left(\|f_1\|_{L_R^2} + \|f_1\|_{L_R^\infty}\right)\|r_1^2\|_{L_R^1}\right) \leq C(M), \tag{29}$$

where we have denoted by $C(M)$ a generic constant. The estimate for P can be obtained in exactly the same way. We thus obtain that $F(X)$ given in (25) is a contraction, and the local existence of solutions follows from the standard contraction argument of ordinary differential equations. $\square$

When proving existence and stability, a priori control is essential. Let us introduce the set

$$A_\varepsilon = \left\{\xi \in R \mid 0 < y_{0,\xi}(\xi) \leq \varepsilon, -\varepsilon \leq U_{0,\xi}(\xi) < 0 \right. \\ \left. \text{or} - \varepsilon \leq V_{0,\xi}(\xi) < 0\right\}, \tag{30}$$

and therefore there exists an $\varepsilon(\overline{M}, T)$ such that

$$0 < y_\xi(\xi) \leq \varepsilon, \qquad -\varepsilon \leq U_\xi(\xi) < 0, \tag{31}$$

for $\xi \in A_\varepsilon$, and in addition

$$A_\varepsilon \subset \{\xi \in R \mid 0 < \tau(\xi) < T\}, \tag{32}$$

and the set

$$K_\gamma = \{\xi \in R \mid h_0(\xi) \geq \gamma\} \tag{33}$$

and therefore, by taking γ large enough depending only on $\overline{M}$ and T, $y_\xi(t,\xi)$ is a decreasing function and $U_\xi(t,\xi)$ are increasing functions of time for $\xi \in K_\gamma$, and in addition

$$K_\gamma \subset \{\xi \in R \mid 0 < \tau(\xi) < T\}. \tag{34}$$

We have $\lim_{\varepsilon \to 0} \text{meas}(A_\varepsilon) = 0$ because $\bigcap_{n>0} A_{1/n} = \emptyset$, and $\text{meas}(K_\gamma) \le 1/\gamma \cdot \int_R h_0(\xi)d\xi \le 1/\gamma \cdot \|h_0\|_{L_R^1}$ which implies $\lim_{\varepsilon \to 0} \text{meas}(K_\gamma) = 0$.

Lemma 3. *Given $X_0 \in B_{M_0}$ for some constant M_0, one denotes the solution of* (20), (21) *by $X = (\varsigma, U, V, N, \varsigma_\xi, U_\xi, V_\xi, N_\xi, h) \in C([0,T], B_M)$ with initial data X_0. Let $\overline{M} = \|P\|_{L_T^\infty L_R^\infty} + \|P_x\|_{L_T^\infty L_R^\infty} + M_0$. Then the following hold.*

(i) *Consider*

$$y_\xi(t,\xi) \ge 0, \qquad h(t,\xi) \ge 0, \tag{35}$$

$$y_\xi h = y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 \tag{36}$$

for all t and almost all ξ. Thus $y_\xi(t,\xi) = 0$ implies $U_\xi(t,\xi) = 0$.

(ii) *Consider*

$$\left\| \frac{1}{(y_\xi + h)} (t, \cdot) \right\|_{L_R^\infty} \le \frac{9}{2} \cdot e^{C(\overline{M},T)T} \left\| \frac{1}{(y_\xi + h)} (0, \cdot) \right\|_{L_R^\infty} \tag{37}$$

for all $t \in [0,T]$ and a constant $C(M,T)$.

Proof. (i) We know that (35)–(37) hold for almost every $\xi \in R$ at $t = 0$ because $X_0 \in \Gamma$. We consider a fixed ξ that we suppress in the notation. On the one hand, it follows from (21) that

$$(y_\xi h)_t = U_\xi h + y_\xi \left[(3U^2 - 2P) U_\xi - 2UP_x y_\xi \right]$$
$$= U_\xi h + 3U^2 U_\xi y_\xi - 2PU_\xi y_\xi - 2UP_x y_\xi^2, \tag{38}$$

and on the other hand

$$\left(y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 \right)_t$$
$$= 2y_\xi y_{\xi t} U^2 + 2y_\xi^2 U U_t$$
$$\quad + 2y_\xi y_{\xi t} V^2 + 2y_\xi^2 V V_t + 2U_\xi U_{\xi t} \tag{39}$$
$$= 3U^2 U_\xi y_\xi - 2UP_x y_\xi^2 + U_\xi h - 2PU_\xi y_\xi.$$

Hence, $(y_\xi h)_t = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)_t$, which together with $y_\xi h(0) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)(0)$ yields $y_\xi h(t) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)(t)$ for all $t \in [0,T]$, and (36) has been proved. It follows directly from the definition of τ and y_ξ that $y_\xi(t) \ge 0$ for $t \ge 0$. For $t \in [0,\tau)$, we have $h(t) \ge 0$ as $h(t) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)/y_\xi(t)$. The continuity (with respect to t) of h implies $h(\tau(\xi)) \ge 0$ and, therefore, $h(t) \ge 0$ for all $t \ge 0$ because the variable does not change for $t \ge \tau(\xi)$.

(ii) We consider a fixed ξ and drop it in the notation. Let us denote the Euclidean norm of $Y = (y_\xi, U_\xi, h)$ by $|Y|_2 = (y_\xi^2 + U_\xi^2 + h^2)^{1/2}$. Since $Y_t = F(X)Y$, we have

$$\frac{d}{dt} \cdot |Y|_2^{-2} = -2|Y|_2^{-4} Y \cdot \frac{dY}{dt}$$
$$= -2|Y|_2^{-4} Y \cdot F(X)Y \tag{40}$$
$$\le 2 \sup_{t \in [0,T]} |F(X)(t)| \, |Y|_2^{-2}.$$

We have $|F(X)(t)| \le K(\|P\|_{L_T^\infty L_R^\infty} + \|P_x\|_{L_T^\infty L_R^\infty} + \|U\|_{L_T^\infty L_R^\infty} + \|V\|_{L_T^\infty L_R^\infty})$ for a constant K depending on the norm chosen for the matrix F. We infer from (25) that

$$\|U\|_{L_T^\infty L_R^\infty} \le U_0 + T\|P_x\|_{L_T^\infty L_R^\infty} \le C\left(\overline{M}, T\right), \tag{41}$$

for a constant depending only on $\overline{M}$ and T. We denote generically by $C(\overline{M}, T)$ such constant. Thus, $|F(X)(t)| \le C(M,T)$. Gronwall's lemma implies that $|Y(t)|_2^{-2} \le e^{2C(\overline{M},T)T} |Y(0)|_2^{-2}$, and therefore

$$\frac{1}{\left(|y_\xi| + |U_\xi| + |h|\right)}(t) \le \frac{3}{\left(|y_\xi| + |U_\xi| + |h|\right)}(0) \cdot e^{C(\overline{M},T)T}. \tag{42}$$

Since $y_\xi > 0$ and $h > 0$, it follows from (36) that $|U_\xi(t)| \le \sqrt{y_\xi(t)h(t)} \le (y_\xi + h)(t)/2$, and therefore $(|y_\xi| + |U_\xi| + |h|)(t) \le 3/2 \cdot (y_\xi + h)(t)$, which yields

$$\frac{1}{\left(|y_\xi| + |h|\right)}(t) \le \frac{9}{2} \cdot e^{C(\overline{M},T)T} \cdot \frac{1}{\left(|y_\xi| + |h|\right)}(0). \tag{43}$$

We show that the local solution can be extended into the global solution.

Theorem 4. *Given initial data $(y_0, U_0, V_0, N_0, h_0) \in \Gamma$, system* (20) *and* (21) *admit a unique global solution $(y, U, V, N, h) \in C(R^+, \Gamma)$.*

Proof. The local solution described in Theorem 2 does not provide any lower bound on the time of existence of the solution. Let us introduce the maximum time of existence $T_{\max}$ defined as

$$T_{\max} = \sup \{t \in R^+ \mid \text{the solutions } X(t) \text{ to } (20) \text{ exists in } [0,t]\}. \tag{44}$$

Let us assume that $T_{\max} < \infty$. To prove global existence of the solution, the basic ingredient is a global bound on the solution X.

