Research Article

Some New Oscillation Criteria for a Class of Nonlinear Fractional Differential Equations with Damping Term

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We are concerned with oscillation of solutions of a class of nonlinear fractional differential equations with damping term. Based on a generalized Riccati function and inequality technique, we establish some new oscillation criteria for it. Some applications are also presented for the established results.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, and one can find their applications in many fields of science and engineering. In the last few decades, research for various aspects of fractional differential equations, for example, the existence, uniqueness, and stability of solutions of fractional differential equations, the numerical methods for fractional differential equations, and so on, has been paid much attention by many authors (e.g., we refer the reader to see [1–8] and the references therein). In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations. Recent results in this direction include Chen's work [9], in which some new oscillation criteria are established for the following fractional differential equation:

$$\left[r(t)(D_-^\alpha x(t))^\eta \right]' - q(t) f\left(\int_t^\infty (v-t)^{-\alpha} y(v)\, dv \right) = 0, \quad t > 0, \tag{1}$$

where r and q are positive functions and η is a quotient of two odd positive numbers.

In this paper, we are concerned with oscillation of solutions of the following nonlinear $(2+\alpha)$-order fractional differential equation with damping term:

$$\left(a(t) \left[\left(r(t) D_-^\alpha x(t) \right)' \right]^\gamma \right)' + p(t) \left[\left(r(t) D_-^\alpha x(t) \right)' \right]^\gamma - q(t) f\left(\int_t^\infty (\xi - t)^{-\alpha} x(\xi)\, d\xi \right) = 0, \quad t \in [t_0, \infty), \tag{2}$$

where $a \in C^1([t_0, \infty), \mathbb{R}_+)$, $r \in C^2([t_0, \infty), \mathbb{R}_+)$, $p, q \in C([t_0, \infty), \mathbb{R}_+)$, $f \in C(\mathbb{R}, \mathbb{R})$ satisfying $xf(x) > 0$, $f(x)/x^\gamma \geq L > 0$ for $x \neq 0$, γ is a quotient of two odd positive integers, $\alpha \in (0, 1)$, $D_-^\alpha x(t)$ denotes the Liouville right-sided fractional derivative of order α of x, and $D_-^\alpha x(t) = -(1/\Gamma(1 - \alpha))(d/dt) \int_t^\infty (\xi - t)^{-\alpha} x(\xi) d\xi$.

A nontrivial solution of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (2) is said to be oscillatory in case all its solutions are oscillatory.

Motivated by the idea in [10], we will establish some new oscillation criteria for (2) by a generalized Riccati function and inequality technique in Section 2, and we will present some applications for our results in Section 3. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0, \infty)$. For more details about the theory of fractional

differential equations, we refer the reader to [11–13]. For the sake of convenience, in the rest of this paper, we set $X(t) = \int_t^\infty (\xi - t)^{-\alpha} x(\xi) d\xi$, $A(t) = \int_{t_0}^t (p(s)/a(s)) ds$,

$$\theta_1(t, a) = \int_a^t \frac{1}{\left[e^{A(s)} a(s)\right]^{1/\gamma}} ds, \qquad \theta_2(t, a) = \int_a^t \frac{\theta_1(s, a)}{r(s)} ds. \tag{3}$$

2. Main Results

The following lemmas are useful for proving our results.

Lemma 1. *Assume that x is a solution of (2). Then, $X'(t) = -\Gamma(1-\alpha) D_-^\alpha x(t)$.*

Lemma 2. *Assume that x is an eventually positive solution of (2) and*

$$\int_{t_0}^\infty \frac{1}{\left[e^{A(s)} a(s)\right]^{1/\gamma}} ds = \infty, \tag{4}$$

$$\int_{t_0}^\infty \frac{1}{r(s)} ds = \infty, \tag{5}$$

$$\int_{t_0}^\infty \frac{1}{r(\xi)} \int_\xi^\infty \left[\frac{1}{e^{A(\tau)} a(\tau)} \int_\tau^\infty e^{A(s)} q(s) ds\right]^{1/\gamma} d\tau \, d\xi = \infty. \tag{6}$$

Then, there exists a sufficiently large T such that

$$\left(r(t) D_-^\alpha x(t)\right)' < 0 \quad on \ [T, \infty) \tag{7}$$

and either $D_-^\alpha x(t) < 0$ on $[T, \infty)$ or $\lim_{t \to \infty} X(t) = 0$.

Proof. Since x is an eventually positive solution of (2), there exists t_1 such that $x(t) > 0$ on $[t_1, \infty)$. So $X(t) > 0$ on $[t_1, \infty)$ and we have

$$\begin{aligned}
&\left(e^{A(t)} a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma\right)' \\
&= e^{A(t)} \left(a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma\right)' \\
&\quad + e^{A(t)} p(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma \\
&= e^{A(t)} \left\{\left(a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma\right)' \right. \\
&\quad \left. + p(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma\right\} \\
&= e^{A(t)} q(t) f(X(t)) \geq L e^{A(t)} q(t) X^\gamma(t) > 0.
\end{aligned} \tag{8}$$

Then, $e^{A(t)} a(t) [(r(t) D_-^\alpha x(t))']^\gamma$ is strictly increasing on $[t_1, \infty)$, and thus $(r(t) D_-^\alpha x(t))'$ is eventually of one sign. We claim that $(r(t) D_-^\alpha x(t))' < 0$ on $[t_2, \infty)$, where $t_2 > t_1$ is sufficiently large. Otherwise, assume there exists a sufficiently

large $t_3 > t_2$ such that $(r(t) D_-^\alpha x(t))' > 0$ on $[t_3, \infty)$. Then, for $t \in [t_3, \infty)$, we have

$$\begin{aligned}
&r(t) D_-^\alpha x(t) - r(t_3) D_-^\alpha x(t_3) \\
&= \int_{t_3}^t \frac{\left[e^{A(s)} a(s)\right]^{1/\gamma} \left(r(s) D_-^\alpha x(s)\right)'}{\left[e^{A(s)} a(s)\right]^{1/\gamma}} ds \\
&\geq \left[e^{A(t_3)} a(t_3)\right]^{1/\gamma} \left(r(t_3) D_-^\alpha x(t_3)\right)' \\
&\quad \times \int_{t_3}^t \frac{1}{\left[e^{A(s)} a(s)\right]^{1/\gamma}} ds.
\end{aligned} \tag{9}$$

