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Research Article

Strong Convergence to a Solution of a Variational Inequality Problem in Banach Spaces

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We consider the variational inequality problem for a family of operators of a nonempty closed convex subset of a 2-uniformly convex Banach space with a uniformly Gâteaux differentiable norm, into its dual space. We assume some properties for the operators and get strong convergence to a common solution to the variational inequality problem by the hybrid method proposed by Haugazeau. Using these results, we obtain several results for the variational inequality problem and the proximal point algorithm.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of all positive integers and the set of all real numbers, respectively. Throughout this paper, E is a real Banach space with norm $\|\cdot\|$ and E^* is the dual of E. For $x \in E$ and $x^* \in E^*$, let $\langle x, x^* \rangle$ be the value of x^* at x. Suppose that C is a nonempty closed convex subset of E and A is a monotone operator of C into E^*; that is, $\langle x - y, Ax - Ay \rangle \geq 0$ holds for all $x, y \in C$. Then, we consider the variational inequality problem [1], that is, the problem of finding an element $z \in C$ such that

$$\langle x - z, Az \rangle \geq 0 \quad \forall x \in C. \tag{1}$$

The set of all solutions to the variational inequality problem for A is denoted by $\mathrm{VI}(C, A)$. For $\alpha > 0$, we say that A is α-inverse strongly monotone [2–5] if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \quad \forall x, y \in C. \tag{2}$$

Haugazeau [6] introduced a sequence $\{x_n\}$ generated by the hybrid method by the following way. Let $\{T_n\}$ be a family of mappings of a real Hilbert space H into itself with $\bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &= x \in H, \\ y_n &= T_n x_n, \\ C_n &= \{z \in H : \langle x_n - y_n, y_n - z \rangle \geq 0\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x) \end{aligned} \tag{3}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. He proved a strong convergence theorem when $T_n = P_{(n \bmod m)+1}$ for every $n \in \mathbb{N}$, where P_i is the metric projection of H onto a nonempty closed convex subset C_i of H for each $i = 1, 2, \ldots, m$ with $\bigcap_{i=1}^{m} C_i \neq \emptyset$. Later, Solodov and Svaiter [7], Bauschke and Combettes [8], Nakajo and Takahashi [9], and many researchers studied the hybrid method in a real Hilbert space. In a real Banach space, Kamimura and Takahashi [10], Ohsawa and Takahashi [11], Kohsaka and Takahashi [12], Matsushita and Takahashi [13], Matsushita et al. [14], Nakajo et al. [15], and several researchers studied the hybrid method.

In a real Hilbert space H, Iiduka et al. [16] considered a sequence $\{x_n\}$ generated by the following hybrid method:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{4}$$

for each $n \in \mathbb{N}$, where A is an α-inverse strongly monotone operator of C into H with $\mathrm{VI}(C, A) \neq \emptyset$, P_C is the metric projection of H onto a nonempty closed convex subset C of H, and $\{\lambda_n\} \subset [0, 2\alpha]$. They proved that $\{x_n\}$ converges strongly to $P_{\mathrm{VI}(C,A)}x$; see also [17, 18]. In a 2-uniformly convex and uniformly smooth Banach space E, Iiduka and Takahashi [19] proved the following.

Theorem 1 (Iiduka and Takahashi [19]). *Let A be an α-inverse strongly monotone operator of E into E^* with $A^{-1}0 \neq \emptyset$ and $\{\lambda_n\} \subset [a, c_1\alpha]$ for some $a \in]0, c_1\alpha[$, where c_1 is a positive constant satisfying that $\|x + y\|^2 \ge \|x\|^2 + 2\langle y, Jx\rangle + c_1\|y\|^2$ for every $x, y \in E$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in E, \\ y_n &= J^{-1}(Jx_n - \lambda_n A x_n), \\ C_n &= \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n &= \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \tag{5}$$

for each $n \in \mathbb{N}$, where $\Pi_{C_n \cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$.

Motivated by [19], we propose a new family of operators and prove strong convergence theorems of the sequence generated by these mappings. Using these results, we get several additional results for the problem of variational inequalities and the proximal point algorithm.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ will symbolize strong convergence. We denote by S_E the unit sphere of a Banach space E; that is, $S_E = \{x \in E : \|x\| = 1\}$.

We define the modulus δ_E of convexity of E as follows: δ_E is a function of $[0, 2]$ into $[0, 1]$ such that

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_E, \|x - y\| \ge \epsilon\right\} \tag{6}$$

for every $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for each $\epsilon > 0$. Let $p > 1$. E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \ge c\epsilon^p$ for every $\epsilon \in [0, 2]$. It is obvious that a p-uniformly convex Banach space is uniformly convex. E is said to be strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in S_E$ with $x \neq y$. We know that a uniformly convex Banach space is strictly convex and reflexive. The duality mapping $J : E \to 2^{E^*}$ of E is defined by

$$J(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\} \tag{7}$$

for every $x \in E$. It is also known that if E is strictly convex and reflexive, then the duality mapping J of E is bijective and $J^{-1} : E^* \to 2^E$ is the duality mapping of E^*. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{8}$$

exists for every $x, y \in S_E$. The norm of E is said to be uniformly Gâteaux differentiable if, for each $y \in S_E$, the limit (8) is attained uniformly for $x \in S_E$. E is said to be uniformly smooth if the limit (8) is attained uniformly for $(x, y) \in S_E \times S_E$. We know that the duality mapping J of E is single-valued if and only if E is smooth. It is also known that if E is uniformly smooth, then the duality mapping J is uniformly continuous on bounded subsets of E and if the norm of E is uniformly Gâteaux differentiable, then J is norm-to-weak* uniformly continuous on bounded subsets of E; see [20, 21] for more details. The following is proved by Xu [22]; see also [23].

Theorem 2 (Xu [22]). *Let E be a smooth Banach space. Then, the following are equivalent.*

(i) *E is 2-uniformly convex.*

(ii) *There exists a constant $c_1 > 0$ such that $\|x + y\|^2 \ge \|x\|^2 + 2\langle y, Jx\rangle + c_1\|y\|^2$ holds for each $x, y \in E$.*

Remark 3. In the case where E is a real Hilbert space, J is the identity mapping and we can choose $c_1 = 1$.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{9}$$

for every $x, y \in E$. It is obvious that $(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2$ for each $x, y \in E$ and $\phi(z, x) + \phi(x, y) = \phi(z, y) + 2\langle x - z, Jx - Jy\rangle$ for all $x, y, z \in E$. It is also known that if E is strictly convex and smooth, then, for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$; see also [13]. We have the following result from Theorem 2.