We begin by showing that, for given $M_0, T_0 > 0$, there exists a constant $M(M_0, T_0)$ such that, for any $X_0 \in B_{M_0}$, if $h_0 \in W(B)$ for some set B, then $\|X\|_{L_T^\infty W(B)}$ is bounded by a constant $C(M, \|h_0\|_{W(B)})$. It follows from (21) that

$$\|h(t, \cdot)\|_{L^1}$$
$$= \lim_{N \to \infty} \int_{-N}^N h(t, \xi) \, d\xi$$

$$= \lim_{N \to \infty} \left(\int_{-N}^{N} h(0, \xi) \, d\xi \right.$$

$$+ \int_{0}^{t} \left((U^3 - 2PU)(t', N) \right.$$

$$\left. - (U^3 - 2PU)(t', -N) \right) dt' \right)$$

$$= \lim_{N \to \infty} \int_{-N}^{N} h(0, \xi) \, d\xi = \|h(0, \cdot)\|_{L^1},$$

$$(45)$$

for all $t \in [0, T]$, where we have used that $\lim_{\xi \to \pm\infty} U(t, \xi) = 0$ as $U(t, \cdot) \in H^1(R)$. We denote by $M(M_0, T_0)$ a general constant and drop t for simplification. We have

$$U^2(\xi) = 2 \int_{-\infty}^{\xi} U(\eta) U_\xi(\eta) \, d\eta \le \int_{R} U(\eta) U_\xi(\eta) \, d\eta$$

$$= \int_{\{y_\xi(\eta) > 0\}} U(\eta) U_\xi(\eta) \, d\eta$$

$$\le \frac{1}{2} \cdot \int_{\{y_\xi(\eta) > 0\}} \left(y_\xi U^2 + \frac{U_\xi^2}{y_\xi} \right)(\eta) \, d\eta \tag{46}$$

$$= \frac{1}{2} \cdot \int_{\{y_\xi(\eta) > 0\}} h(\eta) \, d\eta \le \frac{1}{2} \|h(t, \cdot)\|_{L^1(R)},$$

where we have used the fact that when $y_\xi(\xi) = 0$, $U_\xi(\xi) = 0$. Thus,

$$\|U(t, \cdot)\|_{L^\infty(R)} \le \frac{1}{\sqrt{2}} \|h(t, \cdot)\|_{L^1(R)}^{1/2}$$

$$= \frac{1}{\sqrt{2}} \|h_0\|_{L^1(R)}^{1/2} \le M(M_0, T_0). \tag{47}$$

We can obtain from the governing equation (20) that

$$|\varsigma(t, \xi)| \le |\varsigma(0, \xi)| + \|U(t, \cdot)\|_{L^\infty}, \qquad T \le M(M_0, T_0), \tag{48}$$

and then $\|\varsigma(t, \cdot)\|_{L_R^\infty} \le M(M_0, T_0)$. We can also get from the governing equation (20) that

$$\|V(t, \cdot)\|_{L^\infty(R)} \le M(M_0, T_0),$$

$$\|N(t, \cdot)\|_{L_R^\infty} \le M(M_0, T_0). \tag{49}$$

From the identity $h = (U^2 + M^2 + V^2) y_\xi$, we can deduce that

$$\left| (U^2 + 2V) y_\xi \right| \le \left(U^2 + V^2 + 1 \right) y_\xi \le h + y_\xi, \tag{50}$$

which implies that

$$|P_x| \le \frac{1}{4} \cdot \left| \int_R e^{-|y(\xi) - y(\xi')|} \left[h + (U^2 + 2V) y_\xi \right](\xi') \, d\xi' \right|$$

$$\le \frac{1}{4} \cdot \left| \int_R e^{-|y(\xi) - y(\xi')|} \left[2h + y_\xi \right](\xi') \, d\xi' \right| \tag{51}$$

$$\le C \left(\|h(t, \cdot)\|_{L^1} + \|\varsigma(t, \cdot)\|_{L_R^\infty} \right) \le M(M_0, T_0).$$

Therefore, $\|P_x\|_{L^\infty} \le M(M_0, T_0)$. To summarize, we have established

$$\|\varsigma(t, \cdot)\|_{L_R^\infty} + \|U(t, \cdot)\|_{L_R^\infty} + \|V(t, \cdot)\|_{L_R^\infty}$$

$$+ \|N(t, \cdot)\|_{L_R^\infty} + \|h(t, \cdot)\|_{L_R^\infty} + \|P(t, \cdot)\|_{L_R^\infty} \tag{52}$$

$$+ \|P_x(t, \cdot)\|_{L_R^\infty} \le M(M_0, T_0).$$

We consider a set B such that $\|h_0\|_{W(B)}$ is finite. Thus it is not hard to obtain that

$$\|P_x\|_{L^2(B)} \le \left(\|h(t, \cdot)\|_{L^1(B)} + \|\varsigma(t, \cdot)\|_{L^\infty(B)} \right) \left\| e^{-y(\xi)} \right\|_{L^2(B)}$$

$$\le M(M_0, T_0). \tag{53}$$

Similarly, one can obtain that the same bound holds for P. Let

$$\alpha(t) = \|U(t, \cdot)\|_{W(B)} + \left\| U_\xi(t, \cdot) \right\|_{W(B)}$$

$$+ \|V(t, \cdot)\|_{W(B)} + \left\| \varsigma_\xi(t, \cdot) \right\|_{W(B)} + \|h(t, \cdot)\|_{W(B)}. \tag{54}$$

By taking the $W(B)$-norm on both sides of (20) and (21), it follows that

$$\alpha(t) \le \alpha(0) + M(M_0, T_0) \int_0^t \alpha(t') \, dt', \tag{55}$$

which, by using Gronwall's lemma, yields

$$\alpha(t) \le \alpha(0) e^{M(M_0, T_0)T} \le C, \tag{56}$$

for some constant C depending only on $M(M_0, T_0)$ and $\|h_0\|_{W(B)}$.

It is not hard to check that there exists a constant M such that $X(t) \in B_M$ for all $t \in [0, T_{\max})$. We know that there exists a constant γ such that (34) holds. From (20), we get

$$\left\| \varsigma(t', \cdot) - \varsigma(t, \cdot) \right\|_{L^\infty(R)} \le M \left| t' - t \right|,$$

$$\left\| U(t', \cdot) - U(t, \cdot) \right\|_W \le C(M) \left| t' - t \right|,$$

$$\left\| V(t', \cdot) - V(t, \cdot) \right\|_W \le C(M) \left| t' - t \right|, \tag{57}$$

$$\left\| N(t', \cdot) - N(t, \cdot) \right\|_W \le C(M) \left| t' - t \right|,$$

for a constant $C(M)$ depending only on M. We denote generically by $C(M)$ such constants. From the above proof, we have $\|h(t, \cdot)\|_{W(K_\gamma^c)} \le C(M)$. It follows from (25) that

$$\left\| Y(t', \cdot) - Y(t, \cdot) \right\|_{W(K_\gamma^c)}$$

$$\le \int_t^{t'} \|F(X) Y\|_{W(K_\gamma^c)} \le C(M) \left| t' - t \right|. \tag{58}$$

We obtain from (22c) that

$$
\begin{aligned}
&\left\| h\left(t', \cdot\right) - h\left(t, \cdot\right) \right\|_{L^1(K_\gamma^c)} \\
&= \left\| \left[U^2\left(1 + \varsigma_\xi\right)^2 + U_\xi^2 + V^2\left(1 + \varsigma_\xi\right)^2 - \varsigma_\xi h \right]\left(t', \cdot\right) \right. \\
&\qquad \left. - \left[U^2\left(1 + \varsigma_\xi\right)^2 + U_\xi^2 + V^2\left(1 + \varsigma_\xi\right)^2 - \varsigma_\xi h \right]\left(t, \cdot\right) \right\|_{L^1(K_\gamma^c)} \\
&\leq C\left(M\right)\left(\left\| U\left(t', \cdot\right) - U\left(t, \cdot\right) \right\|_{L^2(K_\gamma^c)} \right. \\
&\qquad\qquad + \left\| V\left(t', \cdot\right) - V\left(t, \cdot\right) \right\|_{L^2(K_\gamma^c)} \\
&\qquad\qquad \left. + \left\| Y\left(t', \cdot\right) - Y\left(t, \cdot\right) \right\|_{L^2(K_\gamma^c)} \right) \\
&\leq C\left(M\right)\left| t' - t \right|,
\end{aligned}
\tag{59}
$$

and, therefore, $\|X(t', \cdot) - X(t, \cdot)\|_{E(K_\gamma^c)} \leq C(M)|t' - t|$. For $\xi \in K_\gamma$, we have from (34) that $(\varsigma_\xi, U_\xi, h)(t', \xi) = (\varsigma_\xi, U_\xi, h)(t, \xi)$ for $t' \geq t \geq T$. Thus we have

$$
\left\| X\left(t', \cdot\right) - X\left(t, \cdot\right) \right\|_E \leq C\left(M\right)\left| t' - t \right|,
\tag{60}
$$

