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Research Article

Zeros for the Gradients of Weakly *A*-Harmonic Tensors

Yuxia Tong,[1,2] Jiantao Gu,[2] and Shenzhou Zheng[1]

[1] *Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China*
[2] *College of Science, Hebei United University, Tangshan, Hebei 063009, China*

Correspondence should be addressed to Yuxia Tong; tongyuxia@126.com

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The Caccioppoli inequality of weakly A-harmonic tensors has been proved, which can be used to consider the weak reverse Hölder inequality, regularity property, and zeros of weakly A-harmonic tensors.

1. Introduction

In this paper, we consider the A-harmonic equation for differential forms

$$d^* A(x, du) = 0, \quad (1)$$

where $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^{l+1}(\mathbf{R}^n)$ satisfies the conditions

$$|A(x,\xi)| \le \beta|\xi|^{p-1}, \qquad \langle A(x,\xi), \xi\rangle \ge \alpha|\xi|^p \quad (2)$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here, $\alpha, \beta > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ is an A-harmonic tensor in Ω if u satisfies (1) in Ω.

There has been remarkable work [1–10] in the study of (1). When u is a 0-form, that is, u is a function, (1) is equivalent to

$$\operatorname{div} A(x, \nabla u) = 0. \quad (3)$$

Lots of results have been obtained in recent years about different versions of the A-harmonic equation; see [11–15].

In 1995, Stroffolini [16] first introduced weakly A-harmonic tensors and gave the higher integrability result of weakly A-harmonic tensors. The word *weak* means that the integrable exponent r of u is smaller than the natural exponent p. In 2010, Gao and Wang [17] gave an alternative proof of the higher integrability result of weakly A-harmonic tensors by introducing the definition of weak WT_2-class of differential forms.

Definition 1 (see [16, 17]). A very weak solution to (1) (also called weakly A-harmonic tensor) is an element u of the Sobolev space $W^{1,r}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with $\max\{1, p-1\} \le r < p$ such that

$$\int_\Omega \langle A(x, du), d\varphi\rangle dx = 0 \quad (4)$$

for all $\varphi \in W^{1,(r/(r-p+1))}(\Omega, \wedge^{l-1})$ with compact support.

Under some conditions, the present paper proves that almost every zero for the gradients of weakly A-harmonic tensor u has infinite order. To do this, we need to give the Caccioppoli inequality and the weak reverse Hölder inequality of weakly A-harmonic tensors.

We keep using the traditional notation.

Let Ω be a connected open subset of $\mathbf{R}^n$, let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbf{R}^n$, and let $\wedge^l = \wedge^l(\mathbf{R}^n)$ be the linear space of l-covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \ldots, n$. Let $\mathbf{R} = \mathbf{R}^1$. The Grassman algebra $\wedge = \oplus\wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta\rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. The Hodge star operator $\star : \wedge \to \wedge$ is denoted by the rules $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta\rangle(\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha\rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$

with $\star : \wedge^l \rightarrow \wedge^{n-l}$ and $\star\star(-1)^{l(n-l)} : \wedge^l \rightarrow \wedge^l$. Balls are denoted by B and ρB is the ball with the same center as B and with $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. We do not distinguish balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$.

Differential forms are important generalizations of real functions and distributions; note that a 0-form is the usual function in $\mathbf{R}^n$. A differential l-form ω on Ω is a Schwartz distribution on ω with values in $\wedge^l(\mathbf{R}^n)$. We use $D'(\Omega, \wedge^l)$ to denote the space of all differential l-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2,\ldots,i_l}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. We write $L^p(\Omega, \wedge^l)$ for the l-forms with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus, $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}. \tag{5}$$

For $\omega \in D'(\Omega, \wedge^l)$, the vector-valued differential form $\nabla\omega = (\partial\omega/\partial x_1, \ldots, \partial\omega/\partial x_n)$ consists of differential forms $\partial\omega/\partial x_i \in D'(\Omega, \wedge^l)$ where the partial differentiations are applied to the coefficients of ω. As usual, $W^{1,p}(\Omega, \wedge^l)$ is used to denote the Sobolev space of l-forms, which equals $L^p(\Omega, \wedge^l) \bigcap L_1^p(\Omega, \wedge^l)$ with norm

$$\begin{aligned} \|\omega\|_{W^{1,p}(\Omega,\wedge^l)} &= \|\omega\|_{W^{1,p}(\Omega,\wedge^l)} \\ &= \operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega} + \|\nabla\omega\|_{p,\Omega}. \end{aligned} \tag{6}$$

