

Research Article

Multiple Periodic Solutions of a Nonautonomous Plant-Hare Model

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Based on Mawhin's coincidence degree theory, sufficient conditions are obtained for the existence of at least *two* positive periodic solutions for a plant-hare model with toxin-determined functional response (nonmonotone). Some new technique is used in this paper, because standard arguments in the literature are not applicable.

1. Introduction

In the past few decades, the classical predator-prey model has been well studied. Such classical predator-prey model has, however, been questioned by several biologists (e.g., see [1, 2]). Based on experimental data, Holling [3] has proposed several types of monotone functional responses $g(x) = c(t)x$, $c(t)x/(m+x)$, $c(t)x^2/(m+x^2)$, $c(t)x/(m+ax+x^2)$ for these and other models. However, this will not be appropriate if we explore the impact of plant toxicity on the dynamics of plant-hare interactions [4]. Recently, Gao and Xia [5] considered a nonautonomous plant-hare dynamical system with a toxin-determined functional response given by

$$\dot{N}(t) = r(t)N(t) \left[1 - \frac{N(t)}{K} \right] - C(N(t))P(t), \quad (1)$$

$$\dot{P}(t) = B(t)C(N(t))P(t) - d(t)P(t),$$

where

$$C(N(t)) = f(N(t)) \left[1 - \frac{f(N(t))}{4G} \right], \quad (2)$$

$$f(N(t)) = \frac{e\delta N(t)}{1 + h e \delta N(t)}.$$

Here, $N(t)$ denotes the density of plant at time t , $P(t)$ denotes the herbivore biomass at time t , $r(t)$ is the plant

intrinsic growth rate at time t , $d(t)$ is the per capita rate of herbivore death unrelated to plant toxicity at time t , $B(t)$ is the conversion rate at time t , e is the encounter rate per unit plant, δ is the fraction of food items encountered that the herbivore ingests, K is the carrying capacity of plant, G measures the toxicity level, and h is the time for handling one unit of plant. The functions $r(t)$, $d(t)$, and $B(t)$ are continuous, positive, and periodic with period ω , and e , δ , K , G , and h are positive real constants. For any continuous ω -periodic function F , we let

$$\bar{F} = \frac{1}{\omega} \int_0^\omega F(t) dt. \quad (3)$$

The topological degree of a mapping has long been known to be a useful tool for establishing the existence of fixed points of nonlinear mappings. In particular, a powerful tool to study the existence of periodic solution of nonlinear differential equations is the coincidence degree theory (see [6]). Many papers study the existence of periodic solutions of biological systems by employing the topological degree theory; see, for example, [7–12] and references cited therein. However, most of them investigated the classical predator-prey model or the models with Holling functional responses; see [7–10]. There is no paper studying the functional responses in model (1) except for [5]. Gao and Xia [5] have obtained some sufficient conditions for the existence of at least *one* positive periodic solution for the system (1). Unlike the traditional Holling Type II functional response, systems with

nonmonotone functional responses are capable of supporting multiple interior equilibria and bistable attractors. Thus, for nonautonomous system (1), it is possible to find two periodic solutions of (1). However, to date there is no work done on the existence of *multiple* periodic solutions of (1). Therefore, in this paper we will establish the existence of at least *two* positive periodic solutions of (1). We will be using the continuation theorem of Mawhin’s coincidence degree theory; to this end some *novel* estimation technique will be employed to obtain a priori bounds of unknown solutions to some operator equation, as the standard estimation techniques used in the literature are not applicable to the system (1) due to the term $C(N(t))$. We will elaborate this in Remark 3.

2. Existence of Multiple Positive Periodic Solutions

In this section, we will establish sufficient conditions for the existence of at least two positive periodic solutions of (1). We will first summarize in the following a few concepts and results from [6] that will be required later.

Let X, Y be normed vector spaces, $L : \text{Dom } L \subset X \rightarrow Y$ a linear mapping, and $N : X \rightarrow Y$ a continuous mapping. The mapping L is called a *Fredholm mapping of index zero* if $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$ and $\text{Im } L$ is closed in Y . If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$. It follows that $L|_{\text{dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the *inverse* of that map by K_p . If Ω is an open bounded subset of X , then the mapping N will be called *L-compact on $\bar{\Omega}$* if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an *isomorphism* $J : \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 1 (see [6]). *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N L-compact on $\bar{\Omega}$. Assume*

- (a) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom } L$, $Lx \neq \lambda Nx$;
- (b) for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$;
- (c) $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