which implies that $\lim_{t \to T_{\max}} X(t)$ exists as E is a Banach space and we denote it by $\widetilde{X}$. We claim that $\widetilde{X} \in \Gamma$. Since $X(t) \in B_M$ for all $t \in [0, T_{\max})$, we have $\|1/(y_\xi + h)(t, \cdot)\| \leq M$ and therefore $\|1/(\widetilde{y}_\xi + \widetilde{h})\| \leq M$ and the condition (22d) holds. Since L^∞ or L^1 convergence implies almost everywhere convergence up to a subsequence, the conditions (22b), (22c), and (22e) are fulfilled. It remains to check (22f). Note that the mapping g defined in (23) is lower-semicontinuous. We consider a sequence $t_n \to T_{\max}$ such that $X(t_n, \xi) \to \widetilde{X}(\xi)$ for almost every ξ. We can check from the definition of g that $g(X) - y_\xi$ is positive as y_ξ and h are positive. Thus, we get by the lower semicontinuity of g that

$$
\begin{aligned}
0 &\leq g\left(\widetilde{X}\left(\xi\right)\right) - \widetilde{y}_\xi\left(\xi\right) \\
&\leq \liminf \left(g\left(X\left(t_n, \cdot\right)\right) - y_\xi\left(t_n, \cdot\right) \right) \leq 2M
\end{aligned}
\tag{61}
$$

and $g(\widetilde{X}) - 1 \in L^\infty(R)$ because $\|g(\widetilde{X}) - 1\|_{L^\infty(R)} \leq \|g(\widetilde{X}) - \widetilde{y}_\xi\|_{L^\infty(R)} + \|\widetilde{\varsigma}_\xi\|_{L^\infty(R)} \leq 3M$. The composition of an increasing lower semicontinuous function with a lower semicontinuous function is also lower semicontinuous. Hence, since $z \to z^2$ is increasing for $z \geq 0$ and $g(X) - y_\xi$ is positive, we get from Fatou's Lemma that

$$
\begin{aligned}
&\int_R \left| g\left(\widetilde{X}\left(\xi\right)\right) - \widetilde{y}_\xi\left(\xi\right) \right|^2 d\xi \\
&\qquad \leq \int_R \liminf \left| g\left(X\left(t_n, \xi\right)\right) - y_\xi\left(t_n, \xi\right) \right|^2 d\xi \\
&\qquad \leq \liminf \int_R \left| g\left(X\left(t_n, \xi\right)\right) - y_\xi\left(t_n, \xi\right) \right|^2 d\xi,
\end{aligned}
\tag{62}
$$

which implies

$$
\left\| g\left(\widetilde{X}\right) - 1 \right\|_{L^2(R)} \leq \left\| g\left(\widetilde{X}\right) - \widetilde{y}_\xi \right\|_{L^2(R)} + \left\| \widetilde{\varsigma}_\xi \right\|_{L^2(R)} \leq 3M
\tag{63}
$$

as

$$
\begin{aligned}
&\left\| g\left(X\left(t_n, \cdot\right)\right) - y_\xi\left(t_n, \cdot\right) \right\|_{L^2(R)} \\
&\qquad \leq \left\| g\left(X\left(t_n, \cdot\right)\right) - 1 \right\|_{L^2(R)} + \left\| \varsigma_\xi\left(t_n, \cdot\right) \right\|_{L^2(R)} \leq 2M.
\end{aligned}
\tag{64}
$$

Hence, $\widetilde{X}$ fulfills (22f) and $\widetilde{X} \in \Gamma$. We get from Theorem 2 the existence of a local solution with initial data $\widetilde{X}$ which, combined with X on $[0, T_{\max})$, gives a solution on $[0, T_{\max} + \delta)$ for some $\delta > 0$. The assumption regarding $T_{\max}$ is contradicted, and we have proved the global existence of solutions. $\qquad\square$

4. Global Dissipative Solutions for the Original System

We show that the global solution of the equivalent system (20) and (21) yields a global dissipative solution of the original system (4), which needs to establish the correspondence between the Lagrangian equivalent system and the original system.

Let us start by introducing the set Γ_0 as $\Gamma_0 = \{X \in \Gamma \mid g(X) = 1\}$. For any $X \in \Gamma$, we define

$$
\varphi\left(\xi\right) = \int_{-\infty}^{\xi} \left(g\left(X\left(\eta\right)\right) - y_\xi\left(\eta\right) \right) d\eta + y\left(\xi\right)
\tag{65}
$$

so that $\varphi_\xi = g(X)$, and

$$
\begin{aligned}
\overline{y}\left(\xi\right) &= y\left(\eta\right), & \overline{U}\left(\xi\right) &= U\left(\eta\right), \\
\overline{V}\left(\xi\right) &= V\left(\eta\right), & \overline{N}\left(\xi\right) &= N\left(\eta\right)
\end{aligned}
\tag{66}
$$

for any η such that $\xi = \varphi(\eta)$. We consider the pushforward of $h d\xi$ by φ and denote it by ν; that is, $\nu = \varphi_\#(h d\xi)$. By the Radon-Nikodym theorem, there exists a unique function $\overline{h}$ in L^1 such that

$$
\varphi_\#\left(h d\xi\right) = \nu = \overline{h} d\xi + \nu_s,
\tag{67}
$$

where ν_s is the singular part of the decomposition of ν and $\overline{h} d\xi$ the absolutely continuous part. From the definitions (65)–(67), it is not hard to check that $\overline{X} = (\overline{\varsigma}, \overline{U}, \overline{V}, \overline{N}, \overline{h}) \in \Gamma_0$. Thus we define the mapping $\prod : \Gamma \to \Gamma_0$ as $\overline{X} = \prod(X)$, which is a continuous mapping with respect to the distance $\widetilde{d}_R$ on bounded set of Γ; that is, for any sequence X^n and X in B_M, we have

$$
\lim_{n \to \infty} \widetilde{d}_R\left(X^n, X\right) = 0 \quad \text{implies} \lim_{n \to \infty} \widetilde{d}_R\left(\prod\left(X^n\right), \prod\left(X\right)\right) = 0.
\tag{68}
$$

The system is invariant with respect to relabeling. Let us explain what we mean by relabeling.

Definition 5. If there exists a ψ which satisfies

$$\psi(\xi) - \xi \in L^\infty(R),$$

$$\psi_\xi - 1 \in L^\infty(R) \cap L^2(R), \quad \psi_\xi \geq 0, \qquad \text{(69a)}$$

$$\lim_{\xi \to -\infty}(\psi(\xi) - \xi) = 0$$

and such that

$$\overline{y} = y \circ \psi, \qquad \overline{U} = U \circ \psi, \qquad \overline{V} = V \circ \psi, \qquad \overline{N} = N \circ \psi, \tag{69b}$$

one says that $\overline{X} \in \Gamma$ is a relabeling of $X \in \Gamma$.

Note that $\overline{X} \in \Gamma$ and $X \in \Gamma$ imply that for almost all $\xi \in R$ such that $\overline{y}_\xi(\xi) \neq 0$,

$$\overline{h}(\xi) = h \circ \psi \psi_\xi(\xi). \tag{70}$$

Thus, for any $t \geq 0$, if $\overline{X}_0$ is a relabeling of X_0, $\overline{X}(t) = S_t(\overline{X}_0)$ is a relabeling of $X(t) = S_t(X_0)$. In the dissipative case, we cannot define an equivalence relation between elements that are equal up to a relabeling. If $\overline{X}$ is a relabeling of X, then X is not necessarily a relabeling of $\overline{X}$, basically because ψ^{-1} is either not well defined or not sufficiently regular. However, we have the result that if X_2 is a relabeling of X_1, then $\prod(X_2) = \prod(X_1)$. Thus we can define an equivalence relation in Γ as follows: X_1 and X_2 are equivalent if $\prod(X_1) = \prod(X_2)$; that is, if X_2 is a relabeling of X_1, then X_1 and X_2 are equivalent. The set of equivalent classes is in bijection with Γ_0 on which we can define a semigroup as $\widetilde{S}_t = \prod \circ S_t$ with the semigroup property

$$\widetilde{S}_{t+t'} = \prod \circ S_{t+t'} = \prod \circ S_t \circ S_{t'} = \prod \circ S_t \circ \prod \circ S_{t'} = \widetilde{S}_t \circ \widetilde{S}_{t'}. \tag{71}$$

We now establish a bijection between Lagrangian equivalent system and original system by introducing two mappings between the original variable $z = (u,v) \in H^1 \times W$ and the Lagrangian variable $X \in \Gamma_0$.