Lemma 4. *Let E be a 2-uniformly convex and smooth Banach space. Then, for each $x, y \in E$, $\phi(x, y) \ge c_1\|x - y\|^2$ holds, where c_1 is a constant in Theorem 2.*

Proof. Let $x, y \in E$. By Theorem 2, we have

$$\phi(x, y) = \|x\|^2 - \|y\|^2 - 2\langle x - y, Jy\rangle \ge c_1\|x - y\|^2, \tag{10}$$

which is the desired result. □

Let C be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space E and let $x \in E$. Then, there exists a unique element $y_0 \in C$ such that

$$\phi(y_0, x) = \inf_{y \in C} \phi(y, x). \tag{11}$$

We denote y_0 by $\Pi_C x$ and call Π_C the generalized projection of E onto C; see [10, 24, 25]. We have the following well-known results [10, 24, 25] for the generalized projection.

Lemma 5. *Let C be a nonempty convex subset of a smooth Banach space E, $x \in E$, and $y_0 \in C$. Then, $\phi(y_0, x) = \inf_{y \in C} \phi(y, x)$ if and only if $\langle y_0 - z, Jx - Jy_0 \rangle \geq 0$ for all $z \in C$.*

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E and let $x \in E$. Then, there exists a unique element $y_0 \in C$ such that $\|y_0 - x\| = \inf_{y \in C} \|y - x\|$. Putting $y_0 = P_C x$, we call P_C the metric projection of E onto C; see [26]. We have the following result for the metric projection; see [20] for more details.

Lemma 6. *Let C be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space E, $x \in E$, and $y_0 \in C$. Then, $y_0 = P_C x$ if and only if $\langle y_0 - z, J(x - y_0) \rangle \geq 0$ for all $z \in C$.*

An operator $T : E \to 2^{E^*}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for every $(x, x^*), (y, y^*) \in T$. Notice that we often identify a set-valued operator with its graph; $x^* \in Tx$ if and only if $(x, x^*) \in T$.

A monotone operator $T \subset E \times E^*$ is said to be maximal if the graph of T is not properly contained in the graph of any other monotone operator. It is easy to see that a monotone operator $T \subset E \times E^*$ is maximal if and only if, for $(u, u^*) \in E \times E^*$, $\langle x - u, x^* - u^* \rangle \geq 0$ for every $(x, x^*) \in T$ implies that $(u, u^*) \in T$. We know the following result.

Theorem 7 (Rockafellar [27]; see also [28]). *Let E be a strictly convex, reflexive, and smooth Banach space and let T be a monotone operator of E into E^*. Then, T is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$, where $R(J + rT)$ is the range of $J + rT$.*

From this fact, we also know that if E is a strictly convex, reflexive, and smooth Banach space and T is a maximal monotone operator of E into E^*, then, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(T)$ such that $J(x_r - x) + rTx_r \ni 0$, where $D(T)$ is the domain of T. We define $J_r : E \to E$ by $J_r x = x_r$ for every $x \in E$ and $r > 0$, and such J_r is called the resolvent of T; see [21, 29] for more details.

3. Main Results

Let C be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space E and $\{A_n\}_{n \in \mathbb{N}}$ a family of operators of C into E^* satisfying the following:

(i) $F = \bigcap_{n=1}^{\infty} \mathrm{VI}(C, A_n) \neq \emptyset$;

(ii) $\langle x - z, A_n x - A_n z \rangle \geq 0$ for all $n \in \mathbb{N}$, $x \in C$, and $z \in F$;

(iii) there exists a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ in $]0, \infty[$ such that $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < \infty$ and $\langle x - z, A_n x \rangle \geq \alpha_n \|A_n x - A_n z\|^2$ for every $n \in \mathbb{N}$, $x \in C$, and $z \in F$;

(iv) for all $z \in F$, $\sup_{n \in \mathbb{N}} \|A_n z\| < \infty$;

(v) for every bounded sequence $\{z_n\} \subset C$, $z \in F$, and $\{r_n\} \subset]0, \infty[$ with $\inf_{n \in \mathbb{N}} r_n > 0$, if $\lim_{n \to \infty} \|z_{n+1} - z_n\| = \lim_{n \to \infty} \|z_n - \Pi_C J^{-1}(Jz_n - r_n A_n z_n)\| = \lim_{n \to \infty} \|A_n z_n - A_n z\| = 0$, then there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in F$.

Let us observe some properties of the mappings and the subsets deduced from the assumptions above.

First, we know that, for any $n \in \mathbb{N}$, the image of F by A_n is a singleton. Indeed, for $z_1, z_2 \in F$, we have $\langle z_1 - z_2, A_n z_1 \rangle \geq \alpha_n \|A_n z_1 - A_n z_2\|^2$ by the condition (iii). On the other hand, since $z_1 \in \mathrm{VI}(C, A_n)$, it follows that $\langle z_2 - z_1, A_n z_1 \rangle \geq 0$. Thus, we get $A_n z_1 = A_n z_2$ for all $z_1, z_2 \in F$.

Next, if we assume $\bigcap_{n=1}^{\infty} A_n^{-1} 0 \neq \emptyset$, then we have $\bigcap_{n=1}^{\infty} A_n^{-1} 0 = \bigcap_{n=1}^{\infty} \mathrm{VI}(C, A_n)$. Indeed, the inclusion $\bigcap_{n=1}^{\infty} A_n^{-1} 0 \subset \bigcap_{n=1}^{\infty} \mathrm{VI}(C, A_n)$ is trivial. To show the opposite inclusion, let $z \in \bigcap_{n=1}^{\infty} \mathrm{VI}(C, A_n)$ and $u \in \bigcap_{n=1}^{\infty} A_n^{-1} 0$. By the condition (iii), we have $0 = \langle u - z, A_n u \rangle \geq \alpha_n \|A_n u - A_n z\|^2 = \alpha_n \|A_n z\|^2$, which implies $A_n z = 0$; that is, $z \in A_n^{-1} 0$ for all $n \in \mathbb{N}$. Hence, we get $\bigcap_{n=1}^{\infty} A_n^{-1} 0 \supset \bigcap_{n=1}^{\infty} \mathrm{VI}(C, A_n)$.

We also know that F is closed and convex. Indeed, for $z_1, z_2 \in F$ and $\beta \in]0, 1[$, let $z = \beta z_1 + (1 - \beta) z_2$. By the condition (iii), $\langle z - z_1, A_n z \rangle \geq \alpha_n \|A_n z - A_n z_1\|^2$ and $\langle z - z_2, A_n z \rangle \geq \alpha_n \|A_n z - A_n z_2\|^2$ hold for all $n \in \mathbb{N}$. Thus we get

$$\begin{aligned} 0 &= \langle z - (\beta z_1 + (1 - \beta) z_2), A_n z \rangle \\ &= \beta \langle z - z_1, A_n z \rangle + (1 - \beta) \langle z - z_2, A_n z \rangle \\ &\geq \alpha_n \left(\beta \|A_n z - A_n z_1\|^2 + (1 - \beta) \|A_n z - A_n z_2\|^2 \right) \\ &\geq 0, \end{aligned} \tag{12}$$

which implies that $A_n z = A_n z_1 = A_n z_2$ for each $n \in \mathbb{N}$. Since $\langle x - z_1, A_n z \rangle = \langle x - z_1, A_n z_1 \rangle \geq 0$ and $\langle x - z_2, A_n z \rangle = \langle x - z_2, A_n z_2 \rangle \geq 0$ for every $n \in \mathbb{N}$ and $x \in C$, we have

$$\langle x - z, A_n z \rangle = \beta \langle x - z_1, A_n z \rangle + (1 - \beta) \langle x - z_2, A_n z \rangle \geq 0 \tag{13}$$

for all $n \in \mathbb{N}$ and $x \in C$; that is, $z \in \mathrm{VI}(C, A_n)$ for each $n \in \mathbb{N}$. Hence, F is convex.