To proceed, we note that (1) is equivalent to

$$\begin{aligned} \dot{N}(t) &= N(t) \left[r(t) \left(1 - \frac{N(t)}{K} \right) - \frac{4Ge\delta P(t) + (4Gh - 1)e^2\delta^2 N(t)P(t)}{4G(1 + he\delta N(t))^2} \right], \\ \dot{P}(t) &= P(t) \left[\frac{4Ge\delta B(t)N(t) + (4Gh - 1)e^2\delta^2 B(t)N^2(t)}{4G(1 + he\delta N(t))^2} - d(t) \right]. \end{aligned} \tag{4}$$

Throughout, we assume the following:

- (A₁) $1/4h < G < 1/3h$;
- (A₂) $4h\bar{d} \exp(2\bar{r}\omega) < \bar{B} < 4G\bar{d}h^2/(4Gh - 1)$.

We further introduce six positive numbers which will be used later as follows:

$$\begin{aligned} h_{\pm} &= \frac{(e\delta\bar{B} \exp(-2\bar{r}\omega) - 2he\delta\bar{d}) \pm \sqrt{\Delta_1}}{2\bar{d}h^2e^2\delta^2}, \\ l_{\pm} &= \frac{[4Gh^2e\delta\bar{B} \exp(2\bar{r}\omega) - 2he\delta(4Gh^2\bar{d} - (4Gh - 1)\bar{B})] \pm \sqrt{\Delta_2}}{2h^2e^2\delta^2[4Gh^2\bar{d} - (4Gh - 1)\bar{B}]}, \\ u_{\pm} &= \frac{(4Ge\delta\bar{B} - 8Ghe\delta\bar{d}) \pm \sqrt{\Delta_3}}{2[4G\bar{d}h^2e^2\delta^2 - (4Gh - 1)e^2\delta^2\bar{B}]}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Delta_1 &= [e\delta\bar{B} \exp(-2\bar{r}\omega) - 2he\delta\bar{d}]^2 - 4\bar{d}^2h^2e^2\delta^2, \\ \Delta_2 &= [4Gh^2e\delta\bar{B} \exp(2\bar{r}\omega) - 2he\delta(4Gh^2\bar{d} - (4Gh - 1)\bar{B})]^2 - 4h^2e^2\delta^2[4Gh^2\bar{d} - (4Gh - 1)\bar{B}]^2, \\ \Delta_3 &= (4Ge\delta\bar{B} - 8Ghe\delta\bar{d})^2 - 16G\bar{d}[4G\bar{d}h^2e^2\delta^2 - (4Gh - 1)e^2\delta^2\bar{B}]. \end{aligned} \tag{6}$$

Under assumptions (A₁) and (A₂), it is not difficult to show that

$$l_- < u_- < h_- < h_+ < u_+ < l_+. \tag{7}$$

Theorem 2. *In addition to (A₁) and (A₂), suppose that*

$$(A_3) \quad 1 - (1/K) \exp(\ln l_+ + 2\bar{r}\omega) > 0.$$

Then system (4) has at least two positive ω -periodic solutions.

Proof. Since we are concerned with positive solutions of system (4), we make use of the change of variables

$$N(t) = \exp(u_1(t)), \quad P(t) = \exp(u_2(t)). \tag{8}$$

Then, system (4) can be rewritten as

$$\begin{aligned} \dot{u}_1(t) &= r(t) - \frac{r(t)}{K} \exp(u_1(t)) - \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2}, \\ \dot{u}_2(t) &= -d(t) + \frac{4Ge\delta B(t) \exp(u_1(t)) + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2}. \end{aligned} \tag{9}$$

Take

$$X = Y = \left\{ x = (u_1, u_2)^T \in C(\mathbb{R}, \mathbb{R}^2) \mid x(t + \omega) = x(t) \right\} \tag{10}$$

and define

$$\begin{aligned} \|x\| &= \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)|, \\ x &= (u_1, u_2)^T \in X \text{ or } Y. \end{aligned} \tag{11}$$

Here $|\cdot|$ denotes the Euclidean norm. Then X and Y are Banach spaces with the norm $\|\cdot\|$. For any $x = (u_1, u_2)^T \in X$, by means of the periodicity assumption, we can easily check that

$$\begin{aligned} r(t) - \frac{r(t)}{K} \exp(u_1(t)) & \\ - \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} & \\ := f_1(t) \in C(\mathbb{R}, \mathbb{R}), & \\ -d(t) + \frac{4Ge\delta B(t) \exp(u_1(t)) + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2} & \\ := f_2(t) \in C(\mathbb{R}, \mathbb{R}) & \end{aligned} \tag{12}$$

are ω -periodic.