The mapping L is defined from original system to Lagrangian equivalent system as follows. Given $(u,v) \in H^1 \times W$, we denote $\overline{X}(x) = (x, u, v, 1, u_x, u^2 + u_x^2 + v^2)$ and define y as

$$\int_{-\infty}^{y(\xi)} \left(g\left(\overline{X}(x)\right) - 1\right) dx + y(\xi) = \xi \tag{72}$$

and set

$$(U, V, N) = (u, v, u_x) \circ y, \qquad h = \left(u^2 + u_x^2 + v^2\right) \circ y y_\xi. \tag{73}$$

Then, $X = (y, U, V, N, h) \in \Gamma_0$. We denote by L the mapping $(u,v) \to X$ from $H^1 \times W$ to Γ_0, which sends bounded set of $H^1 \times W$ into bounded sets of Γ_0; that is, for any $M > 0$ and $(u,v) \in H^1 \times W$, $\|z\|_{H^1} \leq M$ implies $L(z) \in B_{\overline{M}}$ for some constant $\overline{M}$ depending only on M.

Let us introduce the mapping M from Lagrangian equivalent system to original system. Given $X \in \Gamma$, if $x = y(\xi)$, the function

$$(u,v)(x) = (U,V)(\xi) \tag{74}$$

is well defined and belongs to $H^1 \times W$. We denote by M the mapping $X \to (u,v)$ from Γ_0 to $H^1 \times W$ with the property that $M \circ \prod = M$ from which we know that $\prod(X) = \prod(\overline{X})$ implies $M(X) = M(\overline{X})$.

We claim that $H^1 \times W$ is in bijection with Γ_0 as the following theorem shows.

Theorem 6. *Consider the following*

$$M \circ L = Id, \qquad L \circ M = \prod. \tag{75}$$

Proof. Given $(u,v) \in H^1 \times W$, we denote $X = L(u,v)$ and $(\overline{u}, \overline{v}) = M(X)$. We have $(U,V) = (u,v) \circ y$. Since $X \in \Gamma_0$, y is invertible and therefore $(\overline{u}, \overline{v}) = (U,V) \circ y^{-1}$. Hence, $(\overline{u}, \overline{v}) = (u,v)$ and $M \circ L = Id$.

Given $X \in \Gamma_0$, we denote $(u,v) = M(X)$ and $\overline{X} = L(u,v)$. Let $S = \{\xi \in R \mid X(\xi) \in \Omega\}$. It is not hard to check that $\operatorname{meas}(S) < \infty$. As we have known, $g(X) - y_\xi \in L^1$ and since $g(X) = 1$, we have $g(X) - y_\xi = 1 - y_\xi = -\varsigma_\xi$ and $\int_{\xi'}^{\xi}(g(X) - y_\xi)d\eta = -\varsigma(\xi) + \varsigma(\xi')$. Let $\xi' \to -\infty$. We get

$$\int_{-\infty}^{\xi}\left(g(X) - y_\xi\right)d\eta + y(\xi) = \xi \tag{76}$$

as $\lim_{\xi \to -\infty}\varsigma(\xi) = 0$, which can be rewritten as

$$\int_{-\infty}^{\xi}\left(\left(\left|U_\xi\right| + \left(1 + 2U^2 + 2V^2\right)y_\xi\right)\chi_S + h\chi_{S^c}\right)(\xi)\,d\eta + y(\xi)$$
$$= \xi \tag{77}$$

or

$$\int_{-\infty}^{\xi}\left(\left(\left|U_\xi\right| + \left(1 + 2U^2 + 2V^2\right)y_\xi\right)\chi_{y^{-1}(A)} + h\chi_{y^{-1}(A^c)}\right)(\xi)\,d\eta$$
$$+ y(\xi) = \xi. \tag{78}$$

Since $U_\xi = u_x \circ y y_\xi$ and $h = (u^2 + u_x^2 + v^2) \circ y y_\xi$ almost everywhere, after a change of variables in (78), we get

$$\int_{-\infty}^{y(\xi)}\left(\left(\left|u_x\right| + \left(1 + 2u^2 + 2v^2\right)\right)\chi_{(A)} + \left(u^2 + u_x^2 + v^2\right)\chi_{(A^c)}\right)$$
$$\times (\xi)\,d\eta + y(\xi) = \xi. \tag{79}$$

Hence, by definition, y and $\overline{y}$ satisfy (72) and therefore they coincide; that is, $\overline{y} = y$. We have

$$\left(\overline{U}, \overline{V}\right) = (\overline{u}, \overline{v}) \circ \overline{y} = (u,v) \circ y = (U,V),$$

$$\overline{h} = \overline{U}^2\overline{y}_\xi + \frac{\overline{U}_\xi^2}{\overline{y}_\xi} + \overline{V}^2\overline{y}_\xi = U^2 y_\xi + \frac{U_\xi^2}{y_\xi} + V^2 y_\xi = h \tag{80}$$

almost everywhere. $\square$

The bijective mapping L allows us to transport the metric $\tilde{d}_R$ on Γ_0 and the semigroup $\tilde{S}_t$ from Γ_0 to $H^1 \times W$. The metric d_{H^1} on $H^1 \times W$ and the semigroup T_t on H^1 are given by

$$d_{H^1}(z, \bar{z}) = \tilde{d}_R(L(z), L(\bar{z})), \qquad T_t = M \circ \tilde{S}_t \circ L. \quad (81)$$

Then a continuous semigroup of dissipative weak solutions for the two-component Camassa-Holm system is obtained as the following theorem shows.

Theorem 7. *Let $z_0 = (u_0, v_0) \in H^1 \times W$ be given. If one denotes $t \rightarrow z(t) = T_t(z_0)$ to be the corresponding trajectory, then $z = (u, v)$ is a weak solution of the two-component Camassa-Holm system (4), which constructs a continuous semigroup with respect to the metric d_{H^1} on bounded sets of $H^1 \times W$; that is, for any $M > 0$ and any sequence $z_n \in H^1 \times W$ such that $\|z_n\|_{H^1} \le M$, one has that $\lim_{n \to \infty} d_{H^1}(z_n, z) = 0$ implies $\lim_{n \to \infty} d_{H^1}(T_t(z_n), T_t(z)) = 0$.*

Proof. We denote $X_0 = L(z_0)$ and $X(t) = S_t(X_0)$. To prove $z = (u, v)$ is a weak solution of the original system (4), it suffices to show that, for all $\phi \in C^\infty(R^+ \times R)$ with compact support,

$$\int_{R^+ \times R} (-u\phi_t + uu_x\phi)(t, x)\, dx\, dt$$

$$= -\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt,$$

$$\int_{R^+ \times R} (-v\phi_t + uv_x\phi)(t, x)\, dx\, dt \quad (82)$$

$$= -\int_{R^+ \times R} [(v + 1)u_x\phi](t, x)\, dx\, dt,$$

where P_x is given by (4). On the one hand, since $y(t, \xi)$ is Lipschitz and invertible with respect to ξ for almost all t, we then can use the change of variables $x = y(t, \xi)$ and obtain

$$\int_{R^+ \times R} (-u\phi_t + uu_x\phi)(t, x)\, dx\, dt$$

$$= \int_{R^+ \times R} \left[-(Uy_\xi)(t, \xi)\phi_t(t, y(t, \xi)) \right. \quad (83)$$

$$\left. + (UU_\xi)(t, \xi)\phi(t, y(t, \xi)) \right] d\xi\, dt.$$

We have $y_{\xi t}(t, \xi) = \chi_{\{t < \tau(\xi)\}}(\xi)U_\xi(t, \xi)$ and then $y_{\xi t}(t, \xi) = U_\xi(t, \xi)$ as $U_\xi(t, \xi) = 0$ for $t \ge \tau(\xi)$. By using the identity $y_t = U$, and since $y_\xi(t, \xi) = 0$ for $t \ge \tau(\xi)$, it then follows from (20) that

$$\int_{R^+ \times R} \left[-Uy_\xi\phi_t(t, y) + UU_\xi\phi(t, y) \right] d\xi\, dt$$

$$= \frac{1}{4} \cdot \int_{R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}} \left\{ \mathrm{sgn}(\xi - \xi')\, e^{-|y(\xi) - y(\xi')|} \right.$$

$$\left. \times \left[h + (U^2 + 2V)\, y_\xi \right] \right\}(\xi')$$

$$\cdot \phi(t, y(\xi))\, y_\xi(\xi)\, d\xi'\, d\xi\, dt. \quad (84)$$

On the other hand, using the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, and since y is increasing with respect to ξ, we have

$$-\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt$$

$$= \frac{1}{2} \cdot \int_{R^+ \times R^2} \left[\mathrm{sgn}(\xi - \xi')\, e^{-|y(\xi) - y(\xi')|} \right.$$

$$\left. \times \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) \right](t, y(\xi'))$$

$$\cdot \phi(t, y(\xi))\, y_\xi(\xi')\, y_\xi(\xi)\, d\xi'\, d\xi\, dt. \quad (85)$$