By (4), we have

$$\lim_{t \to \infty} r(t) D_-^\alpha x(t) = \infty, \tag{10}$$

which implies, for some sufficiently large $t_4 > t_3$, $D_-^\alpha x(t) > 0$, $t \in [t_4, \infty)$. By Lemma 1, we have

$$\begin{aligned}
X(t) - X(t_4) &= \int_{t_4}^t X'(s) ds \\
&= -\Gamma(1-\alpha) \int_{t_4}^t D_-^\alpha x(s) ds \\
&= -\Gamma(1-\alpha) \int_{t_4}^t \frac{r(s) D_-^\alpha x(s)}{r(s)} ds \\
&\leq -\Gamma(1-\alpha) r(t_4) D_-^\alpha x(t_4) \int_{t_4}^t \frac{1}{r(s)} ds.
\end{aligned} \tag{11}$$

By (5), we obtain $\lim_{t \to \infty} X(t) = -\infty$, which contradicts $X(t) > 0$ on $[t_1, \infty)$. So $(r(t) D_-^\alpha x(t))' < 0$ on $[t_2, \infty)$. Thus, $D_-^\alpha x(t)$ is eventually of one sign. Now, we assume that $D_-^\alpha x(t) > 0$, $t \in [t_5, \infty)$, for some sufficiently $t_5 > t_4$. Then, by Lemma 1, $X'(t) < 0$ for $t \in [t_5, \infty)$. Since $X(t) > 0$, furthermore we have $\lim_{t \to \infty} X(t) = \beta \geq 0$. We claim that $\beta = 0$. Otherwise, assume that $\beta > 0$. Then, $X(t) \geq \beta$ on $[t_5, \infty)$, and, for $t \in [t_5, \infty)$, by (8) we have

$$\begin{aligned}
\left(e^{A(t)} a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma\right)' &\geq L e^{A(t)} q(t) X^\gamma(t) \\
&\geq L \beta^\gamma e^{A(t)} q(t).
\end{aligned} \tag{12}$$

Substituting t with s in (12), an integration for (12) with respect to s from t to ∞ yields

$$\begin{aligned}
&-e^{A(t)} a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma \\
&\geq -\lim_{t \to \infty} e^{A(t)} a(t) \left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\gamma \\
&\quad + L \beta^\gamma \int_t^\infty e^{A(s)} q(s) ds \\
&> L \beta^\gamma \int_t^\infty e^{A(s)} q(s) ds;
\end{aligned} \tag{13}$$

which means

$$\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)' < -L^{1/\gamma}\beta\left[\frac{1}{e^{A(t)}a\left(t\right)}\int_t^\infty e^{A(s)}q\left(s\right)ds\right]^{1/\gamma}.$$

$$(14)$$

Substituting t with τ in (14), an integration for (14) with respect to τ from t to ∞ yields

$$-r\left(t\right)D_-^\alpha x\left(t\right)$$

$$< -\lim_{t\to\infty}r\left(t\right)D_-^\alpha x\left(t\right)$$

$$-L^{1/\gamma}\beta\int_t^\infty\left[\frac{1}{e^{A(\tau)}a\left(\tau\right)}\int_\tau^\infty e^{A(s)}q\left(s\right)ds\right]^{1/\gamma}d\tau \quad (15)$$

$$< -L^{1/\gamma}\beta\int_t^\infty\left[\frac{1}{e^{A(\tau)}a\left(\tau\right)}\int_\tau^\infty e^{A(s)}q\left(s\right)ds\right]^{1/\gamma}d\tau;$$

that is,

$$X'\left(t\right) < -L^{1/\gamma}\Gamma\left(1-\alpha\right)\beta\frac{1}{r\left(t\right)}$$

$$\times\int_t^\infty\left[\frac{1}{e^{A(\tau)}a\left(\tau\right)}\int_\tau^\infty e^{A(s)}q\left(s\right)ds\right]^{1/\gamma}d\tau.$$

$$(16)$$

Substituting t with ξ in (16), an integration for (16) with respect to ξ from t_5 to t yields

$$X\left(t\right) - X\left(t_5\right)$$

$$< -L^{1/\gamma}\Gamma\left(1-\alpha\right)\beta$$

$$\times\int_{t_5}^t\frac{1}{r\left(\xi\right)}\int_\xi^\infty\left[\frac{1}{e^{A(\tau)}a\left(\tau\right)}\int_\tau^\infty e^{A(s)}q\left(s\right)ds\right]^{1/\gamma}d\tau\,d\xi.$$

$$(17)$$

By (6), one can see that $\lim_{t\to\infty}X(t) = -\infty$, which is a contradiction. So the proof is complete. □

Lemma 3. *Assume that x is an eventually positive solution of (2) such that*

$$\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)' < 0, \quad D_-^\alpha x\left(t\right) < 0 \quad on \left[t_1,\infty\right)_\mathbb{T}, \quad (18)$$

where $t_1 \geq t_0$ is sufficiently large. Then, one has

$$X'\left(t\right) \geq -\frac{\Gamma\left(1-\alpha\right)\theta_1\left(t,t_1\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'}{r\left(t\right)},$$

$$X\left(t\right) \geq -\Gamma\left(1-\alpha\right)\theta_2\left(t,t_1\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'.$$

$$(19)$$

Proof. By Lemma 2, we obtain that $e^{A(t)}a(t)[(r(t)D_-^\alpha x(t))']^\gamma$ is strictly increasing on $[t_1,\infty)$. So

$$r\left(t\right)D_-^\alpha x\left(t\right) \leq r\left(t\right)D_-^\alpha x\left(t\right) - r\left(t_1\right)D_-^\alpha x\left(t_1\right)$$