Set

$$L : \text{Dom } L \cap X, \quad L(u_1(t), u_2(t))^T = \left(\frac{du_1(t)}{dt}, \frac{du_2(t)}{dt} \right)^T, \tag{13}$$

where $\text{Dom } L = \{(u_1(t), u_2(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2)\}$. Further, $N : X \rightarrow X$ is defined by

$$N \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}. \tag{14}$$

Define

$$\begin{aligned} P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\omega} \int_0^\omega u_1(t) dt \\ \frac{1}{\omega} \int_0^\omega u_2(t) dt \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X = Y. \end{aligned} \tag{15}$$

It is not difficult to show that

$$\begin{aligned} \text{Ker } L &= \{x \mid x \in X, x = C_0, C_0 \in \mathbb{R}^2\}, \\ \text{Im } L &= \left\{ y \mid y \in Y, \int_0^\omega y(t) dt = 0 \right\} \text{ is closed in } Y, \end{aligned} \tag{16}$$

$$\dim \text{Ker } L = \text{codim } \text{Im } L = 2,$$

and P and Q are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im } (I - Q). \tag{17}$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \text{Im } L \rightarrow \text{Dom } L \cap \text{Ker } P$ exists and is given by

$$K_p(y) = \int_0^t y(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s) ds dt. \tag{18}$$

Then $QN : X \rightarrow Y$ and $K_p(I - Q)N : X \rightarrow X$ are, respectively, defined by

$$\begin{aligned} QNx &= \left(\frac{1}{\omega} \int_0^\omega f_1(t) dt, \frac{1}{\omega} \int_0^\omega f_2(t) dt \right)^T, \\ K_p(I - Q)Nx &= \int_0^t Nx(s) ds \\ &= \int_0^\omega \int_0^t Nx(s) ds dt - \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega Nx(s) ds. \end{aligned} \tag{19}$$

Clearly, QN and $K_p(I - Q)N$ are continuous. By using the Arzelà-Ascoli Theorem, it is not difficult to prove that $K_p(I - Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L -compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Now, we will search for two appropriate open bounded subsets in order to apply the continuation theorem.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} \dot{u}_1(t) &= \lambda \left[r(t) - \frac{r(t)}{K} \exp(u_1(t)) \right. \\ &\quad \left. - \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} \right], \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{u}_2(t) &= \lambda \left[-d(t) \right. \\ &\quad \left. + \frac{4Ge\delta B(t) \exp(u_1(t)) + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2} \right]. \end{aligned} \tag{21}$$

Suppose $x = (u_1(t), u_2(t))^T \in X$ is a solution of (20) and (21) for a certain $\lambda \in (0, 1)$. Integrating (20), (21) over the interval $[0, \omega]$, we obtain

$$\int_0^\omega \frac{r(t)}{K} \exp(u_1(t)) dt + \int_0^\omega \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt = \bar{r}\omega, \tag{22}$$

$$\int_0^\omega \frac{4Ge\delta B(t) \exp(u_1(t)) + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt = \bar{d}\omega. \tag{23}$$

It follows from (A_1) , (20), and (22) that

$$\int_0^\omega |\dot{u}_1(t)| dt = \lambda \int_0^\omega \left[r(t) - \frac{r(t)}{K} \exp(u_1(t)) - \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} \right] dt < \int_0^\omega r(t) dt + \int_0^\omega \frac{r(t)}{K} \exp(u_1(t)) dt + \int_0^\omega \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt = \int_0^\omega r(t) dt + \bar{r}\omega = 2\bar{r}\omega, \tag{24}$$

that is,

$$\int_0^\omega |\dot{u}_1(t)| dt < 2\bar{r}\omega. \tag{25}$$

Similarly, it follows from (A_1) , (21), and (23) that

$$\int_0^\omega |\dot{u}_2(t)| dt < 2\bar{d}\omega. \tag{26}$$

Since $(u_1(t), u_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \min_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \max_{t \in [0, \omega]} u_i(t), \quad i = 1, 2. \tag{27}$$