To see F being closed, let $\{z_m\}$ be a sequence in F such that $z_m \to z$. Since we have $\langle z - z_m, A_n z \rangle \geq \alpha_n \|A_n z - A_n z_m\|^2$ for every $m, n \in \mathbb{N}$ from the condition (iii), we get $\lim_{m \to \infty} \|A_n z - A_n z_m\| = 0$ for all $n \in \mathbb{N}$. Since $\langle x - z_m, A_n z_m \rangle \geq 0$ for each $m, n \in \mathbb{N}$ and $x \in C$, we obtain $\langle x - z, A_n z \rangle \geq 0$ for every $n \in \mathbb{N}$ and $x \in C$; that is, $z \in \mathrm{VI}(C, A_n)$ for all $n \in \mathbb{N}$. Therefore, F is closed.

Now, we get the following result by the hybrid method using the generalized projections.

Theorem 8. *Let C be a nonempty closed convex subset of a 2-uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable, and let $\{A_n\}$ be a sequence of operators of C into E^* satisfying the conditions (i)–(v). Let $\{\lambda_n\}$ be a sequence in $]0,\infty[$ such that $\inf_{n\in\mathbb{N}}\lambda_n > 0$ and $\inf_{n\in\mathbb{N}}(2c_1\alpha_n - \lambda_n) > 0$, where c_1 is the constant in Theorem 2. Let $x \in C$ and $\{x_n\}$ a sequence in C generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n x_n\rangle\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \tag{14}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof. It is obvious that Q_n is closed and convex for every $n \in \mathbb{N}$. Since $\phi(z, y_n) \le \phi(z, x_n) + 2\lambda_n\langle x_n - z, A_n x_n\rangle$ if and only if $2\langle z, Jx_n - Jy_n\rangle + \|y_n\|^2 - \|x_n\|^2 - 2\lambda_n\langle x_n - z, A_n x_n\rangle \le 0$, we have that C_n is closed and convex for all $n \in \mathbb{N}$. Next, we show that, for $n \in \mathbb{N}$, $x_n \in C$ implies that $F \subset C_n$. Let $z \in F$. We have

$$\begin{aligned} &(\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &\quad - (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle) \\ &= (\phi(z, y_n) - \phi(z, x_n)) + 2\lambda_n \langle y_n - x_n, A_n z\rangle \\ &= -\phi(y_n, x_n) + 2\langle y_n - z, Jy_n - Jx_n\rangle \\ &\quad + 2\lambda_n \langle y_n - x_n, A_n z\rangle \end{aligned} \tag{15}$$

for every $n \in \mathbb{N}$. Since $y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n)$ and by Lemma 5, we get $\langle y_n - z, Jx_n - \lambda_n A_n x_n - Jy_n\rangle \ge 0$. Further, by Lemma 4, we obtain

$$\begin{aligned} &(\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &\quad - (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle) \\ &\le -c_1\|y_n - x_n\|^2 - 2\lambda_n \langle y_n - z, A_n x_n\rangle \\ &\quad + 2\lambda_n \langle y_n - x_n, A_n z\rangle \\ &= -c_1\|y_n - x_n\|^2 \\ &\quad - 2\lambda_n (\langle y_n - x_n, A_n x_n\rangle + \langle x_n - z, A_n x_n\rangle) \\ &\quad + 2\lambda_n \langle y_n - x_n, A_n z\rangle. \end{aligned} \tag{16}$$

Using $x_n \in C$ and the condition (iii), we have $\langle x_n - z, A_n x_n\rangle \ge \alpha_n\|A_n x_n - A_n z\|^2$ and thus

$$\begin{aligned} &(\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &\quad - (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle) \\ &\le -c_1\|y_n - x_n\|^2 \\ &\quad - 2\lambda_n (\langle y_n - x_n, A_n x_n\rangle + \alpha_n\|A_n x_n - A_n z\|^2) \\ &\quad + 2\lambda_n \langle y_n - x_n, A_n z\rangle \\ &= -c_1\|y_n - x_n\|^2 - 2\lambda_n\alpha_n\|A_n x_n - A_n z\|^2 \\ &\quad - 2\lambda_n \langle y_n - x_n, A_n x_n - A_n z\rangle \\ &\le -c_1\|y_n - x_n\|^2 - 2\lambda_n\alpha_n\|A_n x_n - A_n z\|^2 \\ &\quad + 2\lambda_n \|y_n - x_n\| \|A_n x_n - A_n z\| \\ &\le -c_1\|y_n - x_n\|^2 - 2\lambda_n\alpha_n\|A_n x_n - A_n z\|^2 \\ &\quad + 2\lambda_n \left(\frac{\beta}{2}\|y_n - x_n\|^2 + \frac{1}{2\beta}\|A_n x_n - A_n z\|^2\right) \\ &= (\beta\lambda_n - c_1)\|y_n - x_n\|^2 \\ &\quad + \lambda_n\left(\frac{1}{\beta} - 2\alpha_n\right)\|A_n x_n - A_n z\|^2 \end{aligned} \tag{17}$$

for any $\beta > 0$. Since a sequence $\{\lambda_n\}$ satisfies $0 < \lambda_n < 2c_1\alpha_n$ for all $n \in \mathbb{N}$, $\limsup_{n\to\infty}\lambda_n < \infty$, and $\inf_{n\in\mathbb{N}}(c_1/\lambda_n - 1/(2\alpha_n)) > 0$, we can choose a positive sequence $\{\beta_n\}$ such that

$$\begin{gathered} \beta_n\lambda_n - c_1 < 0, \quad \lambda_n\left(\frac{1}{\beta_n} - 2\alpha_n\right) < 0 \quad \forall n \in \mathbb{N}, \\ \limsup_{n\to\infty}(\beta_n\lambda_n - c_1) < 0, \qquad \limsup_{n\to\infty}\lambda_n\left(\frac{1}{\beta_n} - 2\alpha_n\right) < 0. \end{gathered} \tag{18}$$

So, we obtain

$$\begin{aligned} &(\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &\quad - (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle) \le 0. \end{aligned} \tag{19}$$