$$= \int_{t_1}^t\frac{\left[e^{A(s)}a\left(s\right)\right]^{1/\gamma}\left[r\left(s\right)D_-^\alpha x\left(s\right)\right]'}{\left[e^{A(s)}a\left(s\right)\right]^{1/\gamma}}ds$$

$$\leq \left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'\int_{t_1}^t\frac{1}{\left[e^{A(s)}a\left(s\right)\right]^{1/\gamma}}ds$$

$$= \theta_1\left(t,t_1\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'.$$

$$(20)$$

Using Lemma 1, we obtain that

$$X'\left(t\right) \geq -\frac{\Gamma\left(1-\alpha\right)\theta_1\left(t,t_1\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'}{r\left(t\right)}.$$

$$(21)$$

Then,

$$X\left(t\right)$$

$$\geq X\left(t\right) - X\left(t_1\right)$$

$$\geq -\int_{t_1}^t\frac{\Gamma\left(1-\alpha\right)\theta_1\left(s,t_1\right)\left[e^{A(s)}a\left(s\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(s\right)\right)'}{r\left(s\right)}ds$$

$$\geq -\Gamma\left(1-\alpha\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'\int_{t_1}^t\frac{\theta_1\left(s,t_1\right)}{r\left(s\right)}ds$$

$$= -\Gamma\left(1-\alpha\right)\theta_2\left(t,t_1\right)\left[e^{A(t)}a\left(t\right)\right]^{1/\gamma}\left(r\left(t\right)D_-^\alpha x\left(t\right)\right)'.$$

$$(22)$$

□

Lemma 4 ([14, Theorem 41]). *Assume that A and B are nonnegative real numbers. Then,*

$$\lambda AB^{\lambda-1} - A^\lambda \leq \left(\lambda - 1\right)B^\lambda, \quad \forall\lambda > 1. \quad (23)$$

Theorem 5. *Assume (4)–(6) hold and there exist two functions $\rho \in C^1([t_0,\infty),\mathbb{R}_+)$ and $\eta \in C^1([t_0,\infty),[0,\infty))$ such that*

$$\int_T^\infty\left\{L\rho\left(s\right)q\left(s\right)e^{A(s)} - \rho\left(s\right)\eta'\left(s\right)\right.$$

$$+ \frac{\rho\left(s\right)\Gamma\left(1-\alpha\right)\theta_1\left(s,T\right)\eta^{1+(1/\gamma)}\left(s\right)}{r\left(s\right)}$$

$$- \left(\left[\left(\gamma+1\right)\eta^{1/\gamma}\left(s\right)\rho\left(s\right)\Gamma\left(1-\alpha\right)\theta_1\left(s,T\right)\right.\right.$$

$$\left.+ r\left(s\right)\rho'\left(s\right)\right]^{\gamma+1}\right)$$

$$\left.\times\left(\left(\gamma+1\right)^{\gamma+1}\left[\Gamma\left(1-\alpha\right)\rho\left(s\right)\theta_1\left(s,T\right)\right]^\gamma r\left(s\right)\right)^{-1}\right\}ds$$

$$= \infty,$$

$$(24)$$

for all sufficiently large T. Then, every solution of (2) is oscillatory or satisfies $\lim_{t\to\infty}X(t) = 0$.

Proof. Assume (2) has a nonoscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 2, we have $(r(t)D^\alpha_- x(t))' < 0, t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $D^\alpha_- x(t) < 0$ on $[t_2, \infty)$ or $\lim_{t \to \infty} X(t) = 0$. Define the generalized Riccati function as follows:

$$\omega(t) = \rho(t) \left\{ -\frac{e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma}{X^\gamma(t)} + \eta(t) \right\}. \quad (25)$$

Then, for $t \in [t_2, \infty)$, we have

$$\omega'(t)$$

$$= -\rho'(t) \frac{e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma}{X^\gamma(t)}$$

$$+ \rho(t) \left\{ -\frac{e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma}{X^\gamma(t)} \right\}'$$

$$+ \rho'(t) \eta(t) + \rho(t) \eta'(t)$$

$$= -\rho(t) \left\{ \left(X^\gamma(t) \left(e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma \right)' \right. \right.$$

$$\left. - \gamma X^{\gamma-1}(t) X'(t) e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma \right)$$

$$\times \left(X^{2\gamma}(t) \right)^{-1} \right\} + \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t)$$

$$= -\rho(t) \left\{ \left(X^\gamma(t) \left\{ e^{A(t)} \left(a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma \right)' \right. \right. \right.$$

$$\left. \left. + e^{A(t)} p(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma \right\} \right)$$

$$\times \left(X^{2\gamma}(t) \right)^{-1}$$

$$\left. - \frac{\gamma X^{\gamma-1}(t) X'(t) e^{A(t)} a(t) \left[(r(t) D^\alpha_- x(t))'\right]^\gamma}{X^{2\gamma}(t)} \right\}$$

$$+ \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t)$$

$$= -\frac{\rho(t) q(t) e^{A(t)} f(X(t))}{X^\gamma(t)}$$

$$+ \frac{\gamma \rho(t) X'(t) a(t) e^{A(t)} \left[(r(t) D^\alpha_- x(t))'\right]^\gamma}{X^{\gamma+1}(t)}$$

$$+ \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t). \quad (26)$$

By Lemma 3 and the definition of f, we get that

$$\omega'(t) \le - L\rho(t) q(t) e^{A(t)}$$

$$- \left(\gamma \rho(t) \Gamma(1 - \alpha) \theta_1(t, t_2) \left[e^{A(t)} a(t)\right]^{1/\gamma} \right.$$

$$\times (r(t) D^\alpha_- x(t))' a(t) e^{A(t)} \left[(r(t) D^\alpha_- x(t))'\right]^\gamma \right)$$

$$\times \left(r(t) X^{\gamma+1}(t) \right)^{-1}$$

$$+ \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t)$$

$$= - L\rho(t) q(t) e^{A(t)} - \frac{\gamma \rho(t) \Gamma(1 - \alpha) \theta_1(t, t_2)}{r(t)}$$

$$\times \left[-\left(\frac{\omega(t)}{\rho(t)} - \eta(t)\right) \right]^{1+(1/\gamma)} + \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t)$$

$$= - L\rho(t) q(t) e^{A(t)} - \frac{\gamma \rho(t) \Gamma(1 - \alpha) \theta_1(t, t_2)}{r(t)}$$

$$\times \left[\frac{\omega(t)}{\rho(t)} - \eta(t) \right]^{1+(1/\gamma)} + \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t). \quad (27)$$