We restrict the integration domain to $R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}$ again because $y_\xi(t, \xi) = 0$ for $t \ge \tau(\xi)$. Then it follows from (22c) that

$$-\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt$$

$$= \frac{1}{4} \cdot \int_{R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}} \left\{ \mathrm{sgn}(\xi - \xi')\, e^{-|y(\xi) - y(\xi')|} \right.$$

$$\left. \times \left[h + (U^2 + 2V)\, y_\xi \right] \right\}(\xi')$$

$$\cdot \phi(t, y(\xi))\, y_\xi(\xi)\, d\xi'\, d\xi\, dt. \quad (86)$$

By comparing (84) and (86), we know that

$$\int_{R^+ \times R} \left[-Uy_\xi\phi_t(t, y) + UU_\xi\phi(t, y) \right] d\xi\, dt \quad (87)$$

$$= -\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt.$$

Hence, the first identity in (82) holds. The second identity in (82) follows in the same way. Given a converging sequence $z_n \in H^1 \times W$ such that $\|z_n\|_{H^1} \le M$, we have, by the definition of L, that $X_n = L(z_n) \rightarrow X = L(z)$ in $\tilde{d}_R$ and $L(z_n) \in B_{\overline{M}}$ for some $\overline{M}$, and $\tilde{S}_t(X_n) \rightarrow \tilde{S}_t(X)$ in $\tilde{d}_R$ and therefore $T_t(z_n) \rightarrow T_t(z)$ in d_{H^1} from the definition of $\tilde{S}_t$. $\qquad\square$

5. Multipeakon Solutions of the Original System

We derive a new system of ordinary differential equations for the multipeakon solutions which is well-posed even when collisions occur in this section, and the variables (y, U, V, N, h) are used to characterize multipeakons in a way that avoids the problems related to blowing up.

Solutions of the two-component Camassa-Holm system may experience wave breaking in the sense that the solution develops singularities in finite time, while keeping the H^1 norm finite. Extending the solution beyond wave breaking

imposes significant challenge as can be illustrated in the case of multipeakons given by

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|}, \tag{88}$$

where $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot{p}_i = \sum_{j=1}^{n} p_i p_j \, \text{sgn}\left(q_i - q_j\right) e^{-|q_i - q_j|},$$

$$\dot{q}_i = \sum_{j=1}^{n} p_j e^{-|q_i - q_j|}. \tag{89}$$

Peakons interact in a way similar to that of solitons of the CH equation, and wave breaking may appear when at least two of the q_i coincide. Clearly, if the q_i remain distinct, the system (89) allows for a global smooth solution. In the case where $p_i(0)$ has the same sign for all $i = 1, 2, \ldots n$, then $q_i(t)$ remain distinct, and (89) admits a unique global solution. In this case, the peakons are traveling in the same direction. However, when two peakons have opposite signs, collisions may occur, and if so, the system (89) blows up.

We consider initial data $\bar{z} = (\bar{u}, \bar{v})$ given by

$$(\bar{u}, \bar{v})(x) = \left(\sum_{i=1}^{n} p_i e^{-|x-\xi_i|}, \sum_{i=1}^{n} r_i e^{-|x-\xi_i|}\right). \tag{90}$$

Without loss of generality, we assume that the p_i and r_i are all nonzero and that the ξ_i are all distinct. From Theorem 7 we know that there exists a unique and global weak solution with initial data (90), and the aim is to characterize this solution explicitly. We consider the following characterization of multipeakons. The multipeakons are given as continuous solutions u defined on intervals $[x_i, x_{i+1}]$ as the solutions of the Dirichlet problem

$$u - u_{xx} = 0, \qquad u(x_i) = u_i, \qquad u(x_{i+1}) = u_{i+1}, \tag{91}$$

where the variables x_i denote the position of the peaks and the variables u_i denote the values of u at the peaks. In the following we will show that this property persists for dissipative solutions.

Let us define $\overline{X} = (\bar{y}, \overline{U}, \overline{V}, \overline{N}, \bar{h})$ as

$$\bar{y}(\xi) = \xi, \tag{92a}$$

$$\overline{U}(\xi) = \bar{u}(\xi), \qquad \overline{V}(\xi) = \bar{v}(\xi), \qquad \overline{N}(\xi) = \bar{u}_x(\xi) \tag{92b}$$

$$\bar{h}(\xi) = \bar{u}^2 + \bar{u}_x^2 + \bar{v}^2, \tag{92c}$$

which is a representative of $z = (u,v)$ in Lagrangian equivalent system; that is, $\overline{X} = L(\bar{z})$.

Let $I = \cup_{i=0}^{n} I_i$ where I_i denote the open interval (ξ_i, ξ_{i+1}) with the conventions that $\xi_0 = -\infty$ and $\xi_{n+1} = \infty$. For each interval I_i, we define $\tau_i = \inf\{\tau(\xi) \mid \xi \in (\xi_i, \xi_{i+1})\}$

such that $\tau_i > 0$ for all $i = 1, \ldots, n$. By the linearity of the governing equations (20) and the bounds which hold on the solution X and P, P_x, it is not hard to check that $y, U, V \in C([0, \tau_i], C^2(I_i))$, while $h \in C([0, \tau_i], C^1(I_i))$.

Thus the existence of multipeakon solutions is given by the following theorem.

Theorem 8. *For any given multipeakon initial data $\bar{z}(x) = (\bar{u}, \bar{v})(x) = (\sum_{i=1}^{n} p_i e^{-|x-\xi_i|}, \sum_{i=1}^{n} r_i e^{-|x-\xi_i|})$, let (y, U, V, N, h) be the solution of system (20), (21) with initial data $(\bar{y}, \overline{U}, \overline{V}, \overline{N}, \bar{h})$ given by (92a), (92b), and (92c). Between adjacent peaks, if $x_i = y(t, \xi_i) \neq x_{i+1} = y(t, \xi_{i+1})$, the solution $u(t,x)$ is twice differentiable with respect to the space variable, and one has*

$$(u - u_{xx})(t, x) = 0 \quad \text{for } x \in (x_i, x_{i+1}). \tag{93}$$

Proof. For a given time t, we consider two adjacent peaks $x_i = y(t, \xi_i)$ and $x_{i+1} = y(t, \xi_{i+1})$. If $x_i = x_{i+1}$, then the two peaks have collided and, since y_ξ is positive, we must have $y_\xi(t, \xi) = 0$ for all $\xi \in I_i$. Hence, $t \geq \tau_i$, which conversely implies that $t < \tau_i$ when $x_i(t) < x_{i+1}(t)$. There exists $\xi \in I_i$ such that $x = y(t, \xi)$ for any $x \in (x_i(t), x_{i+1}(t))$. Since $\xi \in I_i$ and $t < \tau_i$, we have $y_\xi(t, \xi) \neq 0$. It follows from the implicit function theorem that $y(t, \cdot)$ is invertible in a neighborhood of ξ and its inverse is C^2, and therefore $u(t, x') = U(t, y^{-1}(t, x'))$ are C^2 with respect to the spatial variable and the quantity $(u - u_{xx})(t, x)$ is defined in the classical sense.

We now prove that $(u - u_{xx})(t, x) = 0$ for $x \in (x_i, x_{i+1})$. Assuming that $y_\xi(t, \xi) \neq 0$, we have

$$u_x \circ y = N, \qquad u_{xx} \circ y = \frac{N_\xi}{y_\xi} = \frac{\left(U_{\xi\xi} y_\xi - y_{\xi\xi} U_\xi\right)}{y_\xi^3}, \tag{94}$$

and therefore

$$(u - u_{xx}) \circ y = \frac{\left(U y_\xi^3 - U_{\xi\xi} y_\xi + y_{\xi\xi} U_\xi\right)}{y_\xi^3}. \tag{95}$$

We set

$$M = U y_\xi^3 - U_{\xi\xi} y_\xi + y_{\xi\xi} U_\xi. \tag{96}$$

For a given $\xi \in I$ and $t < \tau_i$, differentiating (96) with respect to t, it then follows from (20) and (21) that

$$\frac{dM}{dt} = 3U y_\xi^2 y_{\xi t} + U_t y_\xi^3 - U_{\xi\xi t} y_\xi$$

$$- U_{\xi\xi} y_{\xi t} + y_{\xi\xi t} U_\xi + y_{\xi\xi} U_{\xi t} \tag{97}$$

$$= 2U U_\xi y_\xi^2 - \frac{h_\xi y_\xi}{2} + \frac{h y_{\xi\xi}}{2} - V_\xi y_\xi^2.$$