The notations $W^{1,p}_{\text{loc}}(\Omega, \mathbf{R})$ and $W^{1,p}_{\text{loc}}(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \rightarrow D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \wedge^{l+1}) \rightarrow D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d\star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. A differential l-form $u \in D'(\Omega, \wedge^l)$ is called a closed form if $du = 0$ in Ω. It is called exact if there exists a differential form $\alpha \in D'(\Omega, \wedge^{l-1})$ such that $u = d\alpha$. Poincaré Lemma implies that exact forms are closed.

From [1, 18], if $D \subset \mathbf{R}^n$ is a bounded, convex domain, to each $y \in D$ there corresponds a linear operator $K_y : C^\infty(D, \wedge^l) \rightarrow C^\infty(D, \wedge^{l-1})$ defined by

$$\begin{aligned} &\left(K_y\omega\right)(x; \xi_1, \ldots, \xi_{l-1}) \\ &\quad = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\, dt \end{aligned} \tag{7}$$

and a decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(D, \wedge^l) \rightarrow C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points y in D; that is,

$$T\omega = \int_D \varphi(y) K_y \omega dy, \tag{8}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)dy = 1$. Then, there is also a decomposition

$$\omega = d(T\omega) + T(d\omega). \tag{9}$$

The l-form $\omega_D \in D'(D, \wedge^l)$ is defined by

$$\omega_D = \begin{cases} |D|^{-1} \int_D \omega(y)\, dy & \text{if } l = 0 \\ d(T\omega) & \text{if } l = 1, 2, \ldots, n \end{cases} \tag{10}$$

for all $\omega \in L^p(D, \wedge^l)$. Clearly, ω_D is a closed form and for $l > 0$, ω_D is an exact form.

2. The Caccioppoli Inequality of Weakly A-Harmonic Tensors

We need the following elementary inequality.

Lemma 2 (see [19]). *Suppose X and Y are vectors of an inner product space. Then*

$$\left||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y\right| \leq \frac{1+\varepsilon}{1-\varepsilon} 2^\varepsilon |X - Y|^{1-\varepsilon} \tag{11}$$

for $0 \leq \varepsilon < 1$, and

$$\left||X|^{\varepsilon}X - |Y|^{\varepsilon}Y\right| \leq (1+\varepsilon)\left(|Y| + |X - Y|\right)^\varepsilon |X - Y| \tag{12}$$

for $\varepsilon \geq 0$.

Next is the caccioppoli inequality of weakly A-harmonic tensors.

Theorem 3. *Let $u \in D'(\Omega, \wedge^{l-1})$ be a weakly A-harmonic tensor in a domain $\Omega \in \mathbf{R}^n$ and $du \in L^r(D, \wedge^l)$, $l = 1, \ldots, n$. Then, there exists a constant C, independent of u, such that*

$$\int_{B_\rho} |du|^r dx \leq C\left(n, p, \frac{\beta}{\alpha}\right) \int_{B_R} \left|\frac{u - c}{R - \rho}\right|^r dx \tag{13}$$

for all balls $B \subset \Omega$ and all closed forms c, where $0 < \rho < R$.

Proof. Let $u \in W^{1,r}_{\text{loc}}(\Omega, \wedge^{l-1})$ be a very weak solution of (1). Fix $R_0 : R_0 \leq d = \operatorname{dist}(x_0, \partial\Omega)$ for all $x_0 \in \Omega$. Let $B_R = B_R(x_0) \subset\subset \Omega$ and $0 < R/2 \leq \tau < t \leq R$ be arbitrarily fixed cube. Fix a cutoff function $\eta(x) \in C_0^\infty(B_R)$ such that $\operatorname{supp}\eta \subset B_t$, $0 \leq \eta \leq 1$, $|\nabla\eta| \leq C(n)/(t-\tau)$, and $\eta \equiv 1$ on B_τ. Consider the exact form of $\eta(u - c)$, where $c \in D'(\Omega, \wedge^{l-1})$ with $dc = 0$. With the aid of the Hodge decomposition [18],

$$\left|d\left(\eta(u - c)\right)\right|^{r-p} d\left(\eta(u - c)\right) = d\varphi + h, \tag{14}$$

where $d\varphi, h \in L^{r/(r-p+1)}(B_t, \wedge^l)$, and

$$\|h\|_{r/(r-p+1)} \leq C(n)\left(p - r\right)\|d(\eta(u - c))\|_r^{r-p+1}. \tag{15}$$