Using the following inequality (see [15, Equation (2.17)])

$$(u - v)^{1+(1/\gamma)} \ge u^{1+(1/\gamma)} + \frac{1}{\gamma} v^{1+(1/\gamma)} - \left(1 + \frac{1}{\gamma}\right) v^{1/\gamma} u, \quad (28)$$

we obtain

$$\left[\frac{\omega(t)}{\rho(t)} - \eta(t) \right]^{1+(1/\gamma)} \ge \frac{\omega^{1+(1/\gamma)}(t)}{\rho^{1+(1/\gamma)}(t)} + \frac{1}{\gamma} \eta^{1+(1/\gamma)}(t)$$

$$- \left(1 + \frac{1}{\gamma}\right) \frac{\eta^{1/\gamma}(t) \omega(t)}{\rho(t)}. \quad (29)$$

A combination of (27) and (29) yields the following:

$$\omega'(t)$$

$$\le -L\rho(t) q(t) e^{A(t)} + \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \eta'(t)$$

$$- \frac{\gamma \rho(t) \Gamma(1 - \alpha) \theta_1(t, t_2)}{r(t)}$$

$$\times \left[\frac{\omega^{1+(1/\gamma)}(t)}{\rho^{1+(1/\gamma)}(t)} + \frac{1}{\gamma} \eta^{1+(1/\gamma)}(t) - \left(1 + \frac{1}{\gamma}\right) \frac{\eta^{1/\gamma}(t) \omega(t)}{\rho(t)} \right]$$

$$= -L\rho(t)\,q(t)\,e^{A(t)} + \rho(t)\,\eta'(t)$$

$$- \frac{\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)\,\eta^{1+(1/\gamma)}(t)}{r(t)}$$

$$- \frac{\gamma\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)}{r(t)}\frac{\omega^{1+(1/\gamma)}(t)}{\rho^{1+(1/\gamma)}(t)}$$

$$+ \frac{(\gamma+1)\,\eta^{1/\gamma}(t)\,\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)+r(t)\,\rho'(t)}{r(t)\,\rho(t)}\,\omega(t). \tag{30}$$

Setting

$$\lambda = 1 + \frac{1}{\gamma},$$

$$A^\lambda = \frac{\gamma\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)}{r(t)}\frac{\omega^{1+(1/\gamma)}(t)}{\rho^{1+(1/\gamma)}(t)},$$

$$B^{\lambda-1}$$

$$= \gamma^{1/(\gamma+1)}$$

$$\times \frac{(\gamma+1)\,\eta^{1/\gamma}(t)\,\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)+r(t)\,\rho'(t)}{(\gamma+1)\left[\Gamma(1-\alpha)\,\rho(t)\,\theta_1(t,t_2)\right]^{\gamma/(\gamma+1)}r^{1/(\gamma+1)}(t)}. \tag{31}$$

Using Lemma 4 in (30), we get that

$$\omega'(t)$$

$$\le -L\rho(t)\,q(t)\,e^{A(t)} + \rho(t)\,\eta'(t)$$

$$- \frac{\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)\,\eta^{1+(1/\gamma)}(t)}{r(t)}$$

$$+ \frac{\left[(\gamma+1)\,\eta^{1/\gamma}(t)\,\rho(t)\,\Gamma(1-\alpha)\,\theta_1(t,t_2)+r(t)\,\rho'(t)\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\Gamma(1-\alpha)\,\rho(t)\,\theta_1(t,t_2)\right]^{\gamma}r(t)}. \tag{32}$$

Substituting t with s in (32), an integration for (32) with respect to s from t_2 to t yields

$$\int_{t_2}^{t}\left\{L\rho(s)\,q(s)\,e^{A(s)} - \rho(s)\,\eta'(s)\right.$$

$$+ \frac{\rho(s)\,\Gamma(1-\alpha)\,\theta_1(s,t_2)\,\eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1)\,\eta^{1/\gamma}(s)\,\rho(s)\,\Gamma(1-\alpha)\,\theta_1(s,t_2)\right.\right.$$

$$\left.+r(s)\,\rho'(s)\right]^{\gamma+1}\Big)$$

$$\left.\times\left((\gamma+1)^{\gamma+1}\left[\Gamma(1-\alpha)\,\rho(s)\,\theta_1(s,t_2)\right]^{\gamma}r(s)\right)^{-1}\right\}ds$$

$$\le \omega(t_2) - \omega(t) \le \omega(t_2), \tag{33}$$

which contradicts (24). So the proof is complete. □

Theorem 6. *Assume* (4)–(6) *hold and, for all sufficiently large* T,

$$\int_{T}^{\infty}\left\{L\rho(s)\,q(s)\,e^{A(s)} - \rho(s)\,\eta'(s)\right.$$

$$+ \frac{\gamma\rho(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T)\,\eta^2(s)}{r(s)}$$

$$- \Big(\left\{2\gamma\rho(s)\,\eta(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T)\right.$$

$$\left.+r(s)\,\rho'(s)\right\}^2\Big)$$

$$\left.\times\left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T)\,r(s)\,\rho(s)\right)^{-1}\right\}ds$$

$$= \infty, \tag{34}$$

where ρ *and* η *are defined as in Theorem 5. Then, every solution of* (2) *is oscillatory or satisfies* $\lim_{t\to\infty}X(t) = 0$.

