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Research Article

The Smarandache Curves on S_1^2 and Its Duality on H_0^2

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We introduce special Smarandache curves based on Sabban frame on S_1^2 and we investigate geodesic curvatures of Smarandache curves on de Sitter and hyperbolic spaces. The existence of duality between Smarandache curves on de Sitter space and Smarandache curves on hyperbolic space is shown. Furthermore, we give examples of our main results.

1. Introduction

Curves as a subject of differential geometry have been intriguing for researchers throughout mathematical history and so they have been one of the interesting research fields. Regular curves play a central role in the theory of curves in differential geometry. In the theory of curves, there are some special curves such as Bertrand curves, Mannheim curves, involute and evolute curves, and pedal curves in which differential geometers are interested. A common approach to characterization of curves is to consider the relationship between the corresponding Frenet vectors of two curves. Bertrand and Mannheim curves are excellent examples for such cases. In the study of fundamental theory and the characterizations of space curves, the corresponding relations between the curves are a very fascinating problem. Recently, a new special curve is named according to the Sabban frame in the Euclidean unit sphere; Smarandache curve has been defined by Turgut and Yilmaz in Minkowski space-time [1]. Ali studied Smarandache curves with respect to the Sabban frame in Euclidean 3-space [2]. Then Taşköprü and Tosun studied Smarandache curves on S^2 [3]. Smarandache curves have also been studied by many researchers [1, 4–7]. Smarandache curves are one of the most important tools in Smarandache geometry. Smarandache geometry has an important role in the theory of relativity and parallel universes. There are many results related to Smarandache curves in Euclidean and Minkowski spaces, but Smarandache curves are getting more tedious and complicated when de Sitter space is concerned. A regular curve in Minkowski space-time, whose position vector is associated with Frenet frame vectors on another regular curve, is called a Smarandache curve [1].

In this paper, we define Smarandache curves on de Sitter surface according to the Sabban frame $\{\alpha, t, \eta\}$ in Minkowski 3-space. We obtain the geodesic curvatures and the expressions for the Sabban frame's vectors of special Smarandache curves on de Sitter surface. Furthermore, we give some examples of special de Sitter and hyperbolic Smarandache curves in Minkowski 3-space.

2. Preliminaries

In this section, we prepare some definitions and basic facts. For basic concepts and details of properties, see [8, 9]. Consider $\mathbb{R}^3$ as a three-dimensional vector space. For any vectors $\vec{x} = (x_0, x_1, x_2)$ and $\vec{y} = (y_0, y_1, y_2)$ in $\mathbb{R}^3$ the pseudoscalar product of $\vec{x}$ and $\vec{y}$ is defined by $\langle\cdot,\cdot\rangle_L = -x_0 y_0 + x_1 y_1 + x_2 y_2$. It is called $E_1^3 = (\mathbb{R}^3, \langle\cdot,\cdot\rangle_L)$ Minkowski 3-space. Recall that a nonzero vector $\vec{x} \in E_1^3$ is spacelike if $\langle\vec{x},\vec{x}\rangle_L > 0$, timelike if $\langle\vec{x},\vec{x}\rangle_L < 0$, and null (lightlike) if $\langle\vec{x},\vec{x}\rangle_L = 0$. The norm (length) of a vector $\vec{x} \in E_1^3$ is given by $\|\vec{x}\|_L = \sqrt{|\langle\vec{x},\vec{x}\rangle_L|}$ and two vectors $\vec{x}$ and $\vec{y}$ are said to be orthogonal if $\langle\vec{x},\vec{y}\rangle_L = 0$. Next, we say that an arbitrary curve $\alpha = \alpha(s)$ in

E_1^3 can locally be spacelike, timelike, or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are, respectively, spacelike, timelike, or null (lightlike) for all $s \in I$. If $\|\alpha'(s)\|_L \neq 0$ for every $s \in I$, then α is a regular curve in E_1^3. A spacelike (timelike) regular curve α is parameterized by a pseudoarclength parameter s which is given by $\alpha : I \subset \mathbb{R} \to E_1^3$, and then the tangent vector $\alpha'(s)$ along α has unit length; that is, $\langle \alpha'(s), \alpha'(s) \rangle_L = 1$ ($\langle \alpha'(s), \alpha'(s) \rangle_L = -1$) for all $s \in I$.

Let $\vec{x} = (x_0, x_1, x_2)$, $\vec{y} = (y_0, y_1, y_2) \in E_1^3$. The Lorentzian vector cross-product is defined as follows:

$$\vec{x} \wedge \vec{y} = \begin{vmatrix} -e_0 & e_1 & e_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}, \quad (1)$$

and also the following relations hold:

(i) $\langle \vec{x} \wedge \vec{y}, \vec{z} \rangle_L = \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}$,

(ii) $\vec{x} \wedge (\vec{y} \wedge \vec{z}) = \langle \vec{x}, \vec{y} \rangle_L \vec{z} - \langle \vec{x}, \vec{z} \rangle_L \vec{y}$,

where $\vec{x} = (x_0, x_1, x_2)$, $\vec{y} = (y_0, y_1, y_2)$, $\vec{z} = (z_0, z_1, z_2) \in E_1^3$.

We now define de Sitter 2-space by

$$S_1^2 = \left\{ \vec{x} \in \mathbb{R}_1^3 : -x_0^2 + x_1^2 + x_2^2 = 1 \right\} \quad (2)$$

and hyperbolic space in Minkowski 3-space by

$$H_0^2 = \left\{ \vec{x} \in \mathbb{R}_1^3 : -x_0^2 + x_1^2 + x_2^2 = -1, \ x_0 > 0 \right\}. \quad (3)$$

We can express a new frame different from the Frenet frame for a regular curve. Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed curve lying fully on S_1^2. Then its position vector α is spacelike, which implies that the tangent vector $\alpha' = t$ is the unit timelike, spacelike, or null vector for all $s \in I$.

In our work, we are concerned with the vector $\alpha' = t$ which may be the unit timelike or spacelike.

Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed curve lying fully on S_1^2 for all $s \in I$ and its position vector α a unit spacelike vector; then $\alpha' = t$ is a unit timelike and so η is a unit spacelike vector. In this case, the curve α is called a timelike curve. If $\alpha' = t$ is a unit spacelike vector, then η is a unit timelike vector. In this case, the curve α is called a spacelike curve and we have an orthonormal Sabban frame $\{\alpha(s), t(s), \eta(s)\}$ along the curve α, where $\eta(s) = \alpha(s) \wedge t(s)$ is the unit spacelike or timelike vector. Then Frenet formulas of α are given by

$$\begin{bmatrix} \alpha' \\ t' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \varepsilon & 0 & \varepsilon\kappa_g \\ 0 & \varepsilon\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}, \quad (4)$$

where $\varepsilon = \pm 1$, the curve α is timelike for $\varepsilon = 1$ and spacelike for $\varepsilon = -1$, and $\kappa_g(s)$ is the geodesic curvature of α on S_1^2, which is given by $\kappa_g(s) = \det(\alpha(s), t(s), t'(s))$, where s is the arc length parameter of α. This relation is also given by [10, 11] for $\varepsilon = 1$. In particular, by using equation (ii), the following relations hold:

$$\varepsilon\alpha = t \wedge \eta, \quad t = \alpha \wedge \eta, \quad \eta = \alpha \wedge t. \quad (5)$$

Definition 1. A unit speed regular curve $\beta(\bar{s}(s))$ lying fully in Minkowski 3-space, whose position vector is associated with Sabban frame vectors on another regular curve $\alpha(s)$, is called a Smarandache curve [1].

Based on this definition, if a regular unit speed curve $\alpha : I \subset \mathbb{R} \to S_1^2$ is lying fully on S_1^2 for all $s \in I$ and its position vector α is a unit spacelike, then the Smarandache curve $\beta = \beta(\bar{s}(s))$ of curve α is a regular unit speed curve lying fully on S_1^2 or H_0^2. In this case we have the following:

(a) the Smarandache curve $\beta(\bar{s}(s))$ may be a timelike curve on S_1^2,

(b) the Smarandache curve $\beta(\bar{s}(s))$ may be a spacelike curve on S_1^2, or

(c) the Smarandache curve $\beta(\bar{s}(s))$ is in H_0^2 for all $s \in I$.

Let $\{\alpha, t, \eta\}$ and $\{\beta, t_\beta, \eta_\beta\}$ be the moving Sabban frames of α and β, respectively. Then we have the following definitions and theorems of Smarandache curves $\beta = \beta(\bar{s}(s))$ given in Section 3. In Section 3, we deal with Smarandache curves on de Sitter and hyperbolic spaces for timelike curves. Similar results are given for spacelike curves in the Appendix.

3. De Sitter and Hyperbolic Smarandache Curves for Timelike Curves

In this section we give different Smarandache curves on de Sitter and hyperbolic spaces in Minkowski-space. Let α be a timelike curve on S_1^2; then the Smarandache partner curve of α is either timelike/spacelike or hyperbolic curve. We refer to the hyperbolic Smarandache curve of a timelike curve α as the hyperbolic duality of α.

To avoid repetition we use $\varepsilon = \pm 1$ in the following theorems in this section. If we take $\varepsilon = 1$, then the Smarandache curve β is timelike or spacelike, and if we take $\varepsilon = -1$, then β is hyperbolic.

Definition 2. Let $\alpha = \alpha(s)$ be a unit speed regular timelike curve lying fully on S_1^2. The curve $\beta : I \subset \mathbb{R} \to S_1^2$ ($\beta : I \subset \mathbb{R} \to H_0^2$) of α defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}} \left(c_1 \alpha(s) + c_2 \eta(s) \right) \quad (6)$$

is called the $\alpha\eta$-Smarandache curve of α and fully lies on S_1^2, where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon(c_1^2 + c_2^2) = 2$. If $\varepsilon = -1$; then the hyperbolic $\alpha\eta$-Smarandache curve is undefined since the equation $(c_1^2 + c_2^2) = -2$ has no solution in $\mathbb{R}$.

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ is the $\alpha\eta$-timelike Smarandache curve of α, then the relationships* *between the Sabban frames of α and its $\alpha\eta$-Smarandache curve are given by*

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} \frac{c_1}{\sqrt{2}} & 0 & \frac{c_2}{\sqrt{2}} \\ 0 & \varepsilon & 0 \\ \frac{c_2\varepsilon}{\sqrt{2}} & 0 & \frac{c_1\varepsilon}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}, \tag{7}$$

where $\varepsilon = \pm 1$ and its geodesic curvature κ_g^β is given by

$$\kappa_g^\beta = \frac{c_1\kappa_g - c_2}{\left|c_1 + c_2\kappa_g\right|}. \tag{8}$$

Proof. By taking the derivative of (6) with respect to s and by using (4), we get

$$\beta'(\bar{s}(s)) = \frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_1 + c_2\kappa_g\right)t \tag{9}$$

or equivalently

$$t_\beta \frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_1 + c_2\kappa_g\right)t, \tag{10}$$

where

$$\frac{d\bar{s}}{ds} = \frac{\left|c_1 + c_2\kappa_g\right|}{\sqrt{2}}. \tag{11}$$

Hence, the unit timelike tangent vector of the curve β is given by

$$t_\beta = \varepsilon t, \tag{12}$$

where $\varepsilon = 1$ if $c_1 + c_2\kappa_g > 0$ for all s and $\varepsilon = -1$ if $c_1 + c_2\kappa_g < 0$ for all s. From (6) and (12) we get

$$\eta_\beta = \beta \wedge t_\beta = \frac{\varepsilon}{\sqrt{2}}\left(c_2\alpha + c_1\eta\right). \tag{13}$$

It is easily seen that η_β is a unit spacelike vector. On the other hand, differentiating (12) with respect to s, we find

$$\frac{dt_\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \varepsilon\left(\alpha + \kappa_g\eta\right) \tag{14}$$

and by combining (11) and (14) we have

$$t'_\beta = \frac{\sqrt{2}\varepsilon}{\left|c_1 + c_2\kappa_g\right|}\left(\alpha + \kappa_g\eta\right). \tag{15}$$

Consequently, the geodesic curvature κ_g^β of the curve $\beta = \beta(\bar{s})$ is given by

$$\kappa_g^\beta = \det\left(\beta, t_\beta, t'_\beta\right) = \frac{c_1\kappa_g - c_2}{\left|c_1 + c_2\kappa_g\right|}. \tag{16}$$

□

Corollary 4. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2. Then the $\alpha\eta$-spacelike Smarandache curve $\alpha : I \subset \mathbb{R} \to S_1^2$ of α does not exist.*

Definition 5. Let $\alpha = \alpha(s)$ be a regular unit speed timelike curve lying fully on S_1^2. Then the αt-Smarandache curve $\beta : I \subset \mathbb{R} \to S_1^2$ $(\beta : I \subset \mathbb{R} \to H_0^2)$ of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(c_1\alpha(s) + c_2 t(s)\right), \tag{17}$$

where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon(c_1^2 - c_2^2) = 2$.