Then we have

$$\begin{aligned} \|d\varphi\|_{r/(r-p+1)} &\leq \left\|d\left(\eta(u - c)\right)|^{r-p} d\left(\eta(u - c)\right)\right\|_{r/(r-p+1)} \\ &\quad + \|h\|_{r/(r-p+1)} \\ &\leq \left\|d\left(\eta(u - c)\right)\right\|_r^{r-p+1} \\ &\quad + C(n)\left|p - r\right|\left\|d\left(\eta(u - c)\right)\right\|_r^{r-p+1} \\ &\leq C(n)\left\|d\left(\eta(u - c)\right)\right\|_r^{r-p+1}. \end{aligned} \tag{16}$$

We can use $\varphi \in W^{1,r/(r-p+1)}(\Omega, \wedge^{l-1})$ as a test function for (4). Then, by Definition 1,

$$\int_{B_t} \left\langle A(x, du), \left|d(\eta(u-c))\right|^{r-p} d(\eta(u-c)) - h\right\rangle dx = 0. \tag{17}$$

Let

$$E = \left|d(\eta(u-c))\right|^{r-p} d(\eta(u-c)) - \left|\eta d(u-c)\right|^{r-p} \eta d(u-c); \tag{18}$$

using Lemma 2 yields

$$|E| \le 2^{p-r} \frac{p-r+1}{r-p+1} \left|(u-c)\, d\eta\right|^{r-p+1}. \tag{19}$$

Then (17) becomes

$$\int_{B_t} \left\langle A(x, du), \left|\eta d(u-c)\right|^{r-p} \eta d(u-c)\right\rangle dx \le \int_{B_t} \langle A(x, du), h\rangle\, dx - \int_{B_t} \langle A(x, du), E\rangle\, dx. \tag{20}$$

Noticing that c satisfies $dc = 0$, then by the condition (2) we get

$$\int_{B_t} \left\langle A(x, du), \left|\eta d(u-c)\right|^{r-p} \eta d(u-c)\right\rangle dx = \int_{B_t} \left\langle A(x, du), \left|\eta du\right|^{r-p} \eta du\right\rangle dx \ge \alpha \int_{B_\tau} |du|^r dx. \tag{21}$$

Combining the above inequality with (20), we get

$$\alpha \int_{B_\tau} |du|^r dx \le \int_{B_t} \langle A(x, du), h\rangle\, dx - \int_{B_t} \langle A(x, du), E\rangle\, dx = I_1 + I_2. \tag{22}$$

In the following we will estimate the right side of (22). By (2), the Hölder inequality, and (15),

$$\begin{aligned} \left|I_1\right| &\le \int_{B_t} |A(x, du)|\, |h|\, dx \\ &\le \beta \int_{B_t} |du|^{p-1} |h|\, dx \\ &\le \beta \left(\int_{B_t} |du|^r\right)^{(p-1)/r} \left(\int_{B_t} |h|^{r/(r-p+1)} dx\right)^{(r-p+1)/r} \\ &\le \beta C(n)(p-r) \left(\int_{B_t} |du|^r\right)^{(p-1)/r} \times \left(\int_{B_t} \left|d(\eta^p (u-c))\right|^r dx\right)^{(r-p+1)/r}. \end{aligned} \tag{23}$$

For

$$\begin{aligned} &\left(\int_{B_t} \left|d(\eta^p (u-c))\right|^r dx\right)^{(r-p+1)/r} \\ &\quad = \left(\int_{B_t} \left|\eta^p du + p\eta^{p-1}(u-c)\, d\eta\right|^r dx\right)^{(r-p+1)/r} \\ &\quad \le C(p, r) \left(\int_{B_t} |\eta du|^r dx\right)^{(r-p+1)/r} + C(p, r) \left(\int_{B_t} \left|(u-c)\, d\eta\right|^r dx\right)^{(r-p+1)/r} \\ &\quad \le C(p, r) \left(\int_{B_t} |du|^r dx\right)^{(r-p+1)/r} + C(n, p, r) \left(\int_{B_t} \left|\frac{u-c}{t-\tau}\right|^r dx\right)^{(r-p+1)/r}, \end{aligned} \tag{24}$$