Differentiating (22c) with respect to ξ, we get

$$y_{\xi\xi} h + y_\xi h_\xi = 2 y_\xi y_{\xi\xi} U^2 + 2 y_\xi^2 U U_\xi + 2 U_\xi U_{\xi\xi}$$

$$+ 2 y_\xi y_{\xi\xi} V^2 + 2 y_\xi^2 V V_\xi. \tag{98}$$

We have, after inserting the value of $y_\xi h_\xi$ given by (98) into (97) and multiplying the equation by y_ξ, that

$$y_\xi \cdot \frac{dM}{dt} = UU_\xi y_\xi^3 + U_\xi^2 y_{\xi\xi} - U_\xi U_{\xi\xi} y_\xi. \qquad (99)$$

Since $y_{\xi t} = U_\xi$, it follows from (22c) that

$$y_\xi \cdot \frac{dM}{dt} = y_{\xi t} \cdot M. \qquad (100)$$

For any $\xi \in I$, as $\overline{u}$ is multipeakon initial data, we have $M(0, \xi) = (\overline{u} - \overline{u}_{xx}) \cdot y_\xi^3 = 0$. It thus follows from Gronwall's lemma that $M(t, \xi) = 0$ and therefore $(u - u_{xx})(t, \xi) = 0$ for $t \in [0, \tau_i]$. □

Thus, the system of ordinary differential equations that the dissipative multipeakon solutions satisfy can be derived based on the fact that we have known that the multipeakon structure is preserved by the semigroup of dissipative solutions.

Let us define

$$H_i = \int_{\xi_i}^{\xi_{i+1}} h(\xi) \, d\xi. \qquad (101)$$

For each $i = 1, 2, \ldots, n$, by using (20), we obtain the following system of O.D.E.; namely,

$$\frac{dy_i}{dt} = u_i,$$

$$\frac{du_i}{dt} = -P_{xi},$$

$$\frac{dv_i}{dt} = -(v_i + 1) u_{xi}, \qquad (102)$$

$$\frac{du_{xi}}{dt} = u_i^2 - \frac{u_{xi}^2}{2} + \frac{v_i^2}{2} + v_i - P_i,$$

$$\frac{dH_i}{dt} = u_{i+1}^3 - u_i^3 - 2u_i P_i + 2u_{i+1} P_{i+1},$$

where $(y_i, u_i, v_i, u_{xi}, H_i) = (y, U, V, N, H)(t, \xi_i)$, $P_i = P(t, \xi_i)$, and $P_{xi} = P_x(t, \xi_i)$. We have

$$P_i = \frac{1}{4} \cdot \int_{t < \tau(\xi)} e^{-|y_i - y(\xi')|} \left[h + (U^2 + 2V) y_\xi \right] (\xi') \, d\xi'. \qquad (103)$$

Let us denote $B = \{ \xi' \in R \mid y_\xi(t, \xi') > 0 \}$ and then we get

$$P_i = \frac{1}{2} \cdot \int_B e^{-|y_i - y(\xi')|} \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) \circ y(\xi') y_\xi(\xi') \, d\xi'$$

$$= \frac{1}{2} \cdot \int_{y(B)} e^{-|y_i - x|} \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) (x) \, dx$$

$$= \frac{1}{2} \cdot \int_R e^{-|y_i - x|} \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) dx, \qquad (104)$$

where we have used the fact that $h = (u^2 + u_x^2 + v^2) \circ yy_\xi$ on B and $t < \tau(\xi')$ if and only if $y_\xi(t, \xi') > 0$ and the domain of integration in (103) has been extended to the whole axis as $y_\xi(\xi) = 0$ on B^c which implies that $\text{meas}(y(B^c)) = \int_{y(B^c)} dx = \int_{B^c} y_\xi(\xi) d\xi = 0$. Similarly, we can get that

$$P_{i,x} = -\frac{1}{2} \cdot \int_R \text{sgn}(y_i - x) e^{-|y_i - x|} \left(u^2 + \frac{u_x^2}{2} + \frac{v^2}{2} + v \right) dx. \qquad (105)$$

For $x \in [y_i, y_{i+1}], i = 1, 2, \ldots, n - 1$, we write $z = (u, v)$ as

$$z(x) = (u, v)(x) = (A_i e^x + B_i e^{-x}, C_i e^x + D_i e^{-x}). \qquad (106)$$

The constants A_i, B_i, C_i, and D_i depend on $u_i, u_{i+1}, v_i, v_{i+1}, y_i$, and y_{i+1} and read

$$A_i = \frac{e^{-\overline{y}_i}}{2} \left[\frac{\overline{u}_i}{\cosh(\delta y_i)} + \frac{\delta u_i}{\sinh(\delta y_i)} \right],$$

$$B_i = \frac{e^{\overline{y}_i}}{2} \left[\frac{\overline{u}_i}{\cosh(\delta y_i)} - \frac{\delta u_i}{\sinh(\delta y_i)} \right],$$

$$C_i = \frac{e^{-\overline{y}_i}}{2} \left[\frac{\overline{v}_i}{\cosh(\delta y_i)} + \frac{\delta v_i}{\sinh(\delta y_i)} \right], \qquad (107)$$

$$D_i = \frac{e^{\overline{y}_i}}{2} \left[\frac{\overline{v}_i}{\cosh(\delta y_i)} - \frac{\delta v_i}{\sinh(\delta y_i)} \right],$$

where

$$\overline{y}_i = \frac{1}{2} \cdot (y_i + y_{i+1}), \qquad \delta y_i = \frac{1}{2} \cdot (y_i - y_{i+1}),$$

$$\overline{u}_i = \frac{1}{2} \cdot (u_i + u_{i+1}), \qquad \delta u_i = \frac{1}{2} \cdot (u_i - u_{i+1}), \qquad (108)$$

$$\overline{v}_i = \frac{1}{2} \cdot (v_i + v_{i+1}), \qquad \delta v_i = \frac{1}{2} \cdot (v_i - v_{i+1}).$$

The constants A_i, B_i, C_i, and D_i uniquely determine $z = (u, v)$ on the interval $[y_i, y_{i+1}]$. We now turn to the computation of P_i given by (106). Let us write $z = (u, v)$ as

$$z(t, x) = (u(t, x), v(t, x))$$

$$= \left(\sum_{j=0}^n (A_j e^x + B_j e^{-x}) \chi_{(y_j, y_{j+1})}(x), \right. \qquad (109)$$

$$\left. \sum_{j=0}^n (C_j e^x + D_j e^{-x}) \chi_{(y_j, y_{j+1})}(x) \right).$$

We have set $y_0 = -\infty$, $y_{n+1} = \infty$, $u_0 = u_{n+1} = 0$, $v_0 = v_{n+1} = 0$, $A_0 = u_1 e^{-y_1}$, $B_0 = 0$, $A_n = 0$, $B_n = u_n e^{y_n}$, and $C_0 = v_1 e^{-y_1}$, $D_0 = 0$, $C_n = 0$, $D_n = v_n e^{y_n}$. We have

$$u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}v^2 + v$$

$$= \sum_{j=0}^{n} \left(\frac{3}{2}A_j^2 e^{2x} + A_j B_j + \frac{3}{2}B_j^2 e^{-2x} + \frac{1}{2}C_j^2 e^{2x} \right.$$

$$\left. + C_j D_j + \frac{1}{2}D_j^2 e^{-2x} + C_j e^x + D_j e^{-x} \right) \chi_{(y_j, y_{j+1})}.$$

(110)

Let

$$k_{ij} = \begin{cases} 1 & \text{if } i > j, \\ -1 & \text{if } i \le j. \end{cases}$$

(111)

Inserting (110) into (106), we obtain

$$P_i = \frac{1}{2} \sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)}$$

$$\times \left(\frac{3}{2}A_j^2 e^{2x} + A_j B_j + \frac{3}{2}B_j^2 e^{-2x} \right.$$

$$+ \frac{1}{2}C_j^2 e^{2x} + C_j D_j + \frac{1}{2}D_j^2 e^{-2x}$$

$$\left. + C_j e^x + D_j e^{-x} \right) dx.$$

(112)