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ $(\beta : I \subset \mathbb{R} \to H_0^2)$ is the αt-timelike (hyperbolic) Smarandache curve of α, then its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} \frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} & 0 \\ \frac{c_2}{\sqrt{2 - \varepsilon\left(c_2^2\kappa_g^2\right)}} & \frac{c_1}{\sqrt{2 - \varepsilon\left(c_2^2\kappa_g^2\right)}} & \frac{c_2\kappa_g}{\sqrt{2 - \varepsilon\left(c_2^2\kappa_g^2\right)}} \\ \frac{-c_2^2\kappa_g}{\sqrt{2\left(2 - \varepsilon\left(c_2^2\kappa_g^2\right)\right)}} & \frac{-c_1c_2\kappa_g}{\sqrt{2\left(2 - \varepsilon\left(c_2^2\kappa_g^2\right)\right)}} & \frac{2}{\sqrt{2\left(2 - \varepsilon\left(c_2^2\kappa_g^2\right)\right)}} \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}. \tag{18}$$

The geodesic curvature κ_g^β of curve β is given by

$$\kappa_g^\beta = \frac{1}{\left(2 - \varepsilon\left(c_2^2\kappa_g^2\right)\right)^{5/2}}\left(c_2^2\kappa_g\lambda_1 - c_1c_2\kappa_g\lambda_2 + 2\lambda_3\right), \tag{19}$$

where

$$\begin{aligned} \lambda_1 &= \varepsilon c_2^3\kappa_g\kappa'_g + c_1\left(2 - \varepsilon c_2^2\kappa_g^2\right), \\ \lambda_2 &= \varepsilon c_1c_2^2\kappa_g\kappa'_g + \left(c_2 + c_2\kappa_g^2\right)\left(2 - \varepsilon c_2^2\kappa_g^2\right), \\ \lambda_3 &= \varepsilon c_2^3\kappa_g^2\kappa'_g + \left(c_1\kappa_g + c_2\kappa'_g\right)\left(2 - \varepsilon c_2^2\kappa_g^2\right). \end{aligned} \tag{20}$$

Proof. We take $\varepsilon = 1$. By taking the derivative of (17) with respect to s and by using (4), we get

$$\beta'(\bar{s}(s)) = \frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_2\alpha + c_1 t + c_2\kappa_g\eta\right) \tag{21}$$

or equivalently

$$t_\beta \frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_2\alpha + c_1 t + c_2\kappa_g\eta\right). \tag{22}$$

Taking the Lorentzian inner product in (22) we have

$$\left\langle t_\beta, t_\beta\right\rangle_L \left(\frac{d\bar{s}}{ds}\right)^2 = \frac{1}{2}\left(c_2^2\kappa_g^2 - 2\right). \tag{23}$$

If $c_2^2\kappa_g^2 < 2$, then t_β is a timelike vector. So

$$\frac{d\bar{s}}{ds} = \sqrt{\frac{2 - c_2^2\kappa_g^2}{2}}. \tag{24}$$

Therefore, the unit timelike tangent vector of the curve β is given by

$$t_\beta = \frac{1}{\sqrt{2 - c_2^2\kappa_g^2}}\left(c_2\alpha + c_1 t + c_2\kappa_g\eta\right). \tag{25}$$

On the other hand, from (17) and (25) it can be easily seen that

$$\eta_\beta = \beta \wedge t_\beta = \frac{1}{\sqrt{4 - 2c_2^2\kappa_g^2}}\left(-c_2^2\kappa_g\alpha - c_1c_2\kappa_g t + 2\eta\right) \tag{26}$$

is a unit spacelike vector. Differentiating (25) with respect to s, we obtain

$$\frac{dt_\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\left(2 - c_2^2\kappa_g^2\right)^{3/2}}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right), \tag{27}$$

where

$$\begin{aligned}
\lambda_1 &= c_2^3\kappa_g\kappa_g' + c_1\left(2 - c_2^2\kappa_g^2\right),\\
\lambda_2 &= c_1c_2^2\kappa_g\kappa_g' + \left(c_2 + c_2\kappa_g^2\right)\left(2 - c_2^2\kappa_g^2\right),\\
\lambda_3 &= c_2^3\kappa_g^2\kappa_g' + \left(c_1\kappa_g + c_2\kappa_g'\right)\left(2 - c_2^2\kappa_g^2\right),
\end{aligned} \tag{28}$$

and by combining (24) and (26) we get

$$t_\beta' = \frac{\sqrt{2}}{\left(2 - c_2^2\kappa_g^2\right)^2}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right). \tag{29}$$

Consequently, we have

$$\begin{aligned}
\kappa_g^\beta &= \det\left(\beta, t_\beta, t_\beta'\right)\\
&= \frac{1}{\left(2 - c_2^2\kappa_g^2\right)^{5/2}}\left(c_2^2\kappa_g\lambda_1 - c_1c_2\kappa_g\lambda_2 + 2\lambda_3\right).
\end{aligned} \tag{30}$$

The proof of $\varepsilon = -1$ case is similar. □

The following corollary is proved by the same methods as the above theorem.

Corollary 7. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ is the αt-spacelike Smarandache curve of α, then its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by*

$$\begin{bmatrix}\beta\\ t_\beta\\ \eta_\beta\end{bmatrix} = \begin{bmatrix}
\frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} & 0\\
\frac{c_2}{\sqrt{c_2^2\kappa_g^2 - 2}} & \frac{c_1}{\sqrt{c_2^2\kappa_g^2 - 2}} & \frac{c_2\kappa_g}{\sqrt{c_2^2\kappa_g^2 - 2}}\\
\frac{-c_2^2\kappa_g}{\sqrt{2\left(c_2^2\kappa_g^2 - 2\right)}} & \frac{-c_1c_2\kappa_g}{\sqrt{2\left(c_2^2\kappa_g^2 - 2\right)}} & \frac{2}{\sqrt{2\left(c_2^2\kappa_g^2 - 2\right)}}
\end{bmatrix} \times \begin{bmatrix}\alpha\\ t\\ \eta\end{bmatrix}. \tag{31}$$

The geodesic curvature κ_g^β of curve β is given by

$$\kappa_g^\beta = \frac{1}{\left(c_2^2\kappa_g^2 - 2\right)^{5/2}}\left(c_2^2\kappa_g\lambda_1 - c_1c_2\kappa_g\lambda_2 + 2\lambda_3\right), \tag{32}$$

where λ_1, λ_2, and λ_3 can be calculated as in Theorem 6.

Definition 8. Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2. Then the $t\eta$-Smarandache curve $\beta : I \subset \mathbb{R} \to S_1^2$ ($\beta : I \subset \mathbb{R} \to H_0^2$) of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(c_1 t(s) + c_2\eta(s)\right), \tag{33}$$

where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon(-c_1^2 + c_2^2) = 2$.