Since $z \in VI(C, A_n)$, we have $\langle y_n - z, A_n z\rangle \ge 0$. Using $x_n \in C$ and the condition (ii), we have $\langle x_n - z, A_n z\rangle \le \langle x_n - z, A_n x_n\rangle$. Thus, we get

$$\phi(z, y_n) \le \phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n x_n\rangle; \tag{20}$$

that is, $F \subset C_n$. From this fact, we get that $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$ and $\{x_n\}$ is well defined. Indeed, $x_1 = x \in C$ is given and since $Q_1 = C$, we have $F \subset C_1 \cap Q_1$. Assume that x_k is well defined and $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. There exists a unique element $x_{k+1} = \Pi_{C_k \cap Q_k} x$ and we get $\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0$ for all $z \in C_k \cap Q_k$ by Lemma 5. Since $F \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0$ for every

$z \in F$; that is, $F \subset Q_{k+1}$. Since $x_{k+1} \in C_k \cap Q_k \subset C$, we have $F \subset C_{k+1}$. Hence, we obtain $F \subset C_{k+1} \cap Q_{k+1}$. By mathematical induction, we get $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$ and $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x$ and $F \subset C_n \cap Q_n$, we have $\phi(x_{n+1}, x) \le \phi(\Pi_F x, x)$ for every $n \in \mathbb{N}$, which implies that $\{x_n\}$ is bounded. Further, since $x_{n+1} \in Q_n$, we have

$$\begin{aligned} &\phi(x_{n+1}, x_n) + \phi(x_n, x) \\ &\quad = \phi(x_{n+1}, x) + 2\langle x_n - x_{n+1}, Jx_n - Jx\rangle \\ &\quad \le \phi(x_{n+1}, x) \end{aligned} \tag{21}$$

for all $n \in \mathbb{N}$. Thus, there exists $\lim_{n\to\infty}\phi(x_n, x)$ and

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{22}$$

By Lemma 4, we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{23}$$

Using $x_{n+1} \in C_n$, we have

$$\begin{aligned} \phi(x_{n+1}, y_n) &\le \phi(x_{n+1}, x_n) + 2\lambda_n \langle x_n - x_{n+1}, A_n x_n\rangle \\ &\le \phi(x_{n+1}, x_n) + 2\lambda_n \|x_n - x_{n+1}\| \|A_n x_n\| \end{aligned} \tag{24}$$

for all $n \in \mathbb{N}$. From the condition (iii), we have

$$\begin{aligned} \|x_n - z\| \|A_n x_n\| &\ge \alpha_n (\|A_n x_n\| - \|A_n z\|)^2 \\ &\ge \alpha_n (\|A_n x_n\|^2 - 2\|A_n x_n\| \|A_n z\|), \end{aligned} \tag{25}$$

which implies that

$$\|x_n - z\| \ge \alpha_n \|A_n x_n\| - 2\alpha_n \|A_n z\| \tag{26}$$

for every $n \in \mathbb{N}$ and $z \in F$. Since $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < \infty$, using the condition (iv) and the boundedness of $\{x_n\}$, we get that $\{A_n x_n\}$ is bounded. Since (22)–(24) hold and $\{\lambda_n\}$ is bounded, we have $\lim_{n\to\infty}\phi(x_{n+1}, y_n) = 0$, which implies that

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0 \tag{27}$$

by Lemma 4. From (23) and (27), we also have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Using the facts that

$$\begin{aligned} &(\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &\qquad - (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle) \\ &\quad = 2\langle z, Jx_n - Jy_n\rangle + (\|y_n\|^2 - \|x_n\|^2) \\ &\qquad + 2\lambda_n \langle y_n - x_n, A_n z\rangle \end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, the condition (iv) holds, $\{x_n\}$, $\{y_n\}$, and $\{\lambda_n\}$ are bounded, and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, we obtain

$$\begin{aligned} \lim_{n\to\infty} &((\phi(z, y_n) + 2\lambda_n \langle y_n - z, A_n z\rangle) \\ &- (\phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n z\rangle)) = 0. \end{aligned} \tag{29}$$

Since (17) holds for every $n \in \mathbb{N}$, it follows from (29) and $\lim_{n\to\infty}\|x_n - y_n\| = 0$ that

$$\lim_{n\to\infty}\|A_n x_n - A_n z\| = 0. \tag{30}$$

From the condition (v), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in F$. Since the norm of E is weakly lower semicontinuous, we get

$$\begin{aligned} \phi(z, x) &= \|z\|^2 - 2\langle z, Jx\rangle + \|x\|^2 \\ &\le \liminf_{i\to\infty}\left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx\rangle + \|x\|^2\right) \\ &= \liminf_{i\to\infty}\phi(x_{n_i}, x) = \lim_{n\to\infty}\phi(x_n, x) \le \phi(\Pi_F x, x), \end{aligned} \tag{31}$$

which implies $z = \Pi_F x$ and

$$\lim_{n\to\infty}\phi(x_n, x) = \phi(\Pi_F x, x). \tag{32}$$

Using $x_{n+1} = \Pi_{C_n \cap Q_n} x$, $F \subset C_n \cap Q_n$, and Lemma 5, we have

$$\begin{aligned} 0 &\ge \langle x_{n+1} - \Pi_F x, Jx_{n+1} - Jx\rangle \\ &= \frac{1}{2}(\phi(\Pi_F x, x_{n+1}) + \phi(x_{n+1}, x) - \phi(\Pi_F x, x)) \end{aligned} \tag{33}$$

which implies that

$$\phi(\Pi_F x, x) \ge \phi(\Pi_F x, x_{n+1}) + \phi(x_{n+1}, x) \tag{34}$$

for all $n \in \mathbb{N}$. From (32), we get $\lim_{n\to\infty}\phi(\Pi_F x, x_{n+1}) = 0$ and by Lemma 4, we obtain $x_n \to \Pi_F x$, which is the desired result. □

Next, we have the following result by the hybrid method using the metric projections.

Theorem 9. *Assume that E, C, $\{A_n\}$, $\{\lambda_n\}$, and c_1 are the same as in Theorem 8. Let $x \in C$ and $\{x_n\}$ a sequence in C generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + 2\lambda_n \langle x_n - z, A_n x_n\rangle\}, \\ Q_n &= \{z \in C : \langle x_n - z, J(x - x_n)\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{35}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_F x$.