Proof. Assume (2) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2, we have $(r(t)D_-^\alpha x(t))' < 0, t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_-^\alpha x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty}X(t) = 0$. Let $\omega(t)$ be defined as in Theorem 5. Proceeding as in Theorem 5, we obtain (26). By Lemma 3, we have the following observation:

$$\frac{X'(t)}{X(t)}$$

$$\ge -\frac{\Gamma(1-\alpha)\,\theta_1(t,t_2)\left[e^{A(t)}a(t)\right]^{1/\gamma}(r(t)\,D_-^\alpha x(t))'}{r(t)\,X(t)}$$

$$= -\frac{\Gamma(1-\alpha)\,\theta_1(t,t_2)\left[e^{A(t)}a(t)\right]^{1/\gamma}(r(t)\,D_-^\alpha x(t))'}{r(t)\,X^\gamma(t)}$$

$$\times X^{\gamma-1}(t)$$

$$\ge -\frac{\Gamma(1-\alpha)\,\theta_1(t,t_2)\left[e^{A(t)}a(t)\right]^{1/\gamma}(r(t)\,D_-^\alpha x(t))'}{r(t)\,X^\gamma(t)}$$

$$\times \left\{-\Gamma(1-\alpha)\,\theta_2(t,t_2)\left[e^{A(t)}a(t)\right]^{1/\gamma}(r(t)\,D_-^\alpha x(t))'\right\}^{\gamma-1}$$

$$= -\frac{[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\,\theta_2^{\gamma-1}(t,t_2)}{r(t)}$$

$$\times \left\{\frac{e^{A(t)}a(t)\left[(r(t)\,D_-^\alpha x(t))'\right]^{\gamma}}{X^\gamma(t)}\right\}. \tag{35}$$

Using (35) in (26) we get that

$\omega'(t)$

$$\leq -L\rho(t)q(t)e^{A(t)} - \frac{\gamma\rho(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)}{r(t)}$$

$$\times \left\{ \frac{e^{A(t)}a(t)\left[(r(t)D_-^{\alpha}x(t))'\right]^{\gamma}}{X^{\gamma}(t)} \right\}^2$$

$$+ \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\eta'(t)$$

$$= -L\rho(t)q(t)e^{A(t)} - \frac{\gamma\rho(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)}{r(t)}$$

$$\times \left[\frac{\omega(t)}{\rho(t)} - \eta(t) \right]^2 + \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\eta'(t)$$

$$= -L\rho(t)q(t)e^{A(t)}$$

$$- \frac{\gamma\rho(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)\eta^2(t)}{r(t)}$$

$$- \frac{\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)}{r(t)\rho(t)}\omega^2(t)$$

$$+ \frac{2\gamma\rho(t)\eta(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)+r(t)\rho'(t)}{r(t)\rho(t)}$$

$$\times \omega(t) + \rho(t)\eta'(t)$$

$$\leq -L\rho(t)q(t)e^{A(t)} + \rho(t)\eta'(t)$$

$$- \frac{\gamma\rho(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)\eta^2(t)}{r(t)}$$

$$+ \left(\left\{ 2\gamma\rho(t)\eta(t)[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2) \right.\right.$$

$$\left.\left. +r(t)\rho'(t) \right\}^2 \right)$$

$$\times \left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\theta_2^{\gamma-1}(t,t_2)r(t)\rho(t) \right)^{-1}.$$

$$(36)$$

Substituting t with s in (36), an integration for (36) with respect to s from t_2 to t yields

$$\int_{t_2}^{t} \left\{ L\rho(s)q(s)e^{A(s)} - \rho(s)\eta'(s) \right.$$

$$+ \frac{\gamma\rho(s)[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\theta_2^{\gamma-1}(s,t_2)\eta^2(s)}{r(s)}$$

$$- \left(\left\{ 2\gamma\rho(s)\eta(s)[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\theta_2^{\gamma-1}(s,t_2) \right.\right.$$

$$\left.\left. +r(s)\rho'(s) \right\}^2 \right)$$

$$\times \left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\theta_2^{\gamma-1}(s,t_2)r(s)\rho(s) \right)^{-1} \right\} ds$$

$$\leq \omega(t_2) - \omega(t) \leq \omega(t_2),$$

$$(37)$$

which contradicts (34). So the proof is complete. □

Theorem 7. *Define* $\mathbb{D} = \{(t,s) \mid t \geq s \geq t_0\}$. *Assume* (4)–(6) *hold and there exists a function* $H \in C^1(\mathbb{D}, \mathbb{R})$ *such that*

$$H(t,t) = 0, \quad \text{for } t \geq t_0, \qquad H(t,s) > 0, \quad \text{for } t > s \geq t_0,$$

$$(38)$$

H has a nonpositive continuous partial derivative $H'_s(t,s)$, and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)}$$

$$\times \left\{ \int_{t_0}^{t} H(t,s) \right.$$

$$\times \left\{ L\rho(s)q(s)e^{A(s)} - \rho(s)\eta'(s) \right.$$

$$+ \frac{\rho(s)\Gamma(1-\alpha)\theta_1(s,T)\eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1)\eta^{1/\gamma}(s)\rho(s)\Gamma(1-\alpha)\theta_1(s,T) \right.\right.$$

$$\left.\left. +r(s)\rho'(s) \right]^{\gamma+1} \right)$$

$$\times \left((\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\rho(s)\theta_1(s,T)]^{\gamma} \right.$$

$$\left.\left. \times r(s) \right)^{-1} \right\} ds \right\}$$

$$= \infty,$$

$$(39)$$

for all sufficiently large T, where ρ and η are defined as in Theorem 5. Then, every solution of (2) *is oscillatory or satisfies* $\lim_{t\to\infty}X(t) = 0$.