Theorem 9. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$*

($\beta : I \subset \mathbb{R} \to H_0^2$) is the $t\eta$-timelike (hyperbolic) Smarandache curve of α, then its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{c_1}{\sqrt{2}} & \dfrac{c_2}{\sqrt{2}} \\ \dfrac{c_1}{\sqrt{2\kappa_g^2 - \varepsilon c_1^2}} & \dfrac{c_2\kappa_g}{\sqrt{2\kappa_g^2 - \varepsilon c_1^2}} & \dfrac{c_1\kappa_g}{\sqrt{2\kappa_g^2 - \varepsilon c_1^2}} \\ \dfrac{\varepsilon 2\kappa_g}{\sqrt{2\left(2\kappa_g^2 - \varepsilon c_1^2\right)}} & \dfrac{c_1 c_2}{\sqrt{2\left(2\kappa_g^2 - \varepsilon c_1^2\right)}} & \dfrac{-c_1^2}{\sqrt{2\left(2\kappa_g^2 - \varepsilon c_1^2\right)}} \end{bmatrix} \times \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}. \quad (34)$$

The geodesic curvature κ_g^β of curve β is given by

$$\kappa_g^\beta = \frac{1}{\left(2\kappa_g^2 - \varepsilon c_1^2\right)^{5/2}} \left(-\varepsilon 2\kappa_g \lambda_1 + c_1 c_2 \lambda_2 - c_1^2 \lambda_3\right), \quad (35)$$

where

$$\begin{aligned} \lambda_1 &= -2c_1\kappa_g\kappa_g' + c_2\kappa_g\left(2\kappa_g^2 - \varepsilon c_1^2\right), \\ \lambda_2 &= -2c_2\kappa_g^2\kappa_g' + \left(c_1 + c_2\kappa_g' + c_1\kappa_g^2\right)\left(2\kappa_g^2 - \varepsilon c_1^2\right), \\ \lambda_3 &= -2c_1\kappa_g^2\kappa_g' + \left(c_2\kappa_g^2 + c_1\kappa_g'\right)\left(2\kappa_g^2 - \varepsilon c_1^2\right). \end{aligned} \quad (36)$$

Proof. Let $\varepsilon = 1$. By taking the derivative of (33) with respect to s and by using (4), we get

$$\beta'(\bar{s}(s)) = \frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_1\alpha + c_2\kappa_g t + c_1\kappa_g\eta\right) \quad (37)$$

or equivalently

$$t_\beta \frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(c_1\alpha + c_2\kappa_g t + c_1\kappa_g\eta\right). \quad (38)$$

Taking the Lorentzian inner product in (38) we have

$$\left\langle t_\beta, t_\beta\right\rangle_L \left(\frac{d\bar{s}}{ds}\right)^2 = \frac{1}{2}\left(c_1^2 - 2\kappa_g^2\right) \quad (39)$$

and t_β is a unit timelike vector for $2\kappa_g^2 > c_1^2$. It follows that

$$\frac{d\bar{s}}{ds} = \sqrt{\frac{2\kappa_g^2 - c_1^2}{2}}. \quad (40)$$

Therefore, the unit timelike tangent vector of the curve β is given by

$$t_\beta = \frac{1}{\sqrt{2\kappa_g^2 - c_1^2}}\left(c_1\alpha + c_2\kappa_g t + c_1\kappa_g\eta\right). \quad (41)$$

On the other hand, from (33) and (41) it can be easily seen that

$$\eta_\beta = \beta \wedge t_\beta = \frac{1}{\sqrt{4\kappa_g^2 - 2c_1^2}}\left(2\kappa_g\alpha + c_1c_2 t - c_1^2\eta\right) \quad (42)$$

is a unit spacelike vector. Differentiating (41) with respect to s, we find

$$\frac{dt_\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\left(2\kappa_g^2 - c_1^2\right)^{3/2}}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right), \quad (43)$$

where

$$\begin{aligned} \lambda_1 &= -2c_1\kappa_g\kappa_g' + c_2\kappa_g\left(2\kappa_g^2 - c_1^2\right), \\ \lambda_2 &= -2c_2\kappa_g^2\kappa_g' + \left(c_1 + c_2\kappa_g' + c_1\kappa_g^2\right)\left(2\kappa_g^2 - c_1^2\right), \\ \lambda_3 &= -2c_1\kappa_g^2\kappa_g' + \left(c_2\kappa_g^2 + c_1\kappa_g'\right)\left(2\kappa_g^2 - c_1^2\right), \end{aligned} \quad (44)$$

and by combining (40) and (43) we get

$$t_\beta' = \frac{\sqrt{2}}{\left(2\kappa_g^2 - c_1^2\right)^2}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right). \quad (45)$$

As a result, we have

$$\begin{aligned} \kappa_g^\beta &= \det\left(\beta, t_\beta, t_\beta'\right) \\ &= \frac{1}{\left(2\kappa_g^2 - c_1^2\right)^{5/2}}\left(-2\kappa_g\lambda_1 + c_1c_2\lambda_2 - c_1^2\lambda_3\right). \end{aligned} \quad (46)$$

The proof of $\varepsilon = -1$ case is similar. □

Corollary 10. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ is the $t\eta$-spacelike Smarandache curve of α, then its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{c_1}{\sqrt{2}} & \dfrac{c_2}{\sqrt{2}} \\ \dfrac{c_1}{\sqrt{c_1^2 - 2\kappa_g^2}} & \dfrac{c_2\kappa_g}{\sqrt{c_1^2 - 2\kappa_g^2}} & \dfrac{c_1\kappa_g}{\sqrt{c_1^2 - 2\kappa_g^2}} \\ \dfrac{\varepsilon 2\kappa_g}{\sqrt{2\left(c_1^2 - 2\kappa_g^2\right)}} & \dfrac{c_1 c_2}{\sqrt{2\left(c_1^2 - 2\kappa_g^2\right)}} & \dfrac{-c_1^2}{\sqrt{2\left(c_1^2 - 2\kappa_g^2\right)}} \end{bmatrix} \times \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}. \quad (47)$$

The geodesic curvature κ_g^β of curve β is given by

$$\kappa_g^\beta = \frac{1}{\left(c_1^2 - 2\kappa_g^2\right)^{5/2}} \left(-2\kappa_g\lambda_1 + c_1c_2\lambda_2 - c_1^2\lambda_3\right), \tag{48}$$

where λ_1, λ_2, and λ_3 can be calculated as in Theorem 9.

Definition 11. Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2. Then the $\alpha t\eta$-Smarandache curve $\beta : I \subset \mathbb{R} \to S_1^2$ ($\beta : I \subset \mathbb{R} \to H_0^2$) of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\left(c_1\alpha(s) + c_2 t(s) + c_3\eta(s)\right), \tag{49}$$

where $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon(c_1^2 - c_2^2 + c_3^2) = 3$.