Proof. Q_n is closed convex for every $n \in \mathbb{N}$. As in the proof of Theorem 8, we have that C_n is closed and convex. We also get that, for $n \in \mathbb{N}$, $x_n \in C$ implies that $F \subset C_n$. By this fact, we obtain $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$ and $\{x_n\}$ is well defined. Indeed, $x_1 = x \in C$ is given and $F \subset C_1 \cap Q_1$ since $Q_1 = C$. Assume that x_k is well defined and $F \subset C_k \cap Q_k$ for some

$k \in \mathbb{N}$. There exists a unique element $x_{k+1} = P_{C_k \cap Q_k} x$ and we get $\langle x_{k+1} - z, J(x - x_{k+1}) \rangle \geq 0$ for all $z \in C_k \cap Q_k$ by Lemma 6. Since $F \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, J(x - x_{k+1}) \rangle \geq 0$ for every $z \in F$; that is, $F \subset Q_{k+1}$. Since $x_{k+1} \in C_k \cap Q_k \subset C$, we also have $F \subset C_{k+1}$. Thus, we obtain $F \subset C_{k+1} \cap Q_{k+1}$. By mathematical induction, we get $F \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$ and $\{x_n\}$ is well defined.

Since $x_{n+1} = P_{C_n \cap Q_n} x$ and $F \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|x - P_F x\| \tag{36}$$

for every $n \in \mathbb{N}$ and, hence, $\{x_n\}$ is bounded. Using $x_{n+1} \in Q_n$ and Theorem 2, we have

$$\begin{aligned} &\|x_{n+1} - x\|^2 - \|x_n - x\|^2 \\ &\quad \geq 2 \langle x_{n+1} - x_n, J(x_n - x) \rangle + c_1 \|x_{n+1} - x_n\|^2 \\ &\quad \geq c_1 \|x_{n+1} - x_n\|^2 \end{aligned} \tag{37}$$

for each $n \in \mathbb{N}$, which implies that there exists $\lim_{n\to\infty}\|x_n - x\|$ and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{38}$$

Since $y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n)$, by Lemma 5, we have

$$\langle y_n - x_n, Jx_n - Jy_n - \lambda_n A_n x_n \rangle \geq 0, \tag{39}$$

which implies that

$$\begin{aligned} -c_1 \|x_n - y_n\|^2 &\geq \frac{1}{2} \left(-\phi(y_n, x_n) - \phi(x_n, y_n) \right) \\ &= \langle y_n - x_n, Jx_n - Jy_n \rangle \\ &\geq \lambda_n \langle y_n - x_n, A_n x_n \rangle \\ &\geq -\lambda_n \|y_n - x_n\| \|A_n x_n\| \end{aligned} \tag{40}$$

from Lemma 4. As in the proof of Theorem 8, $\{A_n x_n\}$ is bounded. Thus, we get that $\{\|y_n - x_n\|\}$ is also bounded by the boundedness of $\{\lambda_n\}$ and so is $\{y_n\}$. Since $x_{n+1} \in C_n$, we have $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + 2\lambda_n \langle x_n - x_{n+1}, A_n x_n \rangle$; that is,

$$\begin{aligned} &\phi(x_n, y_n) \\ &\quad \leq 2 \langle x_n - x_{n+1}, Jx_n - Jy_n \rangle + 2\lambda_n \langle x_n - x_{n+1}, A_n x_n \rangle \end{aligned} \tag{41}$$

for all $n \in \mathbb{N}$. By the boundedness of $\{x_n\}$, $\{y_n\}$, $\{\lambda_n\}$, and $\{A_n x_n\}$ with (38) and Lemma 4, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{42}$$

As in the proof of Theorem 8, using (17) and (29), we have

$$\lim_{n\to\infty} \|A_n x_n - A_n z\| = 0. \tag{43}$$

From the condition (v), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in F$. Since the norm of E is weakly lower semicontinuous and (36) holds, we get

$$\begin{aligned} \|z - x\| &\leq \liminf_{i\to\infty} \|x_{n_i} - x\| \\ &= \lim_{n\to\infty} \|x_n - x\| \leq \|P_F x - x\| \end{aligned} \tag{44}$$

which implies that $z = P_F x$ and

$$\lim_{n\to\infty} \|x_n - x\| = \|P_F x - x\|. \tag{45}$$

Using the facts that $x_{n+1} = P_{C_n \cap Q_n} x$ and $F \subset C_n \cap Q_n$ and by Theorem 2, we obtain

$$\begin{aligned} &\|x - P_F x\|^2 \\ &\quad \geq \|x - x_{n+1}\|^2 + 2 \langle x_{n+1} - P_F x, J(x - x_{n+1}) \rangle \\ &\qquad + c_1 \|x_{n+1} - P_F x\|^2 \\ &\quad \geq \|x - x_{n+1}\|^2 + c_1 \|x_{n+1} - P_F x\|^2 \end{aligned} \tag{46}$$

for all $n \in \mathbb{N}$. By (45), we have

$$\lim_{n\to\infty} \|x_{n+1} - P_F x\| = 0, \tag{47}$$

which is the desired result. □

Remark 10. Even though we replace the definition of C_n in Theorems 8 and 9 with

$$C_n = \begin{cases} \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\} \\ \quad (A_n z = 0 \ \forall z \in F), \\ \{z \in C : \phi(z, y_n) \leq \phi(z, x_n) \\ \quad + 2\lambda_n \langle x_n - z, A_n x_n \rangle\} \\ \quad (A_n z \neq 0 \text{ for some } z \in F), \end{cases} \tag{48}$$

the theorems are still valid.

4. The Variational Inequality Problem for Monotone Operators

Let I be a countable set and $i : \mathbb{N} \to I$ a mapping. Nakajo et al. [30] propose the condition (NST) as follows: i satisfies the condition (NST) if there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that, for any $j \in I$, there is $M_j \in \mathbb{N}$ with $j \in \{i(n_k), i(n_k + 1), \ldots, i(n_k + M_j - 1)\}$ for all sufficiently large $k \in \mathbb{N}$. Using the condition (NST), we get the following result for the variational inequality problem by Theorem 8.

Theorem 11. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E, I a countable set, and $\{B_i\}_{i\in I}$ a family of operators of C into E^* such that*

(i) $F = \bigcap_{i\in I} VI(C, B_i) \neq \emptyset$;

(ii) *B_i is an inverse strongly monotone operator for each $i \in I$; that is, there exists $\{\beta_i : i \in I\} \subset]0, \infty[$ such that for every $i \in I$ and $x, y \in C$, the inequality $\langle x - y, B_i x - B_i y \rangle \geq \beta_i \|B_i x - B_i y\|^2$ holds;*

(iii) *for all $z \in F$, $\sup_{i\in I} \|B_i z\| < \infty$.*

Suppose that the index mapping $i : \mathbb{N} \to I$ satisfies the condition (NST) and $0 < \liminf_{n\to\infty} \beta_{i(n)} \leq \limsup_{n\to\infty} \beta_{i(n)} < \infty$. Let $\{\lambda_n\}$ be a sequence in $]0, \infty[$ such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$ and

$\inf_{n\in\mathbb{N}}(2c_1\beta_{i(n)}-\lambda_n)>0$, *where* c_1 *is the constant in Theorem 2. Let* $x\in C$ *and* $\{x_n\}$ *a sequence in* C *generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= \Pi_C J^{-1}\left(Jx_n-\lambda_n B_{i(n)}x_n\right), \\ C_n &= \{z\in C:\phi(z,y_n)\le\phi(z,x_n) \\ &\qquad +2\lambda_n\left\langle x_n-z,B_{i(n)}x_n\right\rangle\}, \\ Q_n &= \{z\in C:\left\langle x_n-z,Jx-Jx_n\right\rangle\ge 0\}, \\ x_{n+1} &= \Pi_{C_n\cap Q_n}x \end{aligned} \tag{49}$$

for each $n\in\mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_F x$.