Proof. Assume (2) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2, we have either $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$, for some sufficiently large $t_2 > t_1$, or $\lim_{t\to\infty}X(t) = 0$. Now we assume $D_-^{\alpha}x(t) < 0$. Let $\omega(t)$ be defined as in Theorem 5. By (32), we have

$$L\rho(t)q(t)e^{A(t)} - \rho(t)\eta'(t)$$

$$+ \frac{\rho(t)\Gamma(1-\alpha)\theta_1(t,t_2)\eta^{1+(1/\gamma)}(t)}{r(t)}$$

$$- \frac{\left[(\gamma+1)\eta^{1/\gamma}(t)\rho(t)\Gamma(1-\alpha)\theta_1(t,t_2) + r(t)\rho'(t) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[\Gamma(1-\alpha)\rho(t)\theta_1(t,t_2)]^{\gamma}r(t)}$$

$$\leq -\omega'(t).$$

$$(40)$$

Substituting t with s in (40), multiplying both sides by $H(t,s)$, and then integrating with respect to s from t_2 to t yield

$$\int_{t_2}^{t} H(t,s)$$

$$\times \left\{ L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\left. \times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} r(s) \right)^{-1} \right\} ds$$

$$\leq - \int_{t_2}^{t} H(t,s) \omega'(s) ds = H(t,t_2) \omega(t_2)$$

$$+ \int_{t_2}^{t} H_s'(t,s) \omega(s) \Delta s$$

$$\leq H(t,t_2) \omega(t_2) \leq H(t,t_0) \omega(t_2).$$

$$(41)$$

Then

$$\int_{t_0}^{t} H(t,s)$$

$$\times \left\{ L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\left. \times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} r(s) \right)^{-1} \right\} ds$$

$$= \int_{t_0}^{t_2} H(t,s)$$

$$\times \left\{ L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\left. \times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} r(s) \right)^{-1} \right\} ds$$

$$+ \int_{t_2}^{t} H(t,s)$$

$$\times \left\{ L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\left. \times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} r(s) \right)^{-1} \right\} ds$$

$$\leq H(t,t_0) \omega(t_2) + H(t,t_0)$$

$$\times \int_{t_0}^{t_2} \left| L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\left. \times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} r(s) \right)^{-1} \right| ds.$$

$$(42)$$

So

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)}$$

$$\times \left\{ \int_{t_0}^{t} H(t,s) \right.$$

$$\times \left\{ L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1) \eta^{1/\gamma}(s) \rho(s) \Gamma(1-\alpha) \theta_1(s,t_2) \right. \right.$$

$$\left. + r(s) \rho'(s) \right]^{\gamma+1} \right)$$

$$\times \left((\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s,t_2) \right]^{\gamma} \right.$$

$$\left. \left. \left. \times r(s) \right)^{-1} \right\} ds \right\}$$

$$\leq \omega(t_2)$$

$$+ \int_{t_0}^{t_2} \left| L\rho(s) q(s) e^{A(s)} - \rho(s) \eta'(s) \right.$$

$$+ \frac{\rho(s)\,\Gamma(1-\alpha)\,\theta_1(s,t_2)\,\eta^{1+(1/\gamma)}(s)}{r(s)}$$

$$- \left(\left[(\gamma+1)\,\eta^{1/\gamma}(s)\,\rho(s)\,\Gamma(1-\alpha)\,\theta_1(s,t_2) \right. \right.$$

$$\left. + r(s)\,\rho'(s) \right]^{\gamma+1} \right)$$

$$\times \left. \left((\gamma+1)^{\gamma+1} [\Gamma(1-\alpha)\,\rho(s)\,\theta_1(s,t_2)]^{\gamma}\,r(s) \right)^{-1} \right| ds$$

$$< \infty,$$

$$(43)$$

which contradicts (39). So the proof is complete. $\qquad\square$

Theorem 8. *Let H, ρ, and η be defined as in Theorem 7. If (4)–(6) hold and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}$$

$$\times \int_{t_0}^{t} H(t,s)$$

$$\times \left\{ L\rho(s)\,q(s)\,e^{A(s)} - \rho(s)\,\eta'(s) \right.$$

$$+ \frac{\gamma\rho(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T)\,\eta^2(s)}{r(s)}$$

$$- \left(\left\{ 2\gamma\rho(s)\,\eta(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T) \right. \right.$$

$$\left. + r(s)\,\rho'(s) \right\}^2 \right)$$

$$\times \left. \left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,T)\,\theta_2^{\gamma-1}(s,T)\,r(s)\,\rho(s) \right)^{-1} \right\} ds$$

$$= \infty,$$

$$(44)$$

for all sufficiently large T, then every solution of (2) is oscillatory or satisfies $\lim_{t\to\infty} X(t) = 0$.

Proof. Assume (2) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2, we have either $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$, for some sufficiently large $t_2 > t_1$, or $\lim_{t\to\infty} X(t) = 0$. Now, we assume $D_-^{\alpha}x(t) < 0$. Let $\omega(t)$ be defined as in Theorem 5. By (36), we have

$$L\rho(t)\,q(t)\,e^{A(t)} - \rho(t)\,\eta'(t)$$

$$+ \frac{\gamma\rho(t)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\,\theta_2^{\gamma-1}(t,t_2)\,\eta^2(t)}{r(t)}$$

$$- \frac{\left\{ 2\gamma\rho(t)\,\eta(t)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\,\theta_2^{\gamma-1}(t,t_2) + r(t)\,\rho'(t) \right\}^2}{4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(t,t_2)\,\theta_2^{\gamma-1}(t,t_2)\,r(t)\,\rho(t)}$$

$$\leq -\omega'(t).$$

$$(45)$$

Substituting t with s in (45), multiplying both sides by $H(t,s)$, and then integrating with respect to s from t_2 to t yield

$$\int_{t_2}^{t} H(t,s)$$

$$\times \left\{ L\rho(s)\,q(s)\,e^{A(s)} - \rho(s)\,\eta'(s) \right.$$

$$+ \frac{\gamma\rho(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\,\theta_2^{\gamma-1}(s,t_2)\,\eta^2(s)}{r(s)}$$

$$- \left(\left\{ 2\gamma\rho(s)\,\eta(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\,\theta_2^{\gamma-1}(s,t_2) \right. \right.$$

$$\left. + r(s)\,\rho'(t) \right\}^2 \right)$$

$$\times \left. \left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\theta_2^{\gamma-1}(s,t_2)r(s)\rho(s) \right)^{-1} \right\} ds$$

$$\leq -\int_{t_2}^{t} H(t,s)\,\omega'(s)\,ds = H(t,t_2)\,\omega(t_2)$$

$$+ \int_{t_2}^{t} H_s'(t,s)\,\omega(s)\,\Delta s$$

$$\leq H(t,t_2)\,\omega(t_2) \leq H(t,t_0)\,\omega(t_2).$$

$$(46)$$

Then, similar to the process of Theorem 7, we get that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}$$