From (108) and (110), we get

$$A_j^2 = \frac{e^{-2\bar{y}_j}}{\sinh^2(2\delta y_j)}$$

$$\times \left[\bar{u}_j^2 \sinh^2(\delta y_j) + 2\bar{u}_j \delta u_j \sinh(\delta y_j) \cosh(\delta y_j) \right.$$

$$\left. + \delta u_j^2 \cosh^2(\delta y_j) \right]$$

$$= \frac{e^{-2\bar{y}_j}}{2\sinh(2\delta y_j)}$$

$$\times \left[\bar{u}_j^2 \tanh(\delta y_j) + 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right],$$

$$B_j^2 = \frac{e^{2\bar{y}_j}}{2\sinh(2\delta y_j)}$$

$$\times \left[\bar{u}_j^2 \tanh(\delta y_j) - 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right],$$

$$A_j B_j = \frac{1}{4\sinh(2\delta y_j)} \left[2\bar{u}_j^2 \tanh(\delta y_j) - 2\delta u_j^2 \coth(\delta y_j) \right],$$

$$C_j^2 = \frac{e^{-2\bar{y}_j}}{2\sinh(2\delta y_j)}$$

$$\times \left[\bar{v}_j^2 \tanh(\delta y_j) + 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right],$$

$$D_j^2 = \frac{e^{2\bar{y}_j}}{2\sinh(2\delta y_j)}$$

$$\times \left[\bar{v}_j^2 \tanh(\delta y_j) - 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right],$$

$$C_j D_j = \frac{1}{4\sinh(2\delta y_j)} \left[2\bar{v}_j^2 \tanh(\delta y_j) - 2\delta v_j^2 \coth(\delta y_j) \right].$$

(113)

It then follows from (113) that

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} A_j^2 e^{2x} dx$$

$$= \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(2 + k_{ij})\sinh(2\delta y_j)} \sinh\left((2 + k_{ij})\delta y_j\right)$$

$$\times \left[\bar{u}_j^2 \tanh(\delta y_j) + 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right],$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} B_j^2 e^{-2x} dx$$

$$= \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(k_{ij} - 2)\sinh(2\delta y_j)} \sinh\left((k_{ij} - 2)\delta y_j\right)$$

$$\times \left[\bar{u}_j^2 \tanh(\delta y_j) - 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right],$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} C_j^2 e^{2x} dx$$

$$= \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(2 + k_{ij})\sinh(2\delta y_j)} \sinh\left((2 + k_{ij})\delta y_j\right)$$

$$\times \left[\bar{v}_j^2 \tanh(\delta y_j) + 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right],$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} D_j^2 e^{-2x} dx$$

$$= \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(k_{ij} - 2)\sinh(2\delta y_j)} \sinh\left((k_{ij} - 2)\delta y_j\right)$$

$$\times \left[\bar{v}_j^2 \tanh(\delta y_j) - 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right].$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} A_j B_j dx$$

$$= \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{2\sinh(2\delta y_j)} \sinh(\delta y_j)$$

$$\times \left[2\bar{u}_j^2 \tanh(\delta y_j) - 2\delta u_j^2 \coth(\delta y_j) \right],$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} C_j D_j dx$$

$$= \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{2\sinh\left(2\delta y_j\right)} \sinh\left(\delta y_j\right)$$
$$\times \left[2\bar{v}_j^2 \tanh\left(\delta y_j\right) - 2\delta v_j^2 \coth\left(\delta y_j\right)\right].$$

$$(114)$$

Therefore, the above formulas in (114) imply that

$$P_i = \sum_{j=0}^{n} \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{4\cosh\left(\delta y_j\right)}$$
$$\times \left\{\left[\bar{u}_j^2 \tanh\left(\delta y_j\right) + \delta u_j^2 \coth\left(\delta y_j\right)\right]\right.$$
$$\cdot 2\cosh^2\left(\delta y_j\right) + 4k_{ij}\bar{u}_j\delta u_j\sinh^2\left(\delta y_j\right)$$
$$+ 2\bar{u}_j^2 \tanh\left(\delta y_j\right) + \frac{2}{3}\sinh^2\left(\delta y_j\right)$$
$$\cdot \left[\bar{v}_j^2 \tanh\left(\delta y_j\right) + \delta v_j^2 \coth\left(\delta y_j\right)\right]$$
$$+ \frac{4}{3}k_{ij}\bar{v}_j\delta v_j\sinh^2\left(\delta y_j\right) + 2\bar{v}_j^2$$
$$\cdot \tanh\left(\delta y_j\right) + 2\left[\bar{v}_j - k_{ij}\delta v_j \coth\left(\delta y_j\right)\right]$$
$$\cdot \delta y_j + \left[\bar{v}_j \sinh\left(\delta y_j\right) + k_{ij}\delta v_j \cosh\left(\delta y_j\right)\right]$$
$$\left. \cdot 2\cosh\left(\delta y_j\right)\right\},$$

$$(115)$$

which can also be written in the following form:

$$P_i = \sum_{j=0}^{n} P_{ij} \qquad (116)$$

with

$$P_{ij} = \begin{cases} \dfrac{1}{4}u_1^2 e^{y_1-y_i} + \dfrac{1}{12}v_1^2 e^{y_1-y_i} + \dfrac{1}{4}v_1 e^{y_1-y_i}, \\ \qquad \text{for } j = 0, \\[4pt] \dfrac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{4\cosh\left(\delta y_j\right)} \left\{\left[\bar{u}_j^2 \tanh\left(\delta y_j\right) + \delta u_j^2 \coth\left(\delta y_j\right)\right]\right. \\ \cdot 2\cosh^2\left(\delta y_j\right) + 4k_{ij}\bar{u}_j\delta u_j\sinh^2\left(\delta y_j\right) \\ + 2\bar{u}_j^2 \tanh\left(\delta y_j\right) + \dfrac{2}{3}\sinh^2\left(\delta y_j\right) \\ \cdot \left[\bar{v}_j^2 \tanh\left(\delta y_j\right) + \delta v_j^2 \coth\left(\delta y_j\right)\right] \\ + \dfrac{4}{3}k_{ij}\bar{v}_j\delta v_j\sinh^2\left(\delta y_j\right) + 2\bar{v}_j^2 \tanh\left(\delta y_j\right) \\ + 2\left[\bar{v}_j - k_{ij}\delta v_j \coth\left(\delta y_j\right)\right] \\ \cdot \delta y_j + \left[\bar{v}_j \sinh\left(\delta y_j\right) + k_{ij}\delta v_j \cosh\left(\delta y_j\right)\right] \\ \left. \cdot 2\cosh\left(\delta y_j\right)\right\}, \\ \qquad \text{for } j = 1, \dots, n-1, \\[4pt] \dfrac{1}{4}u_n^2 e^{y_i-y_n} + \dfrac{1}{12}v_n^2 e^{y_i-y_n} + \dfrac{1}{4}v_n e^{y_i-y_n}, \\ \qquad \text{for } j = n. \end{cases}$$

$$(117)$$

We compute P_{ix} in the same way and obtain

$$P_{i,x} = -\sum_{j=0}^{n} k_{ij}P_{ij}. \qquad (118)$$

Now we can summarize the result as follows.

Theorem 9. *Let* $\bar{y}_i = \xi_i$, $(\bar{u}_i, \bar{v}_i, \bar{u}_{xi}) = (\bar{u}, \bar{v}, \bar{u}_x)(\xi_i)$, *and* $\bar{H}_i = \int_{-\infty}^{\xi_i} (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2)dx$ *for* $i = 1, \dots, n$ *with multipeakon initial data* $\bar{z} = (\bar{u}, \bar{v})$ *given by* (90). *Then, with initial data* $(\bar{y}_i, \bar{u}_i, \bar{v}_i, \bar{u}_{xi}, \bar{H}_i)$, *there exists a global solution* $(y_i, u_i, v_i, u_{xi}, H_i)$ *of* (105), (116), *and* (118). *For each time* t, $u(t, x)$ *is defined as the solution of the Dirichlet problem* $u - u_{xx} = 0$ *with boundary conditions* $u(t, y_i(t)) = u_i(t)$, $u(t, y_{i+1}(t)) = u_{i+1}(t)$ *on each interval* $[y_i(t), y_{i+1}(t)]$. *Thus* $z = (u, v)$ *is a dissipative solution of the two-component Camassa-Holm system, which is the dissipative multipeakon solution.*

In the following, we give the examples with cases $n = 1, 2$ with collision and without collision.

6. Numerical Examples

Two examples are considered here to illustrate the property of the system solution. The first is when $n = 1$, and the second is when $n = 2$.

(i) Let $n = 1$. From (116) and (118), we can compute that $P_1 = \sum_{j=0,1} P_{1j} = u_1^2/2 + v_1^2/6 + v_1/2$ and $P_{1,x} = -\sum_{j=0,1} k_{1j}P_{1j} = 0$. Thus from (102), we obtain that $u_1 = c$ and $y_1 = ct + a$ with c, a some constants. There is no collision and we find the one peakon $u(t, x) = ce^{-|x-ct-a|}$.