Theorem 12. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ ($\beta : I \subset \mathbb{R} \to H_0^2$) is the $\alpha t\eta$-timelike (hyperbolic) Smarandache curve of α, then its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} \dfrac{c_1}{\sqrt{3}} & \dfrac{c_2}{\sqrt{3}} & \dfrac{c_3}{\sqrt{3}} \\ \dfrac{c_2}{\sqrt{\varepsilon A}} & \dfrac{c_1 + c_3\kappa_g}{\sqrt{\varepsilon A}} & \dfrac{c_2\kappa_g}{\sqrt{\varepsilon A}} \\ \dfrac{-c_2^2\kappa_g + c_3\left(c_1 + c_3\kappa_g\right)}{\sqrt{3\varepsilon A}} & \dfrac{-c_1c_2\kappa_g + c_2c_3}{\sqrt{3\varepsilon A}} & \dfrac{c_1\left(c_1 + c_3\kappa_g\right) - c_2^2}{\sqrt{3\varepsilon A}} \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}, \tag{50}$$

where $A = (c_1 + c_3\kappa_g)^2 - c_2^2 - c_2^2\kappa_g^2$. If we take $\varepsilon = 1$ or -1, then the Smarandache curve β is timelike or hyperbolic, respectively. Furthermore, the geodesic curvature κ_g^β of curve β is given by

$$\begin{aligned} \kappa_g^\beta = {} & \left(\left(c_2^2\kappa_g - c_3^2\kappa_g - c_1c_3\right)\lambda_1 + \left(-c_1c_2\kappa_g + c_2c_3\right)\lambda_2 \right. \\ & \left. + \left(c_1^2 + c_1c_3\kappa_g - c_2^2\right)\lambda_3\right) \\ & \times \left(\left(\varepsilon\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)\right)^{5/2}\right)^{-1}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \lambda_1 &= \varepsilon\Big(c_2\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(c_1 + c_3\kappa_g\right)\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)\Big), \\ \lambda_2 &= \varepsilon\Big(\left(c_1 + c_3\kappa_g\right)\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(c_2 + c_3\kappa_g' + c_2\kappa_g^2\right)\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)\Big), \\ \lambda_3 &= \varepsilon\Big(c_2\kappa_g\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(\kappa_g\left(c_1 + c_3\kappa_g\right) + c_2\kappa_g'\right) \\ &\quad \times \left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)\Big). \end{aligned} \tag{52}$$

Proof. We take $\varepsilon = 1$. By taking the derivative of (49) with respect to s and using (4), we get

$$\beta'(\bar{s}(s)) = \frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{3}}\left(c_2\alpha + \left(c_1 + c_3\kappa_g\right)t + c_2\kappa_g\eta\right) \tag{53}$$

or equivalently

$$t_\beta \frac{d\bar{s}}{ds} = \frac{1}{\sqrt{3}}\left(c_2\alpha + \left(c_1 + c_3\kappa_g\right)t + c_2\kappa_g\eta\right). \tag{54}$$

Taking the Lorentzian inner product in (54) we have

$$\left\langle t_\beta, t_\beta\right\rangle_L \left(\frac{d\bar{s}}{ds}\right)^2 = \frac{1}{3}\left(c_2^2 + c_2^2\kappa_g^2 - \left(c_1 + c_3\kappa_g\right)^2\right). \tag{55}$$

For $(c_1 + c_3\kappa_g)^2 > c_2^2 + c_2^2\kappa_g^2$, t_β is a unit timelike vector. It follows that

$$\frac{d\bar{s}}{ds} = \sqrt{\frac{\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2}{3}}. \tag{56}$$

Therefore, the unit timelike tangent vector of the curve β is given by

$$t_\beta = \frac{1}{\sqrt{\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2}}\left(c_2\alpha + \left(c_1 + c_3\kappa_g\right)t + c_2\kappa_g\eta\right). \tag{57}$$

On the other hand, taking the cross-product of (49) with (57) it can be easily seen that

$$\begin{aligned}\eta_\beta &= \beta \wedge t_\beta \\ &= \left(\left(-c_2^2\kappa_g + c_3\left(c_1 + c_3\kappa_g\right)\right)\alpha + \left(c_2c_3 - c_1c_2\kappa_g\right)t \right.\\ &\quad \left. + \left(c_1\left(c_1 + c_3\kappa_g\right) - c_2^2\right)\eta\right) \\ &\quad \times \left(\sqrt{3\left(c_1 + c_3\kappa_g\right)^2 - 3c_2^2 - 3c_2^2\kappa_g^2}\right)^{-1}.\end{aligned} \tag{58}$$

This means that the η_β is a unit spacelike vector. In order to obtain the tangent vector of β let us differentiate (57) with respect to s. We find

$$\frac{dt_\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)^{3/2}}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right), \tag{59}$$

where

$$\begin{aligned}\lambda_1 &= c_2\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(c_1 + c_3\kappa_g\right)\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right), \\ \lambda_2 &= \left(c_1 + c_3\kappa_g\right)\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(c_2 + c_3\kappa_g' + c_2\kappa_g^2\right)\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right), \\ \lambda_3 &= c_2\kappa_g\left(-c_3\kappa_g'\left(c_1 + c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ &\quad + \left(\kappa_g\left(c_1 + c_3\kappa_g\right) + c_2\kappa_g'\right) \\ &\quad \times \left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right),\end{aligned} \tag{60}$$

and by combining (56) and (59) we get

$$t_\beta' = \frac{\sqrt{3}}{\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)^2}\left(\lambda_1\alpha + \lambda_2 t + \lambda_3\eta\right). \tag{61}$$

Finally, the geodesic curvature κ_g^β of the curve $\beta = \beta(\bar{s}(s))$ is given by

$$\begin{aligned}\kappa_g^\beta &= \det\left(\beta, t_\beta, t_\beta'\right) \\ &= \left(\left(c_2^2\kappa_g - c_3^2\kappa_g - c_1c_3\right)\lambda_1 + \left(-c_1c_2\kappa_g + c_2c_3\right)\lambda_2 \right.\\ &\quad \left. + \left(c_1^2 + c_1c_3\kappa_g - c_2^2\right)\lambda_3\right) \\ &\quad \times \left(\left(\left(c_1 + c_3\kappa_g\right)^2 - c_2^2 - c_2^2\kappa_g^2\right)^{5/2}\right)^{-1}.\end{aligned} \tag{62}$$