Proof. We apply Theorem 8 with $A_n=B_{i(n)}$ for all $n\in\mathbb{N}$. Then, the conditions (i)–(iv) are satisfied, and we will verify the condition (v). Let $\{z_n\}$ be a bounded sequence in C with

$$\begin{aligned}\lim_{n\to\infty}\|z_{n+1}-z_n\| &= \lim_{n\to\infty}\left\|z_n-\Pi_C J^{-1}\left(Jz_n-r_nA_nz_n\right)\right\| \\ &= 0,\end{aligned} \tag{50}$$

$\{r_n\}\subset]0,\infty[$ with $\liminf_{n\to\infty}r_n>0$, and $z\in F$ such that $\lim_{n\to\infty}\|A_nz_n-A_nz\|=0$. By the condition (NST), there exists a weakly convergent subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that, for any $i\in I$, there is $M_i\in\mathbb{N}$ with $i\in\{i(n_k),i(n_k+1),\dots,i(n_k+M_i-1)\}$ for all sufficiently large $k\in\mathbb{N}$. Let $z_{n_k}\rightharpoonup u$ and fix $i\in I$. There exists $j_k\in\{0,1,\dots,M_i-1\}$ such that $i(n_k+j_k)=i$ for every sufficiently large $k\in\mathbb{N}$. We consider a subsequence of $\{n_k+j_k\}$ for all $k\in\{k\in\mathbb{N}:n_k+j_k<n_{k+1}+j_{k+1}\}$ and denote it by $\{n_k+j_k\}$ again. We have

$$\left\|z_{n_k+j_k}-z_{n_k}\right\|\le\sum_{l=n_k}^{n_k+M_i-1}\|z_{l+1}-z_l\| \tag{51}$$

for all $k\in\mathbb{N}$, which implies that $z_{n_k+j_k}\rightharpoonup u$. Let $y_n=\Pi_C J^{-1}(Jz_n-r_nA_nz_n)$. By Lemma 5, we have

$$\begin{aligned} &\left\|y_{n_k+j_k}-y\right\|\left\|Jz_{n_k+j_k}-Jy_{n_k+j_k}\right\| \\ &\quad\ge\left\langle y_{n_k+j_k}-y,Jz_{n_k+j_k}-Jy_{n_k+j_k}\right\rangle \\ &\quad\ge r_{n_k+j_k}\left\langle y_{n_k+j_k}-y,A_{n_k+j_k}z_{n_k+j_k}\right\rangle\end{aligned} \tag{52}$$

for every sufficiently large $k\in\mathbb{N}$ and $y\in C$. Since $A_{n_k+j_k}=B_{i(n_k+j_k)}=B_i$ for each sufficiently large $k\in\mathbb{N}$, $y_{n_k+j_k}\rightharpoonup u$, $\|z_{n_k+j_k}-y_{n_k+j_k}\|\to 0$, $\liminf_{n\to\infty}r_n>0$, $\lim_{k\to\infty}\|B_iz_{n_k+j_k}-B_iz\|=0$, and the duality mapping J is uniformly continuous on bounded subset of E, we have

$$\left\langle y-u,B_iz\right\rangle\ge 0 \tag{53}$$

for all $y\in C$. Since B_i is inverse strongly monotone, we have

$$\left\langle z_{n_k+j_k}-u,B_iz_{n_k+j_k}-B_iu\right\rangle\ge\beta_i\left\|B_iz_{n_k+j_k}-B_iu\right\|^2 \tag{54}$$

for every $k\in\mathbb{N}$, which implies that

$$0=\langle u-u,B_iz-B_iu\rangle\ge\beta_i\|B_iz-B_iu\|^2; \tag{55}$$

that is, $B_iz=B_iu$. From (53),

$$\left\langle y-u,B_iu\right\rangle\ge 0 \tag{56}$$

for all $y\in C$. Therefore, $u\in\mathrm{VI}(C,B_i)$ for every $i\in I$; that is, $u\in F$. Hence, the condition (v) is satisfied. Consequently, we obtain $x_n\to\Pi_F x$ by Theorem 8. □

As in the proof of Theorem 11, we get the following result for the variational inequality problem by Theorem 9.

Theorem 12. *Assume that* E, C, I, $\{B_i\}$, F, c_1, i, $\{\beta_i\}$, *and* $\{\lambda_n\}$ *are the same as Theorem 11. Let* $x\in C$ *and* $\{x_n\}$ *a sequence in* C *generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= \Pi_C J^{-1}\left(Jx_n-\lambda_n B_{i(n)}x_n\right), \\ C_n &= \{z\in C:\phi(z,y_n)\le\phi(z,x_n) \\ &\qquad +2\lambda_n\left\langle x_n-z,B_{i(n)}x_n\right\rangle\}, \\ Q_n &= \{z\in C:\left\langle x_n-z,J(x-x_n)\right\rangle\ge 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n}x \end{aligned} \tag{57}$$

for each $n\in\mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_F x$.

Remark 13. In Theorems 11 and 12, under the assumption that $\bigcap_{i\in I}B_i^{-1}0\neq\emptyset$, we have $\mathrm{VI}(C,B_i)=B_i^{-1}0$ for all $i\in I$. Indeed, $B_i^{-1}0\subset\mathrm{VI}(C,B_i)$ is trivial. Let $i\in I$, $u\in\mathrm{VI}(C,B_i)$, and $z\in B_i^{-1}0$. From the condition (ii), we have $\langle z-u,B_iz-B_iu\rangle\ge\beta_i\|B_iz-B_iu\|^2$ which implies that $-\langle z-u,B_iu\rangle\ge\beta_i\|B_iu\|^2$. On the other hand, $\langle z-u,B_iu\rangle\ge 0$ from $u\in\mathrm{VI}(C,B_i)$. So, we obtain $u\in B_i^{-1}0$; that is, $\mathrm{VI}(C,B_i)\subset B_i^{-1}0$. Therefore, $B_i^{-1}0=\mathrm{VI}(C,B_i)$ for all $i\in I$. Now suppose that $F=\bigcap_{i\in I}B_i^{-1}0\neq\emptyset$ instead of the condition (i) and $C_n=\{z\in C:\phi(z,y_n)\le\phi(z,x_n)\}$. By the argument mentioned above and Remark 10, Theorems 11 and 12 hold under the conditions (i) and (ii) and we get the result of [19].