$$\times \left\{ \int_{t_0}^{t} H(t,s) \right.$$

$$\times \left\{ L\rho(s)\,q(s)\,e^{A(s)} - \rho(s)\,\eta'(s) \right.$$

$$+ \frac{\gamma\rho(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\,\theta_2^{\gamma-1}(s,t_2)\,\eta^2(s)}{r(s)}$$

$$- \left(\left\{ 2\gamma\rho(s)\,\eta(s)\,[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2) \right. \right.$$

$$\left. \times \theta_2^{\gamma-1}(s,t_2) + r(s)\,\rho'(s) \right\}^2 \right)$$

$$\times \left(4\gamma[\Gamma(1-\alpha)]^{\gamma}\theta_1(s,t_2)\,\theta_2^{\gamma-1} \right.$$

$$\left. \left. \times (s,t_2)r(s)\rho(s) \right)^{-1} \right\} ds \right\} < \infty,$$

$$(47)$$

which contradicts (44). So the proof is complete. $\qquad\square$

Remark 9. In Theorems 7 and 8, if we take $H(t, s)$ for some special functions such as $(t-s)^m$ or $\ln(t/s)$, then we can obtain some corollaries, which are omitted here.

Remark 10. The established oscillation criteria for (2) above are new results so far in the literature to the best of our knowledge.

3. Applications

In this section, we will present some applications for the established results above.

Example 1. Consider the following:

$$
\left(t^{5/3} \left[\left(D_-^\alpha x(t) \right)' \right]^{5/3} \right)' + t^{-8/3} \left[\left(D_-^\alpha x(t) \right)' \right]^{5/3}
$$
$$
- t^{-8/3} \left[M + e^{\int_t^\infty (\xi - t)^{-\alpha} x(\xi) d\xi} \right]
$$
$$
\times \left(\int_t^\infty (\xi - t)^{-\alpha} x(\xi) d\xi \right)^{5/3} = 0, \quad t \in [2, \infty),
\tag{48}
$$

where $M > 0$ is a constant.

We have in (2) $\gamma = 5/3$, $a(t) = t^{5/3}$, $p(t) = q(t) = t^{-8/3}$, $f(x) = x^{5/3}[e^x + M]$, $r(t) = 1$, $t_0 = 2$. Then, $f(x)/x^\gamma \geq M = L$. Moreover, $1 \leq e^{A(t)} = e^{\int_2^t (p(s)/a(s)) ds} = e^{\int_2^t s^{-13/3} ds} = e^{-(3/10)[t^{-10/3} - 2^{-10/3}]} \leq e$. Then, we have

$$
\int_{t_0}^\infty \frac{1}{\left[e^{A(s)} a(s) \right]^{1/\gamma}} ds = \int_2^\infty \frac{1}{\left[e^{A(s)} a(s) \right]^{3/5}} ds
$$
$$
\geq e^{-3/5} \int_2^\infty \frac{1}{s} ds = \infty,
\tag{49}
$$
$$
\int_{t_0}^\infty \frac{1}{r(s)} ds = \infty.
$$

Furthermore,

$$
\int_{t_0}^\infty \frac{1}{r(\xi)} \int_\xi^\infty \left[\frac{1}{e^{A(\tau)} a(\tau)} \int_\tau^\infty e^{A(s)} q(s) ds \right]^{1/\gamma} d\tau\, d\xi
$$
$$
= \int_2^\infty \int_\xi^\infty \left[\frac{1}{e^{A(\tau)} \tau^{5/3}} \int_\tau^\infty e^{A(s)} s^{-8/3} ds \right]^{3/5} d\tau\, d\xi
$$
$$
\geq e^{-3/5} \int_2^\infty \int_\xi^\infty \left[\frac{1}{\tau^{5/3}} \int_\tau^\infty s^{-8/3} ds \right]^{3/5} d\tau\, d\xi
$$
$$
= \left(\frac{5e}{3} \right)^{-3/5} \int_2^\infty \left[\int_\xi^\infty \frac{1}{\tau^2} d\tau \right] d\xi = \left(\frac{5e}{3} \right)^{-3/5} \int_2^\infty \frac{1}{\xi} d\xi = \infty.
\tag{50}
$$

On the other hand, for a sufficiently large T, we have

$$
\theta_1(t, T) = \int_T^t \frac{1}{\left[e^{A(s)} a(s) \right]^{1/\gamma}} ds
$$
$$
= \int_T^t \frac{1}{\left[e^{A(s)} s^{5/3} \right]^{3/5}} ds \geq e^{-3/5} \int_T^t \frac{1}{s} ds \longrightarrow \infty.
\tag{51}
$$

So we can take $T^* > T$ such that $\theta_1(t, T) > 1$ for $t \in [T^*, \infty)$. Taking $\rho(t) = t^{5/3}$ and $\eta(t) = 0$ in (24), we get that

$$
\int_T^\infty \left\{ L\rho(s) q(s) \right.
$$
$$
\left. - \frac{\left[r(s) \rho'(s) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s, T) \right]^\gamma r(s)} \right\} ds
$$
$$
= \int_T^\infty \left\{ M - \left(\frac{5}{8} \right)^{8/3} \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^{5/3}} \right\} \frac{1}{s} ds
$$
$$
= \int_T^{T^*} \left\{ M - \left(\frac{5}{8} \right)^{8/3} \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^{5/3}} \right\} \frac{1}{s} ds
$$
$$
+ \int_{T^*}^\infty \left\{ M - \left(\frac{5}{8} \right)^{8/3} \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^{5/3}} \right\} \frac{1}{s} ds
$$
$$
\geq \int_T^{T^*} \left\{ M - \left(\frac{5}{8} \right)^{8/3} \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^{5/3}} \right\} \frac{1}{s} ds
$$
$$
+ \int_{T^*}^t \left\{ M - \left(\frac{5}{8} \right)^{8/3} \frac{1}{\left[\Gamma(1-\alpha) \right]^{5/3}} \right\} \frac{1}{s} ds \longrightarrow \infty,
\tag{52}
$$

provided that $M > (5/8)^{8/3} (1/[\Gamma(1-\alpha)]^{5/3})$. So (4)–(6) and (24) all hold, and by Theorem 5 we deduce that every solution of (48) is oscillatory or satisfies $\lim_{t \to \infty} X(t) = 0$ under the condition $M > (5/8)^{8/3} (1/[\Gamma(1-\alpha)]^{5/3})$.