(23) and (24) with Young's inequality yield

$$\begin{aligned} \left|I_1\right| &\le \beta C(n, p, r)(p-r) \times \int_{B_t} |du|^r dx + \beta C(n, p, r)(p-r)\varepsilon \int_{B_t} |du|^r dx \\ &\quad + \beta C(\varepsilon, n, p, r)(p-r) \int_{B_t} \left|\frac{u-c}{t-\tau}\right|^r dx. \end{aligned} \tag{25}$$

Next we estimate I_2. By (2), the Hölder inequality, (19), and Young's inequality,

$$\begin{aligned} \left|I_2\right| &\le \int_{B_t} |A(x, du)|\, |E|\, dx \\ &\le 2^{p-r} \frac{p-r+1}{r-p+1} \beta \int_{B_t} |du|^{p-1} \left|(u-c)\, d\eta\right|^{r-p+1} dx \\ &\le \beta C(n, p, r) \left(\int_{B_t} |du|^r dx\right)^{(p-1)/r} \times \left(\int_{B_t} \left|(u-c)\, d\eta\right|^r dx\right)^{(r-p+1)/r} \\ &\le \varepsilon \beta C(n, p, r) \int_{B_t} |du|^r dx + \beta C(n, p, r, \varepsilon) \times \int_{B_t} \left|\frac{u-c}{t-\tau}\right|^r dx. \end{aligned} \tag{26}$$

Combining (22), (25), and (26), we get

$$\begin{aligned} \alpha \int_{B_\tau} |du|^r dx &\le \beta C(n, p, r)((p-r)(1+\varepsilon)+\varepsilon) \int_{B_t} |du|^r dx \\ &\quad + \beta C(\varepsilon, n, p, r)((p-r)+1) \int_{B_t} \left|\frac{u-c}{t-\tau}\right|^r dx. \end{aligned} \tag{27}$$

Let $p-r$ and ε small enough to let

$$\frac{\beta C(n,p,r)((p-r)(1+\varepsilon)+\varepsilon)}{\alpha}=\theta<1; \tag{28}$$

then we have

$$\int_{B_\tau}|du|^r dx\le\theta\int_{B_t}|du|^r dx+C\left(n,p,r,\frac{\beta}{\alpha}\right)\int_{B_t}\left|\frac{u-c}{t-\tau}\right|^r dx. \tag{29}$$

Next we will refine the inequality (29). Let

$$0<\rho,t<R,\quad f(t)=\int_{B_t}|du|^r dx,\quad A=\int_{B_R}\left|\frac{u-c}{R-\rho}\right|^r dx. \tag{30}$$

Choosing $\varepsilon\in(0,1)$ satisfied $\varepsilon^r>\theta$. Let

$$t_0=\rho,\quad t_{i+1}=t_i+(1-\varepsilon)\varepsilon^i(R-\rho),\quad i=0,1,2,\ldots \tag{31}$$

then when $k\to\infty$, $t_k\to R$. We deduce from (29) that

$$\begin{aligned}f(\rho)&=f(t_0)\\&\le\theta f(t_1)+C\int_{B_{t_1}}\left|\frac{u-c}{(1-\varepsilon)\varepsilon^0(R-\rho)}\right|^r dx\\&\le\theta f(t_1)+\frac{CA}{(1-\varepsilon)^r}\\&\le\theta^k f(t_k)+\frac{CA}{(1-\varepsilon)^r}\sum_{i=0}^{k-1}(\theta\varepsilon^{-r})^i.\end{aligned} \tag{32}$$

Let $k\to\infty$ yield

$$\int_{B_\rho}|du|^r dx\le C\left(n,p,r,\frac{\beta}{\alpha}\right)\int_{B_R}\left|\frac{u-c}{R-\rho}\right|^r dx. \tag{33}$$

Finally, in our case, r is sufficiently close to p; we can estimate $C(n,p,r,\beta/\alpha)$ independently of r. □

Especially, let $\rho=R/2$, $c=u_{B_R}$, and then (13) becomes

$$\int_{B_{R/2}}|du|^r dx\le C\left(n,p,\frac{\beta}{\alpha}\right)\int_{B_R}\left|\frac{u-u_{B_R}}{R}\right|^r dx, \tag{34}$$

or

$$\begin{aligned}&\left(\fint_{B_{R/2}}|du|^r dx\right)^{1/r}\\&\quad\le\frac{C(n,p,\beta/\alpha)}{R}\left(\fint_{B_R}\left|u-u_{B_R}\right|^r dx\right)^{1/r}.\end{aligned} \tag{35}$$

3. Zeros for the Gradients of Weakly A-Harmonic Tensors

We need the following Poincaré inequality.