From (A_1) and (23), we see that

$$\bar{d}\omega \leq \int_0^\omega \frac{4Ge\delta B(t) \exp(u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt + \int_0^\omega \frac{(4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4Gh^2e^2\delta^2 \exp(2u_1(t))} dt, \tag{28}$$

which implies

$$\bar{d} \leq \frac{e\delta\bar{B} \exp(u_1(\eta_1))}{(1 + he\delta \exp(u_1(\xi_1)))^2} + \frac{(4Gh - 1)\bar{B}}{4Gh^2}. \tag{29}$$

So

$$u_1(\eta_1) \geq \ln \frac{[4Gh^2\bar{d} - (4Gh - 1)\bar{B}](1 + he\delta \exp(u_1(\xi_1)))^2}{4Gh^2e\delta\bar{B}}. \tag{30}$$

This, combined with (25), gives

$$u_1(t) \geq u_1(\eta_1) - \int_0^\omega |\dot{u}_1(t)| dt > \ln \frac{[4Gh^2\bar{d} - (4Gh - 1)\bar{B}](1 + he\delta \exp(u_1(\xi_1)))^2}{4Gh^2e\delta\bar{B}} - 2\bar{r}\omega. \tag{31}$$

In particular, we have

$$u_1(\xi_1) > \ln \frac{[4Gh^2\bar{d} - (4Gh - 1)\bar{B}](1 + he\delta \exp(u_1(\xi_1)))^2}{4Gh^2e\delta\bar{B}} - 2\bar{r}\omega, \tag{32}$$

or

$$[4Gh^2\bar{d} - (4Gh - 1)\bar{B}] h^2 e^2 \delta^2 \exp(2u_1(\xi_1)) - [4Gh^2 e\delta\bar{B} \exp(2\bar{r}\omega) - 2he\delta(4Gh^2\bar{d} - (4Gh - 1)\bar{B})] \exp(u_1(\xi_1)) + [4Gh^2\bar{d} - (4Gh - 1)\bar{B}] < 0. \tag{33}$$

In view of (A_2) , we have

$$\ln L_- < u_1(\xi_1) < \ln L_+. \tag{34}$$

Similarly, it follows from (A_1) and (23) that

$$\bar{d}\omega \geq \int_0^\omega \frac{4Ge\delta B(t) \exp(u_1(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt, \tag{35}$$

which implies

$$\bar{d} \geq \frac{e\delta\bar{B} \exp(u_1(\xi_1))}{(1 + he\delta \exp(u_1(\eta_1)))^2}. \tag{36}$$

So

$$u_1(\xi_1) \leq \ln \frac{\bar{d}(1 + he\delta \exp(u_1(\eta_1)))^2}{e\delta\bar{B}}. \tag{37}$$

This, combined with (25), gives

$$\begin{aligned}
 u_1(t) &\leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)| dt \\
 &< \ln \frac{\bar{d}(1 + he\delta \exp(u_1(\eta_1)))^2}{e\delta\bar{B}} + 2\bar{r}\omega.
 \end{aligned}
 \tag{38}$$

In particular, we have

$$u_1(\eta_1) < \ln \frac{\bar{d}(1 + he\delta \exp(u_1(\eta_1)))^2}{e\delta\bar{B}} + 2\bar{r}\omega, \tag{39}$$

or

$$\begin{aligned}
 &\bar{d}h^2 e^2 \delta^2 \exp(2u_1(\eta_1)) \\
 &- (e\delta\bar{B} \exp(-2\bar{r}\omega) - 2he\delta\bar{d}) \exp(u_1(\eta_1)) + \bar{d} > 0.
 \end{aligned}
 \tag{40}$$

It follows from (A₂) that

$$u_1(\eta_1) < \ln h_- \quad \text{or} \quad u_1(\eta_1) > \ln h_+. \tag{41}$$

From (25) and (34), we find

$$u_1(t) \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)| dt < \ln l_+ + 2\bar{r}\omega \triangleq H_{11}. \tag{42}$$

On the other hand, it follows from (A₁), (22), and (42) that

$$\bar{r}\omega \geq \int_0^\omega \frac{4Ge\delta \exp(u_2(\xi_2))}{4G(1 + he\delta \exp(\ln l_+ + 2\bar{r}\omega))^2} dt, \tag{43}$$

$$\begin{aligned}
 \bar{r}\omega &\leq \int_0^\omega \frac{r(t)}{K} \exp(\ln l_+ + 2\bar{r}\omega) dt \\
 &+ \int_0^\omega e\delta \exp(u_2(\eta_2)) dt \\
 &+ \int_0^\omega \frac{e\delta \exp(u_2(\eta_2))}{2} dt.
 \end{aligned}
 \tag{44}$$