Remark 14. We know that, for a continuously Fréchet differentiable and convex functional f on a Banach space E, if ∇f is Lipschitz continuous with constant $1/\alpha$, then ∇f is α-inverse strongly monotone operator; see [2, 19]. So, we can apply Theorems 11 and 12 and Remark 13 to such a functional; see [19].

5. The Proximal Point Algorithm

Let E be a strictly convex, reflexive, and smooth Banach space, $T\subset E\times E^*$ a maximal monotone operator with $T^{-1}0\neq\emptyset$, $r>0$, and $Ax=J(x-J_rx)$ for all $x\in E$, where J_r is the resolvent of T. Then, A is well defined as a mapping of E into E^* for all $r>0$. We also have

$$\begin{gathered}\mathrm{VI}(E,A)=T^{-1}0, \\ \langle x-u,Ax\rangle\ge\|Ax\|^2\quad\forall x\in E,u\in T^{-1}0.\end{gathered} \tag{58}$$

In fact, $\mathrm{VI}(E, A) = T^{-1}0$ since $u \in T^{-1}0$ is equivalent to $J_r u = u$ and $Au = 0$. Let $x \in E$ and $u \in T^{-1}0$. Since $(1/r)J(x - J_r x) \in TJ_r x$ and $0 \in Tu$, we have $\langle J_r x - u, J(x - J_r x)\rangle \geq 0$ which implies that $\langle x - u, Ax\rangle \geq \|Ax\|^2$. By Theorem 8 and Remark 10, we get the following result using the index mapping which satisfies the condition (NST).

Theorem 15. *Let I be a countable set, E a 2-uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, and $\{T_i\}_{i\in I}$ a family of maximal monotone operators of E into E^* such that $F = \bigcap_{i\in I} T_i^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0,\infty[$ with $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$ such that $\inf_{n\in\mathbb{N}}\lambda_n > 0$ and $\inf_{n\in\mathbb{N}}(2c_1 - \lambda_n) > 0$, where c_1 is the constant in Theorem 2. Let $x \in E$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in E generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= J^{-1}\left(Jx_n - \lambda_n J\left(x_n - J_{r_n}^{T_{i(n)}} x_n\right)\right), \\ C_n &= \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n\cap Q_n} x \end{aligned} \tag{59}$$

for each $n \in \mathbb{N}$, where the index mapping $i : \mathbb{N} \to I$ satisfies the condition (NST) and $J_{r_n}^{T_{i(n)}}$ is the resolvent of $T_{i(n)}$. Then, $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof. Suppose that $A_n x = J(x - J_{r_n}^{T_{i(n)}} x)$ for every $n \in \mathbb{N}$ and $x \in E$ in Theorem 8. Then, we have that A_n is a mapping of E into E^* with $\bigcap_{n\in\mathbb{N}} \mathrm{VI}(E, A_n) = F \neq \emptyset$, the condition (iii) is satisfied with $\alpha_n = 1$ for all $n \in \mathbb{N}$, and the conditions (ii) and (iv) hold by $A_n z = 0$ for all $n \in \mathbb{N}$ and all $z \in F$. Let $\{z_n\}$ be a bounded sequence in E, $z \in F$, and $\{r_n\} \subset]0,\infty[$ with $\liminf_{n\to\infty} r_n > 0$. Assume that

$$\begin{aligned} \lim_{n\to\infty} \|z_{n+1} - z_n\| &= \lim_{n\to\infty} \left\|z_n - J^{-1}(Jz_n - r_n A_n z_n)\right\| \\ &= \lim_{n\to\infty} \|A_n z_n - A_n z\| = 0. \end{aligned} \tag{60}$$

By the condition (NST), there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that, for any $i \in I$, there is $M_i \in \mathbb{N}$ with $i \in \{i(n_k), i(n_k + 1), \ldots, i(n_k + M_i - 1)\}$ for all sufficiently large $k \in \mathbb{N}$. Let $z_{n_k} \rightharpoonup u$ and $i \in I$. As in the proof of Theorem 11, there exists $j_k \in \{0, 1, \ldots, M_i - 1\}$ such that $i(n_k + j_k) = i$ for every sufficiently large $k \in \mathbb{N}$ and we get $z_{n_k+j_k} \rightharpoonup u$. Let $(v, v^*) \in T_i$. Since

$$\begin{gathered} \left\langle J_{r_{n_k+j_k}}^{T_i} z_{n_k+j_k} - v, \frac{1}{r_{n_k+j_k}} J\left(z_{n_k+j_k} - J_{r_{n_k+j_k}}^{T_i} z_{n_k+j_k}\right) - v^*\right\rangle \geq 0, \\ \lim_{k\to\infty} \left\|z_{n_k+j_k} - J_{r_{n_k+j_k}}^{T_i} z_{n_k+j_k}\right\| = 0, \end{gathered} \tag{61}$$

we obtain

$$\langle u - v, -v^*\rangle \geq 0 \tag{62}$$

for each $(v, v^*) \in T_i$. As T_i is a maximal monotone operator, $u \in T_i^{-1}0$ for every $i \in I$. So, we get $u \in F$. Therefore, the condition (v) holds. So, we get conclusion by Theorem 8 and Remark 10. □

As in the proof of Theorem 15, we get the following result from Theorem 9 and Remark 10.

Theorem 16. *Assume that E, I, $\{T_i\}$, F, $\{r_n\}$, $\{\lambda_n\}$, c_1, i, and $J_{r_n}^{T_{i(n)}}$ are the same as Theorem 15. Let $x \in E$ and let $\{x_n\}$ be a sequence in E generated by*

$$\begin{aligned} x_1 &= x, \\ y_n &= J^{-1}\left(Jx_n - \lambda_n J\left(x_n - J_{r_n}^{T_{i(n)}} x_n\right)\right), \\ C_n &= \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in E : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n} x \end{aligned} \tag{63}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_F x$.

Let $f : E \to]-\infty,\infty]$ be a proper, lower semicontinuous, and convex function. Then, it is known that the subdifferential ∂f of f defined by

$$\begin{aligned} \partial f(x) = \{x^* \in E^* : f(y) \\ \geq f(x) + \langle y - x, x^*\rangle \ \forall y \in E\} \end{aligned} \tag{64}$$

for all $x \in E$ is a maximal monotone operator [31, 32]. Moreover, when E is strictly convex, reflexive, and smooth, we know that, for the resolvent of ∂f,

$$J_r^{\partial f} x = \operatorname*{argmin}_{y\in E} \left(f(y) + \frac{1}{2r}\|y - x\|^2\right) \tag{65}$$

for every $r > 0$ and $x \in E$ and $\partial f^{-1}0 = \operatorname{argmin}_{y\in E} f(y)$; see [21] for more details. Now, we have the following results from Theorems 15 and 16.