Lemma 4 (see [16]). *Let D be a cube or a ball, and $\omega\in L^s(D,\wedge^l)$ with $d\omega\in L^s(D,\wedge^{l+1})$. Then*

$$\begin{aligned}&\frac{1}{\operatorname{diam}D}\left(\fint_D|\omega-\omega_D|^s\right)^{1/s}\\&\quad\le C(n,s)\left(\fint_D|d\omega|^{ns/(n+s-1)}\right)^{(n+s-1)/ns}.\end{aligned} \tag{36}$$

Here we denote by $\fint_D$ the integral mean over D.

Using the Caccioppoli inequality (13) and Lemma 4, we can get the weak-reverse Hölder inequality of weakly A-harmonic tensors.

Theorem 5. *Let $u\in D'(\Omega,\wedge^{l-1})$ be a weakly A-harmonic tensor in a domain $\Omega\in\mathbf{R}^n$, and $du\in L^r(D,\wedge^l)$, $l=1,\ldots,n$. Then there exists a constant C, independent of u and R, such that*

$$\begin{aligned}&\left(\fint_{B_{R/2}}|du|^r dx\right)^{1/r}\\&\quad\le C\left(n,p,r,\frac{\beta}{\alpha}\right)\left(\fint_{B_R}|du|^{nr/(n+r-1)}dx\right)^{(n+r-1)/nr}\end{aligned} \tag{37}$$

for all balls $B\subset\Omega$.

Proof. By Lemma 4,

$$\begin{aligned}&\left(\fint_{B_R}|u-u_c|^r dx\right)^{1/r}\\&\quad\le C(n,r)R\left(\fint_{B_R}|du|^{nr/(n+r-1)}dx\right)^{(n+r-1)/nr}.\end{aligned} \tag{38}$$

Then, by (35), we get

$$\begin{aligned}&\left(\fint_{B_{R/2}}|du|^r dx\right)^{1/r}\\&\quad\le C\left(n,p,r,\frac{\beta}{\alpha}\right)\left(\fint_{B_R}|du|^{nr/(n+r-1)}dx\right)^{(n+r-1)/nr}.\end{aligned} \tag{39}$$

□

Next we consider the main results of this paper.

Definition 6 (see [20]). A point $x_0\in\Omega$ is said to be an essential zero of a function $h\in L^1_{\mathrm{loc}}(\Omega)$ if

$$\lim_{R\to0}\frac{1}{R^n}\int_{Q(x_0,R)}|h(x)|\,dx=0, \tag{40}$$

where $Q(x_0, R)$ denotes the cube centered at x_0 of side length $2R$. The order of the essential zero is defined to be

$$N(x_0) = \sup\left\{\alpha : \lim_{R\to 0}\frac{1}{R^{n+\alpha}}\int_{Q(x_0,R)}|h(x)|\,dx = 0\right\}. \quad (41)$$

Lemma 7 (see [20]). *Let $h \in L^1_{loc}(\Omega)$ satisfy the weak-reverse Hölder inequality*

$$\frac{1}{|Q|}\int_Q h^p dx \le A_p\left(\frac{1}{|2Q|}\int_{2Q} h dx\right)^p \quad (42)$$

for all cubes $Q \subset 2Q \subset \Omega$ and some $1 < p < \infty$, with a constant A_P independent of the cube. Then almost every zero of h has infinite order.

Theorem 8. *There exist exponents $r_0 = r_0(n, p, \beta/\alpha) \in (1, 2)$; if $r > r_0$, then for the weakly A-harmonic tensor $u \in W^{1,r}_{loc}(\Omega, \wedge^{l-1})$ almost every zero of du has infinite order.*

Proof. By the weak-reverse Hölder inequality (37) and Lemma 7, we get the desired result. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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