(ii) Let $n = 2$. We consider the case of an antisymmetric pair of peakons when the two peakons collide. We take the initial conditions as

$$y_2(0) = -y_1(0) = \bar{y}, \qquad u_2(0) = -u_1(0) = \bar{u},$$
$$v_2(0) = -v_1(0) = \bar{v}, \qquad \delta H_1(0) = E^2$$

$$(119)$$

for some strictly positive constants $\bar{y}, \bar{u}, \bar{v}$ and E (the initial total energy of the system). We denote by $\tau = \tau_1$ the time of collision. For $t < \tau$, the solution is identical to the conservative case. After collision, for $t \geq \tau$, the solution remains antisymmetric. Let us assume this for the moment and write

$$y = y_2 = -y_1, \qquad u = u_2 = -u_1, \qquad h = \delta H_1,$$
$$P = P_1 = P_2, \qquad P_x = -P_{1,x} = P_{2,x}.$$

$$(120)$$

Using (116) and (118) we can compute P and P_x and obtain after some calculations that

$$P = \sum_{j=0,2} P_{1,j} = \frac{1}{12} \cdot \left(3u^2 + v^2\right)\left(e^{-2y} + 1\right)$$

$$+ \frac{1}{4} \cdot v\left(e^{-2y} - 1\right),$$

$$P_x = -\sum_{j=0,2} k_{1,j} P_{1,j} = \frac{1}{12} \cdot \left(3u^2 + v^2\right)\left(e^{-2y} - 1\right) \qquad (121)$$

$$+ \frac{1}{4} \cdot v\left(e^{-2y} + 1\right).$$

Thus we are led to the following system of ordinary differential equations:

$$y_t = u,$$

$$u_t = \frac{1}{12} \cdot \left(3u^2 + v^2\right)\left(1 - e^{-2y}\right) - \frac{1}{4} \cdot v\left(1 + e^{-2y}\right), \qquad (122)$$

$$h_t = \left(u^3 + uv\right)\left(1 - e^{-2y}\right) - \frac{1}{3} \cdot uv^2 \left(1 + e^{-2y}\right).$$

Note that this system holds before collision. The solution of (122) with the initial condition $y(\tau) = u(\tau) = v(\tau) = 0$ is $y(t) = u(t) = v(t) = 0$ and $h(t) = h(\tau)$. It means that the multipeakon solution remains identically equal to zero after the collision.

We can get from (116) that

$$P_{1,0} = \frac{1}{4} \cdot \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right),$$

$$P_{1,2} = \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$P_{2,0} = \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right), \qquad (123)$$

$$P_{2,2} = \frac{1}{4} \cdot \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right).$$

Thus, for $t \geq \tau$, we have

$$P_1 = P_{1,0} + P_{1,2} = \frac{1}{4} \cdot \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right)$$

$$+ \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$P_2 = P_{2,0} + P_{2,2} = \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right)$$

$$+ \frac{1}{4} \cdot \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$P_{1,x} = -P_{1,0} + P_{1,2} = -\frac{1}{4} \cdot \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right) \qquad (124)$$

$$+ \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$P_{2,x} = -P_{2,0} + P_{2,2} = -\frac{1}{4} \cdot e^{y_1 - y_2} \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right)$$

$$+ \frac{1}{4} \cdot \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

and therefore the system (102) rewrites

$$\frac{dy_1}{dt} = u_1, \qquad \frac{dy_2}{dt} = u_2,$$

$$\frac{du_1}{dt} = \frac{1}{4} \cdot \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right)$$

$$- \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$\frac{du_2}{dt} = \frac{1}{4} \cdot e^{y_1 - y_2} \left(u_1^2 + \frac{1}{3} \cdot v_1^2 + v_1\right) \qquad (125)$$

$$- \frac{1}{4} \cdot \left(u_2^2 + \frac{1}{3} \cdot v_2^2 + v_2\right),$$

$$\frac{dH_1}{dt} = u_2^3 - u_1^3 - 2u_1 P_1 + 2u_2 P_2.$$

If we consider a more general case with two colliding peakons by using the Hamiltonian system before collision, we have

$$y_1 = \ln\left(\frac{c_1 - c_2}{c_1 e^{-c_1(t-\tau)} - c_2 e^{-c_2(t-\tau)}}\right),$$

$$y_2 = \ln\left(\frac{c_1 e^{c_1(t-\tau)} - c_2 e^{c_2(t-\tau)}}{c_1 - c_2}\right),$$

$$u_1 = \frac{c_1^2 - c_2^2 e^{(c_1 - c_2)(t-\tau)}}{c_1 - c_2 e^{(c_1 - c_2)(t-\tau)}}, \qquad (126)$$

$$u_2 = \frac{c_2^2 - c_1^2 e^{(c_1 - c_2)(t-\tau)}}{c_2 - c_1 e^{(c_1 - c_2)(t-\tau)}},$$

where c_1, c_2 denote the speed of the peaks y_1 and y_2. At collision time, we have

$$y_1(\tau) = y_2(\tau) = 0,$$

$$u_1(\tau) = u_2(\tau) = c_1 + c_2. \qquad (127)$$

Thus, for the initial data given by (127), the solution of (125) is

$$u_1(t) = u_2(t) = c_1 + c_2,$$

$$y_1 = y_2 = (c_1 + c_2)t, \qquad H_1(t) = H_1(\tau) \qquad (128)$$

and after the collision we obtain a single peakon traveling at speed $c_1 + c_2$. Figures 1 and 2 represent the solution for $c_1 = 15$, $c_2 = -5$, and $\tau = 1$, respectively.

7. Conclusion

Considered in this paper is the dissipative property of the two-component Camassa-Holm system after wave breaking. By using a new approach, we obtain the global dissipative solutions of the two-component Camassa-Holm system and then the multipeakon dissipative solutions, a useful result for understanding the inevitable multipeakon phenomenon near wave breaking.
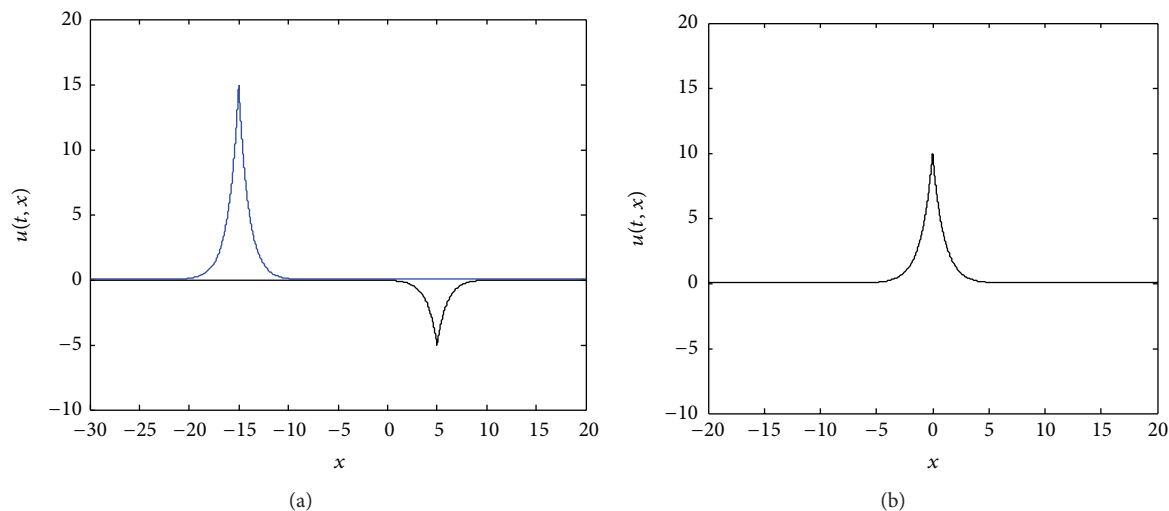
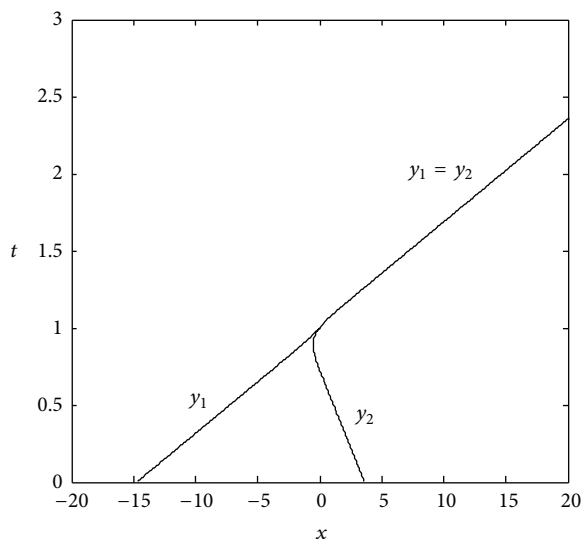Figure 1: (a) Initial configuration. (b) The two peakons collide.

Figure 2: Position of the peakons.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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