Theorem 17. *Let I be a countable set, E a 2-uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, and $\{f_i\}_{i\in I}$ a family of proper, lower semicontinuous, and convex functions of E into $]-\infty,\infty]$ such that $F = \bigcap_{i\in I} \operatorname{argmin}_{y\in E} f_i(y) \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0,\infty[$ with $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$ such that $\inf_{n\in\mathbb{N}}\lambda_n > 0$ and $\inf_{n\in\mathbb{N}}(2c_1 - \lambda_n) > 0$, where c_1 is the constant in Theorem 2. Let $x \in E$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E generated by*

$$\begin{aligned} x_1 &= x, \\ u_n &= \operatorname*{argmin}_{y\in E} \left(f_{i(n)}(y) + \frac{1}{2r_n}\|y - x_n\|^2\right), \\ y_n &= J^{-1}(Jx_n - \lambda_n J(x_n - u_n)), \end{aligned}$$

$$C_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\},$$
$$Q_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \ge 0\},$$
$$x_{n+1} = \Pi_{C_n \cap Q_n} x \tag{66}$$

for each $n \in \mathbb{N}$, where the index mapping $i : \mathbb{N} \to I$ satisfies the condition (NST). Then, $\{x_n\}$ converges strongly to $\Pi_F x$.

Theorem 18. *Assume that E, I, $\{f_i\}$, F, $\{r_n\}$, $\{\lambda_n\}$, c_1, and i are the same as Theorem 17. Let $x \in E$ and let $\{x_n\}$ be a sequence in E generated by*

$$\begin{aligned} x_1 &= x, \\ u_n &= \operatorname*{argmin}_{y \in E} \left(f_{i(n)}(y) + \frac{1}{2r_n} \|y - x_n\|^2 \right) \\ y_n &= J^{-1}(Jx_n - \lambda_n J(x_n - u_n)), \\ C_n &= \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n &= \{z \in E : \langle x_n - z, J(x - x_n) \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{67}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_F x$.

At the end of this section, we make a remark about a result of the problem of image recovery due to [33, 34]. Let i_C be the indicator function of a nonempty closed convex subset C of E. We know that $i_C : E \to]-\infty, \infty[$ is proper lower semicontinuous and convex and, for a nonempty closed convex subset C of a strictly convex, reflexive, and smooth Banach space E, $(\partial i_C)^{-1} 0 = C$ and $J_r^{\partial i_C} x = P_C(x)$ for every $r > 0$ and $x \in E$; see [15]. So, when $f_j = i_{C_j}$ for nonempty closed convex subset C_j of E for every $j \in I$ with $\bigcap_{j \in I} C_j \ne \emptyset$ in Theorems 17 and 18, we get the strong convergence to a common point of $\{C_j\}_{j \in I}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J.-L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 493–519, 1967.

[2] J.-B. Baillon and G. Haddad, "Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones," *Israel Journal of Mathematics*, vol. 26, no. 2, pp. 137–150, 1977.

[3] J. C. Dunn, "Convexity, monotonicity, and gradient processes in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 53, no. 1, pp. 145–158, 1976.

[4] F. Liu and M. Z. Nashed, "Regularization of nonlinear ill-posed variational inequalities and convergence rates," *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.

[5] K. Nakajo and W. Takahashi, "Strong and weak convergence theorems by an improved splitting method," *Communications on Applied Nonlinear Analysis*, vol. 9, no. 2, pp. 99–107, 2002.

[6] Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes [Ph.D. thesis]*, Université de Paris, Paris, France, 1968.

[7] M. V. Solodov and B. F. Svaiter, "Forcing strong convergence of proximal point iterations in a Hilbert space," *Mathematical Programming*, vol. 87, no. 1, pp. 189–202, 2000.

[8] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Mathematics of Operations Research*, vol. 26, no. 2, pp. 248–264, 2001.

[9] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[10] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[11] S. Ohsawa and W. Takahashi, "Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces," *Archiv der Mathematik*, vol. 81, no. 4, pp. 439–445, 2003.

[12] F. Kohsaka and W. Takahashi, "Iterative scheme for finding a common point of infinitely many convex sets in a Banach space," *Journal of Nonlinear and Convex Analysis*, vol. 5, no. 3, pp. 407–414, 2004.

[13] S. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[14] S. Matsushita, K. Nakajo, and W. Takahashi, "Strong convergence theorems obtained by a generalized projections hybrid method for families of mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 6, pp. 1466–1480, 2010.

[15] K. Nakajo, K. Shimoji, and W. Takahashi, "Strong convergence theorems by the hybrid method for families of mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 812–818, 2009.

[16] H. Iiduka, W. Takahashi, and M. Toyoda, "Approximation of solutions of variational inequalities for monotone mappings," *Panamerican Mathematical Journal*, vol. 14, no. 2, pp. 49–61, 2004.

[17] K. Nakajo, K. Shimoji, and W. Takahashi, "Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 339–360, 2006.

[18] K. Nakajo, K. Shimoji, and W. Takahashi, "On strong convergence by the hybrid method for families of mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 112–119, 2009.

[19] H. Iiduka and W. Takahashi, "Strong convergence studied by a hybrid type method for monotone operators in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3679–3688, 2008.

[20] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its applications, Yokohama Publishers, Yokohama, Japan, 2000.

[21] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, vol. 2, Yokohama Publishers, Yokohama, Japan, 2000 (Japanese).

[22] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[23] C. Zălinescu, "On uniformly convex functions," *Journal of Mathematical Analysis and Applications*, vol. 95, no. 2, pp. 344–374, 1983.

[24] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Dekker, New York, NY, USA, 1996.

[25] Y. I. Alber and S. Reich, "An iterative method for solving a class of nonlinear operator equations in Banach spaces," *Panamerican Mathematical Journal*, vol. 4, no. 2, pp. 39–54, 1994.

[26] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1984.

[27] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[28] F. E. Browder, "Nonlinear maximal monotone operators in Banach space," *Mathematische Annalen*, vol. 175, pp. 89–113, 1968.

[29] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei Republicii Socialiste România, Bucharest, Romania, 1976.

[30] K. Nakajo, K. Shimoji, and W. Takahashi, "Approximations for nonlinear mappings by the hybrid method in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 7025–7032, 2011.

[31] R. T. Rockafellar, "Characterization of the subdifferentials of convex functions," *Pacific Journal of Mathematics*, vol. 17, pp. 497–510, 1966.

[32] R. T. Rockafellar, "On the maximal monotonicity of subdifferential mappings," *Pacific Journal of Mathematics*, vol. 33, pp. 209–216, 1970.

[33] L. M. Bregman, "The method of successive projection for nding a common point of convex sets," *Soviet Mathematics. Doklady*, vol. 6, pp. 688–692, 1965.

[34] G. Crombez, "Image recovery by convex combinations of projections," *Journal of Mathematical Analysis and Applications*, vol. 155, no. 2, pp. 413–419, 1991.