The proof of $\varepsilon = -1$ case is similar. □

Corollary 13. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed timelike curve lying fully on S_1^2 with the Sabban frame $\{\alpha, t, \eta\}$ and geodesic curvature κ_g. If $\beta : I \subset \mathbb{R} \to S_1^2$ is the $\alpha t\eta$-spacelike Smarandache curve of α, then, for $A = c_2^2 + c_2^2\kappa_g^2 - (c_1 + c_3\kappa_g)^2$, its frame $\{\beta, t_\beta, \eta_\beta\}$ is given by*

$$\begin{bmatrix}\beta \\ t_\beta \\ \eta_\beta\end{bmatrix} = \begin{bmatrix}\dfrac{c_1}{\sqrt{3}} & \dfrac{c_2}{\sqrt{3}} & \dfrac{c_3}{\sqrt{3}} \\ \dfrac{c_2}{\sqrt{A}} & \dfrac{c_1 + c_3\kappa_g}{\sqrt{A}} & \dfrac{c_2\kappa_g}{\sqrt{A}} \\ \dfrac{-c_2^2\kappa_g + c_3\left(c_1 + c_3\kappa_g\right)}{\sqrt{3A}} & \dfrac{-c_1c_2\kappa_g + c_2c_3}{\sqrt{3A}} & \dfrac{c_1\left(c_1 + c_3\kappa_g\right) - c_2^2}{\sqrt{3A}}\end{bmatrix}\begin{bmatrix}\alpha \\ t \\ \eta\end{bmatrix}. \tag{63}$$

Furthermore, the geodesic curvature κ_g^β of curve β is given by

$$\begin{aligned}\kappa_g^\beta &= \left(\left(c_2^2\kappa_g - c_3^2\kappa_g - c_1c_3\right)\lambda_1 + \left(-c_1c_2\kappa_g + c_2c_3\right)\lambda_2 \right.\\ &\quad \left. + \left(c_1^2 + c_1c_3\kappa_g - c_2^2\right)\lambda_3\right) \\ &\quad \times \left(\left(c_2^2 + c_2^2\kappa_g^2 - \left(c_1 + c_3\kappa_g\right)^2\right)^{5/2}\right)^{-1},\end{aligned} \tag{64}$$

where λ_1, λ_2, and λ_3 can be calculated as in Theorem 12.

Example 14. Let us consider a unit speed timelike curve α on S_1^2 defined by

$$\alpha(s) = \left(\sqrt{2}\sinh s, \cosh s, \sinh s\right). \tag{65}$$

Then the orthonormal Sabban frame $\{\alpha(s), t(s), \eta(s)\}$ of α can be calculated as follows:

$$\begin{aligned}\alpha(s) &= \left(\sqrt{2}\sinh s, \cosh s, \sinh s\right), \\ t(s) &= \left(\sqrt{2}\cosh s, \sinh s, \cosh s\right), \\ \eta(s) &= \left(-1, 0, -\sqrt{2}\right).\end{aligned} \tag{66}$$

The geodesic curvature of α is 0. In terms of the definitions, we obtain Smarandache curves according to Sabban frame on S_1^2.

Firstly, when we take $c_1 = 1$ and $c_2 = 1$, then the timelike $\alpha\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\sinh s - 1, \cosh s, \sinh s - \sqrt{2}\right) \quad (67)$$

and the Sabban frame of the $\alpha\eta$-Smarandache curve is given by

$$\begin{bmatrix} \beta \\ t_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \eta \end{bmatrix}, \quad (68)$$

and its geodesic curvature κ_g^β is -1. Here the hyperbolic $\alpha\eta$-Smarandache curve is undefined.

Secondly, when we take $c_1 = 3$ and $c_2 = \sqrt{7}$, then the timelike αt-Smarandache curve is given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\Big(&3\sqrt{2}\sinh s + \sqrt{14}\cosh s, \\ &3\cosh s + \sqrt{7}\sinh s, 3\sinh s + \sqrt{7}\cosh s\Big),\end{aligned} \quad (69)$$

and if we take $c_1 = \sqrt{7}$ and $c_2 = 3$, then the hyperbolic αt-Smarandache curve is given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\Big(&\sqrt{14}\sinh s + 3\sqrt{2}\cosh s, \\ &\sqrt{7}\cosh s + 3\sinh s, \sqrt{7}\sinh s + 3\cosh s\Big).\end{aligned} \quad (70)$$

Thirdly, when we take $c_1 = \sqrt{2}$ and $c_2 = 2$, then the spacelike $t\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \left(\sqrt{2}(\cosh s - 1), \sinh s, \cosh s - 2\right) \quad (71)$$

and if we take $c_1 = 2$ and $c_2 = \sqrt{2}$, then the hyperbolic $t\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \left(2\cosh s - 1, \sqrt{2}\sinh s, \sqrt{2}(\cosh s - 1)\right). \quad (72)$$

Finally, when we take $c_1 = 2$, $c_2 = \sqrt{2}$, and $c_3 = 1$, then the $\alpha t\eta$-Smarandache curve is a timelike curve and given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\Big(&2\sqrt{2}\sinh s + 2\cosh s - 1, \\ &2\cosh s + \sqrt{2}\sinh s, \\ &2\sinh s + \sqrt{2}\cosh s - \sqrt{2}\Big),\end{aligned} \quad (73)$$

and if we take $c_1 = \sqrt{2}$, $c_2 = 2\sqrt{2}$, and $c_3 = \sqrt{3}$, then the $\alpha t\eta$-Smarandache curve is a hyperbolic curve and given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\Big(&2\sinh s + 4\cosh s - \sqrt{3}, \\ &\sqrt{2}\cosh s + 2\sqrt{2}\sinh s, \\ &\sqrt{2}\sinh s + 2\sqrt{2}\cosh s - \sqrt{6}\Big).\end{aligned} \quad (74)$$

On the other hand, in the last case, if we take $c_1 = 1$, $c_2 = \sqrt{2}$, and $c_3 = 2$ (i.e., $c_1 < c_2 < c_3$), then the $\alpha t\eta$-Smarandache curve is spacelike and given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\Big(&\sqrt{2}\sinh s + 2\cosh s - 2, \\ &\cosh s + \sqrt{2}\sinh s, \\ &\sinh s + \sqrt{2}\cosh s - 2\sqrt{2}\Big).\end{aligned} \quad (75)$$

The Sabban frames and geodesic curvatures of αt, $t\eta$, and $\alpha t\eta$-Smarandache curves can be easily obtained by using a similar way to the above. Also we give the curve α and its Smarandache partners in Figure 1.

Appendix

In this section, we give as a table different Smarandache curves on de Sitter space or on hyperbolic space for spacelike curves in Minkowski-space. We give the theorem about undefined Smarandache curve below. The other $\alpha\eta$-, αt-, $t\eta$-, and $\alpha t\eta$-Smarandache curves on S_1^2 and $\alpha\eta$-, $t\eta$-, $\alpha t\eta$-Smarandache curves on H_0^2 and their corresponding Sabban frames and geodesic curvatures are similar to those in the previous section.