Example 2. Consider the following:

$$
\left(t^3 \left[\left(D_-^\alpha x(t) \right)' \right]^3 \right)' + t^{-4} \left[\left(D_-^\alpha x(t) \right)' \right]^3
$$
$$
- Mt^{-4} \left[\int_t^\infty (\xi - t)^{-\alpha} x(\xi) d\xi \right]^3 = 0, \quad t \in [2, \infty),
\tag{53}
$$

where $\alpha \in (0, 1)$ and $M > 0$ is a constant.

We have in (2) $\gamma = 3$, $a(t) = t^3$, $p(t) = q(t) = t^{-4}$, $f(x) = Mx^3$, $r(t) = 1$, $t_0 = 2$. Then, $f(x)/x^\gamma \geq M = L$. Moreover, $1 \leq e^{A(t)} = e^{\int_2^t (p(s)/a(s)) ds} = e^{\int_2^t s^{-7} ds} = e^{-(1/6)[t^{-6} - 2^{-6}]} \leq e^{1/384}$. Then, we have

$$
\int_{t_0}^\infty \frac{1}{\left[e^{A(s)} a(s) \right]^{1/\gamma}} ds = \int_2^\infty \frac{1}{\left[e^{A(s)} a(s) \right]^{1/3}} ds
$$
$$
\geq e^{-1/1152} \int_2^\infty \frac{1}{s} ds = \infty,
\tag{54}
$$
$$
\int_{t_0}^\infty \frac{1}{r(s)} ds = \infty.
$$

Furthermore,

$$\int_{t_0}^{\infty} \frac{1}{r(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{e^{A(\tau)} a(\tau)} \int_{\tau}^{\infty} e^{A(s)} q(s) ds \right]^{1/\gamma} d\tau\, d\xi$$

$$= \int_{2}^{\infty} \int_{\xi}^{\infty} \left[\frac{1}{e^{A(\tau)} \tau^3} \int_{\tau}^{\infty} e^{A(s)} s^{-4} ds \right]^{1/3} d\tau\, d\xi$$

$$\geq e^{-1/1152} \int_{2}^{\infty} \int_{\xi}^{\infty} \left[\frac{1}{\tau^3} \int_{\tau}^{\infty} s^{-4} ds \right]^{1/3} d\tau\, d\xi$$

$$= \frac{e^{-1/1152}}{\sqrt[3]{3}} \int_{2}^{\infty} \left[\int_{\xi}^{\infty} \frac{1}{\tau^2} d\tau \right] d\xi = \frac{e^{-1/1152}}{\sqrt[3]{3}} \int_{2}^{\infty} \frac{1}{\xi} d\xi = \infty.$$

$$(55)$$

On the other hand, for a sufficiently large T, we have

$$\theta_1(t, T) = \int_{T}^{t} \frac{1}{\left[e^{A(s)} a(s) \right]^{1/\gamma}} ds$$

$$= \int_{T}^{t} \frac{1}{\left[e^{A(s)} s^3 \right]^{1/3}} ds \geq e^{-1/1152} \int_{T}^{t} \frac{1}{s} ds \longrightarrow \infty.$$

$$(56)$$

So we can take $T^* > T$ such that $\theta_1(t, T) > 1$ for $t \in [T^*, \infty)$. Taking $\rho(t) = t^3, \eta(t) = 0, H(t, s) = t - s$ in (39), we get that

$$\lim_{t \to \infty} \sup \frac{1}{t - t_0}$$

$$\times \left\{ \int_{t_0}^{t} (t - s) \right.$$

$$\times \left\{ L\rho(s) q(s) \right.$$

$$\left. \left. - \frac{\left[r(s) \rho'(s) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\Gamma(1-\alpha) \rho(s) \theta_1(s, T) \right]^{\gamma} r(s)} \right\} ds \right\}$$

$$= \lim_{t \to \infty} \sup \frac{1}{t - 2}$$

$$\times \left\{ \int_{2}^{t} (t - s) \right.$$

$$\times \left\{ M - \left(\frac{3}{4} \right)^4 \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^3} \right\} \frac{1}{s} ds \right\}$$

$$= \lim_{t \to \infty} \sup \frac{1}{t - 2}$$

$$\times \left\{ \int_{2}^{T^*} (t - s) \right.$$

$$\times \left\{ M - \left(\frac{3}{4} \right)^4 \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^3} \right\} \frac{1}{s} ds$$

$$+ \int_{T^*}^{t} (t - s)$$

$$\times \left\{ M - \left(\frac{3}{4} \right)^4 \frac{1}{\left[\Gamma(1-\alpha) \theta_1(s, T) \right]^3} \right\} \frac{1}{s} ds \right\}$$

$$\geq \lim_{t \to \infty} \sup \frac{1}{t - 2}$$

$$\times \left\{ \int_{2}^{T^*} (t - s) \left\{ M - \left(\frac{3}{4} \right)^4 \frac{1}{\left[\Gamma(1-\alpha) \right]^3} \right\} \frac{1}{s} ds$$

$$+ \int_{T^*}^{t} (t - s) \left\{ M - \left(\frac{3}{4} \right)^4 \frac{1}{\left[\Gamma(1-\alpha) \right]^3} \right\} \frac{1}{s} ds \right\} = \infty,$$

$$(57)$$

provided that $M > (3/4)^4 (1/[\Gamma(1 - \alpha)]^3)$. So (4)–(6) and (39) all hold, and by Theorem 7 we deduce that every solution of (48) is oscillatory or satisfies $\lim_{t \to \infty} X(t) = 0$ under the condition $M > (3/4)^4 (1/[\Gamma(1 - \alpha)]^3)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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