It follows from (43) that

$$u_2(\xi_2) \leq \ln \frac{\bar{r}(1 + he\delta \exp(\ln l_+ + 2\bar{r}\omega))^2}{e\delta}. \tag{45}$$

This, combined with (26), gives

$$\begin{aligned}
 u_2(t) &\leq u_2(\xi_2) + \int_0^\omega |\dot{u}_2(t)| dt \\
 &< \ln \frac{\bar{r}(1 + he\delta \exp(\ln l_+ + 2\bar{r}\omega))^2}{e\delta} + 2\bar{d}\omega \triangleq H_{21}.
 \end{aligned}
 \tag{46}$$

Moreover, because of (A₃), it follows from (44) that

$$u_2(\eta_2) \geq \ln \frac{2\bar{r}(1 - (1/K) \exp(\ln l_+ + 2\bar{r}\omega))}{3e\delta}. \tag{47}$$

This, combined with (26) again, gives

$$\begin{aligned}
 u_2(t) &\geq u_2(\eta_2) - \int_0^\omega |\dot{u}_2(t)| dt \\
 &> \ln \frac{2\bar{r}(1 - (1/K) \exp(\ln l_+ + 2\bar{r}\omega))}{3e\delta} - 2\bar{d}\omega \triangleq H_{22}.
 \end{aligned}
 \tag{48}$$

It follows from (46) and (48) that

$$\max_{t \in [0, \omega]} u_2(t) < \max\{|H_{21}|, |H_{22}|\} \triangleq H_2. \tag{49}$$

Now, let us consider QN x with $x = (u_1, u_2)^T \in \mathbb{R}^2$. Note that

$$\begin{aligned}
 &QN(u_1, u_2)^T \\
 &= \left(\bar{r} - \frac{\bar{r}}{K} \exp(u_1) \right. \\
 &\quad \left. - \frac{4Ge\delta \exp(u_2) + (4Gh - 1) e^2 \delta^2 \exp(u_1 + u_2)}{4G(1 + he\delta \exp(u_1))^2}, \right. \\
 &\quad \left. -\bar{d} + \frac{4Ge\delta\bar{B} \exp(u_1) + (4Gh - 1) e^2 \delta^2 \bar{B} \exp(2u_1)}{4G(1 + he\delta \exp(u_1))^2} \right)^T.
 \end{aligned}
 \tag{50}$$

Noting (A₁), (A₂), and (A₃), we can show that the equation QN(u₁, u₂)^T = 0 has two distinct solutions:

$$\begin{aligned}
 \tilde{u} &= \left(\ln u_-, \ln \frac{4G(\bar{r} - (\bar{r}/K) u_-)(1 + he\delta u_-)^2}{4Ge\delta + (4Gh - 1) e^2 \delta^2 u_-} \right), \\
 \hat{u} &= \left(\ln u_+, \ln \frac{4G(\bar{r} - (\bar{r}/K) u_+)(1 + he\delta u_+)^2}{4Ge\delta + (4Gh - 1) e^2 \delta^2 u_+} \right).
 \end{aligned}
 \tag{51}$$

Choose $C > 0$ such that

$$\begin{aligned}
 C > \max \left\{ \left| \ln \frac{4G(\bar{r} - (\bar{r}/K) u_-)(1 + he\delta u_-)^2}{4Ge\delta + (4Gh - 1) e^2 \delta^2 u_-} \right|, \right. \\
 \left. \left| \ln \frac{4G(\bar{r} - (\bar{r}/K) u_+)(1 + he\delta u_+)^2}{4Ge\delta + (4Gh - 1) e^2 \delta^2 u_+} \right| \right\}.
 \end{aligned}
 \tag{52}$$

We are now ready to define two open bounded subsets in order to apply the continuation theorem. Let

$$\Omega_1 = \left\{ x = (u_1, u_2)^T \in X \mid u_1(t) \in (\ln l_-, \ln h_-), \right.$$

$$\left. \max_{t \in [0, \omega]} |u_2(t)| < H_2 + C \right\},$$

$$\Omega_2 = \left\{ x = (u_1, u_2)^T \in X \mid \min_{t \in [0, \omega]} u_1(t) \in (\ln l_-, \ln l_+), \right.$$

$$\left. \max_{t \in [0, \omega]} u_1(t) \in (\ln h_+, H_{11}), \max_{t \in [0, \omega]} |u_2(t)| < H_2 + C \right\}. \tag{53}$$