Theorem A.1. *Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. Then the αt-Smarandache curve $\beta : I \subset \mathbb{R} \to H_0^2$ of α does not exist.*

Proof. Let $\alpha : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. Then the αt-Smarandache curve $\beta : I \subset \mathbb{R} \to H_0^2$ of α can be written as follows:

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(c_1\alpha(s) + c_2 t(s)\right), \quad (\text{A.1})$$

$c_1, c_2 \in \mathbb{R} \setminus \{0\}$, and $c_1^2 + c_2^2 = -2$ which is contradiction. □

The Smarandache curves of a regular unit spacelike curve α are given in Table 1.

Example A.2. Let us consider a unit speed spacelike curve α on S_1^2 defined by

$$\alpha(s) = \left(\frac{(s-1)^2}{2}, \frac{(s-1)^2}{2} - 1, s - 1\right). \quad (\text{A.2})$$

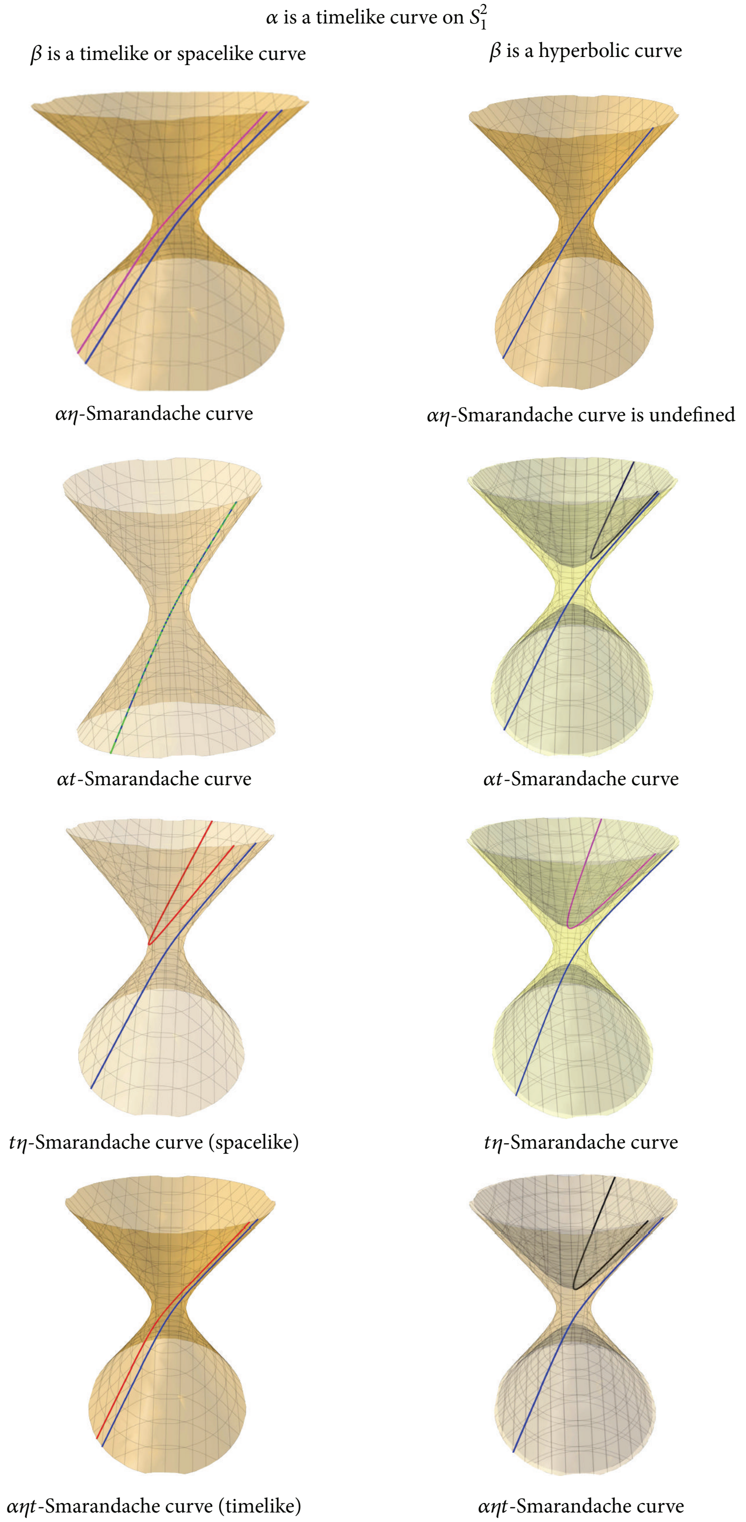

Figure 1: Smarandache curves of a timelike curve α.

Table 1: Classification of the Smarandache curves for spacelike curve α.

	α is a spacelike curve on S_1^2	
	β is a spacelike or timelike curve	β is a hyperbolic curve
$\alpha\eta$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1\alpha + c_2\eta)$, $c_1^2 - c_2^2 = 2$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1\alpha + c_2\eta)$, $c_1^2 - c_2^2 = -2$
αt	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1\alpha + c_2 t)$, $c_1^2 + c_2^2 = 2$	undefined
$t\eta$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1 t + c_2\eta)$, $c_1^2 - c_2^2 = 2$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1 t + c_2\eta)$, $c_1^2 - c_2^2 = -2$
$\alpha t\eta$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}(c_1\alpha + c_2 t + c_3\eta)$, $c_1^2 + c_2^2 - c_3^2 = 3$	$\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}(c_1\alpha + c_2 t + c_3\eta)$, $c_1^2 + c_2^2 - c_3^2 = -3$

Then the orthonormal Sabban frame $\{\alpha(s), t(s), \eta(s)\}$ of α can be calculated as follows:

$$\alpha(s) = \left(\frac{(s-1)^2}{2}, \frac{(s-1)^2}{2} - 1, s-1\right),$$
$$t(s) = (s-1, s-1, 1), \tag{A.3}$$
$$\eta(s) = \left(\frac{(s-1)^2}{2} + 1, \frac{(s-1)^2}{2}, s-1\right).$$

The geodesic curvature of α is expressed as

$$\kappa_g(s) = -1. \tag{A.4}$$

In terms of the definitions, we obtain Smarandache curves according to Sabban frame on S_1^2. Firstly, we take $c_1 = 2$ and $c_2 = \sqrt{2}$; then the spacelike $\alpha\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \left(\left(\frac{\sqrt{2}+1}{2}\right)(s-1)^2 + 1, \left(\frac{\sqrt{2}+1}{2}\right)(s-1)^2 - \sqrt{2}, (s-1)\left(\sqrt{2}+1\right)\right), \tag{A.5}$$

and also when we take $c_1 = \sqrt{2}$ and $c_2 = 2$, then the hyperbolic $\alpha\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \left(\left(\frac{\sqrt{2}+1}{2}\right)(s-1)^2 + \sqrt{2}, \left(\frac{\sqrt{2}+1}{2}\right)(s-1)^2 - 1, (s-1)\left(\sqrt{2}+1\right)\right). \tag{A.6}$$

The $\{\beta, t_\beta, \eta_\beta\}$ Sabban frames and geodesic curvatures κ_g^β are similar to the above section. Secondly, we take $c_1 = 1$ and $c_2 = 1$; then the spacelike Smarandache-αt curve is given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\frac{(s-1)^2}{2} + s - 1, \frac{(s-1)^2}{2} + s - 2, s\right). \tag{A.7}$$

Here the hyperbolic αt-Smarandache curve is undefined.