Then both Ω_1 and Ω_2 are bounded open subsets of X . It follows from (4) and (52) that $\bar{u} \in \Omega_1$ and $\hat{u} \in \Omega_2$. With the help of (4), (34), (41), (42), (49), and (52), it is easy to see that $\Omega_1 \cap \Omega_2 = \emptyset$ and Ω_i satisfies the requirement (a) in Lemma 1 for $i = 1, 2$. Moreover, $QNx \neq 0$ for $x \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^2$. A direct computation gives $\text{deg}\{JQN, \Omega_i \cap \text{Ker } L, 0\} \neq 0$. Here, J is taken as the identity mapping since $\text{Im } Q = \text{Ker } L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 1. Hence, (4) has at least two ω -periodic solutions. This completes the proof of Theorem 2. \square

Remark 3. In the proof of Theorem 2, we have employed some new technique to obtain a priori bounds for u_1 . Here, the standard arguments in the literature (see, e.g., [7–12]) do not work. Indeed, from (23) in the proof it follows that

$$\bar{d}\omega \leq \frac{4Ge\delta\bar{B}\omega \exp(u_1(\eta_1)) + 4Ghe^2\delta^2\bar{B}\omega \exp(2u_1(\eta_1))}{4G(1 + he\delta \exp(u_1(\xi_1)))^2}. \tag{54}$$

If we were to use the standard arguments in the literature, then we have

$$\begin{aligned} & \left[4\bar{d}h^3 e^2\delta^2\bar{B} - 4h^2 e^2\delta^2\bar{B}^2 \exp(4\bar{r}\omega) \right] \exp(2u_1(\xi_1)) \\ & + \left[8\bar{d}h^2 e\delta\bar{B} - 4he\delta\bar{B}^2 \exp(2\bar{r}\omega) \right] \exp(u_1(\xi_1)) \tag{55} \\ & + 4\bar{d}h\bar{B} < 0, \end{aligned}$$

where $u_1(\xi_1) = \min_{t \in [0, \omega]} u_1(t)$ and $u_1(\eta_1) = \max_{t \in [0, \omega]} u_1(t)$. It follows from (55) that

$$\bar{l}_- < \exp(u_1(\xi_1)) < \bar{l}_+, \tag{56}$$

where \bar{l}_- and \bar{l}_+ are the roots of the following equation in x :

$$\begin{aligned} & \left[4\bar{d}h^3 e^2\delta^2\bar{B} - 4h^2 e^2\delta^2\bar{B}^2 \exp(4\bar{r}\omega) \right] x^2 \\ & + \left[8\bar{d}h^2 e\delta\bar{B} - 4he\delta\bar{B}^2 \exp(2\bar{r}\omega) \right] x \tag{57} \\ & + 4\bar{d}h\bar{B} = 0. \end{aligned}$$

We claim that (57) has at least a negative root; that is, at least one of \bar{l}_-, \bar{l}_+ is negative. Otherwise, if both \bar{l}_- and \bar{l}_+ are positive, then from (57) we see that

$$\bar{l}_+ \cdot \bar{l}_- = \frac{4\bar{d}h\bar{B}}{4\bar{d}h^3 e^2\delta^2\bar{B} - 4h^2 e^2\delta^2\bar{B}^2 \exp(4\bar{r}\omega)} > 0, \tag{58}$$

which implies

$$h\bar{d} > \bar{B} \exp(4\bar{r}\omega). \tag{59}$$

On the other hand, it follows from (57) and (58) that

$$\bar{l}_+ + \bar{l}_- = -\frac{8\bar{d}h^2 e\delta\bar{B} - 4he\delta\bar{B}^2 \exp(2\bar{r}\omega)}{4\bar{d}h^3 e^2\delta^2\bar{B} - 4h^2 e^2\delta^2\bar{B}^2 \exp(4\bar{r}\omega)} < 0, \tag{60}$$

which contradicts the positivity of \bar{l}_- and \bar{l}_+ . Therefore, at least one of \bar{l}_-, \bar{l}_+ is negative. However, to use the standard arguments in the literature we need both \bar{l}_- and \bar{l}_+ to be positive. Hence, we have illustrated that standard arguments in the literature are not applicable to the system (4) and some new technique should be used. To see how this problem is handled, the reader may refer to (27)–(34) in the proof of Theorem 2.

Conflict of Interest

No conflict of interests exists in the submission of this paper, and the paper is approved by all authors for publication. The authors would like to declare that the work described was original research that has not been published previously and not under consideration for publication elsewhere.

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