Thirdly, we take $c_1 = 2$ and $c_2 = \sqrt{2}$; then the spacelike $t\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}(s-1)^2 + 2s - 2 + \sqrt{2}, \frac{\sqrt{2}}{2}(s-1)^2 + 2s - 2, \sqrt{2}(s-1) + 2\right), \tag{A.8}$$

and also when we take $c_1 = \sqrt{2}$ and $c_2 = 2$, then the hyperbolic $t\eta$-Smarandache curve is given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left((s-1)^2 + \sqrt{2}(s-1) + 2, (s-1)^2 + \sqrt{2}(s-1), 2(s-1) + \sqrt{2}\right). \tag{A.9}$$

Finally, when $c_1 = 2$, $c_2 = \sqrt{2}$, and $c_3 = \sqrt{3}$, then the $\alpha t\eta$-Smarandache curve is a spacelike curve and is given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\left((s-1)^2\left(\frac{\sqrt{3}}{2}+1\right) + \sqrt{2}(s-1) + \sqrt{3}, (s-1)^2\left(\frac{\sqrt{3}}{2}+1\right) + \sqrt{2}(s-1) - 2, (s-1)\left(\sqrt{3}+2\right) + \sqrt{2}\right) \tag{A.10}$$

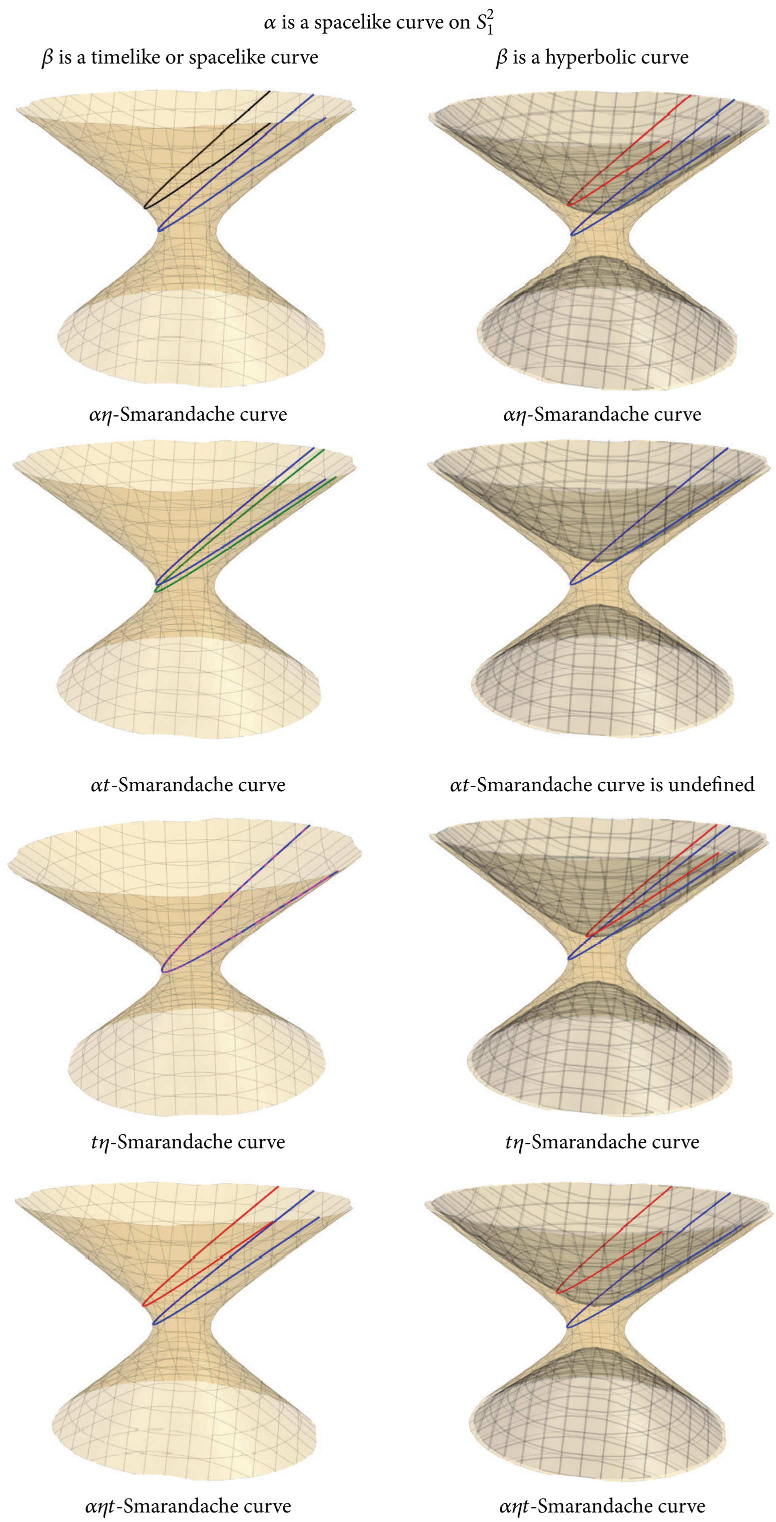

FIGURE 2: Smarandache curves of a spacelike curve α.

and also when we take $c_1 = 2$, $c_2 = \sqrt{2}$, and $c_3 = 3$, then the hyperbolic $\alpha t\eta$-Smarandache curve is given by

$$\begin{aligned}\beta(\bar{s}(s)) = \frac{1}{\sqrt{3}}\bigg(&\frac{5}{2}(s-1)^2 + \sqrt{2}(s-1) + 3,\\ &\frac{5}{2}(s-1)^2 + \sqrt{2}(s-1) - 3,\\ &5(s-1) + \sqrt{2}\bigg).\end{aligned} \tag{A.11}$$

The Sabban frames and geodesic curvatures of the $\alpha\eta$, αt, $t\eta$ and $\alpha t\eta$-Smarandache curves can be easily obtained by using methods similar to those in the previous section. Furthermore, we give curve the α and its Smarandache partners in Figure 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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