Research Article

On Characterizations of Fourier Frames and Tilings

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We give some characterizations of Fourier frames and tilings and obtain a more general form of characterizations of spectra and tilings.

1. Introduction

A countable family of elements $\{f_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space H is called a frame if there are positive constants A, B such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \tag{1}$$

for all $f \in H$. A and B are called frame bounds. The sequence is called a tight frame if $A = B$. The sequence is called Bessel if the second inequality above holds. In this case, B is called the Bessel bound. Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series, and today they have applications in a wide range of areas. A frame can be considered as a generalized basis in the sense that every element in H can be written as a linear combination of the frame elements.

In this paper, we consider Fourier frames for a special separable Hilbert space. Let $\Omega \subset \mathbb{R}^d$ have positive Lebesgue measure $m(\Omega) > 0$ and let Λ be a discrete subset of $\mathbb{R}^d$. The inner product and the norm on $L^2(\Omega)$ are

$$\langle f(x), g(x)\rangle_\Omega = \frac{1}{m(\Omega)} \int_\Omega f(x) \overline{g(x)}\, dx,$$
$$\|f\|_\Omega^2 = \frac{1}{m(\Omega)} \int_\Omega |f(x)|^2\, dx. \tag{2}$$

We write

$$e_\lambda(x) := e^{2\pi i\langle\lambda,x\rangle} \quad \text{for } x \in \mathbb{R}^d,$$
$$\mathscr{E}(\Lambda) := \{e_\lambda(x) : \lambda \in \Lambda\}. \tag{3}$$

If $\mathscr{E}(\Lambda)$ is a frame or an orthonormal basis for $L^2(\Omega)$, then $\mathscr{E}(\Lambda)$ and (Ω, Λ) are called a Fourier frame and a spectral pair, respectively. In the case of the spectral pair, the Λ is then called a spectrum for Ω and Ω is called a spectral set. We follow the terminology of [2] and consider the packing and tiling in $\mathbb{R}^d$ by compact set Ω of the following kind.

A compact set Ω in $\mathbb{R}^d$ is a regular region if it has positive Lebesgue measure, is the closure of its interior $\Omega°$, and has a boundary $\partial\Omega = \Omega \setminus \Omega°$ of measure zero. If Ω is a regular region, then a discrete set Λ is a packing set for Ω if the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ have disjoint interiors or the intersections $(\Omega + \lambda) \cap (\Omega + \mu)$ for $\lambda \neq \mu$ in Λ have measure zero. It is a tiling set if, further, the translates $\{\Omega + \lambda : \lambda \in \Lambda\}$ cover $\mathbb{R}^d$ up to measure zero. In these cases, we say that $\Omega + \Lambda$ is a packing or tiling of $\mathbb{R}^d$, respectively. Equivalently, we call (Ω, Λ) a packing pair or a tiling pair, respectively.

It is well known that spectral sets and tilings are connected by the following conjecture of Fuglede [3].

Spectral Set Conjecture. A set Ω in $\mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translations.

Many people attempt to prove the spectral set conjecture for some special sets, although the conjecture is false in many

cases (see [4–7]). For example, Jorgensen and Pedersen [8] conjectured that $([0, 1]^n, \Lambda)$ is a spectral pair if and only if $([0, 1]^n, \Lambda)$ is a tiling pair. They established the conjecture for dimension $n \leq 3$ and for all n when Λ is a discrete periodic set. Iosevich and Pedersen [9] simultaneously and independently established the above-mentioned conjecture by a different approach based on a geometric argument. Kolountzakis [10] gave an alternative proof of this fact, which is based on a characterization of translational tiling by a Fourier analytic criterion. Lagarias et al. [2] related the spectra of sets Ω to tiling in the Fourier space and obtained the following characterization of spectra and tilings.

Theorem 1. *Let Ω be a regular region in $\mathbb{R}^d$ and let Λ be such that the set of exponentials $\mathscr{E}(\Lambda)$ is orthogonal for $L^2(\Omega)$. Suppose that D is a regular region with $m(\Omega)m(D) = 1$ such that $D + \Lambda$ is a packing of $\mathbb{R}^d$. Then, Λ is a spectrum for Ω if and only if $D + \Lambda$ is a tiling of $\mathbb{R}^d$.*

Li [11] presented an elementary approach to obtain a more general form of Theorem 1. Enlightened by the ideas from [11, 12], we give some characterizations of Fourier frames and tilings and extend several results in [2] and [11].

2. Main Results and Their Proofs

Throughout this section, let Ω and D be two regular regions in $\mathbb{R}^d$. By the definition of frames, we may get that the following lemma.

Lemma 2. *Let $\Delta \subset \mathbb{R}^d$ be a discrete set. If $\mathscr{E}(\Delta)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, then*

$$A(m(\Omega))^2 \leq \sum_{\delta \in \Delta} \left| \widehat{\chi_\Omega}(t - \delta) \right|^2 \leq B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d, \quad (4)$$

where

$$\widehat{\chi_\Omega}(u) = \int_{\mathbb{R}^d} \chi_\Omega(x) e^{-2\pi i \langle u, x \rangle} dx, \quad u \in \mathbb{R}^d \quad (5)$$

is the Fourier transform of the characteristic function $\chi_\Omega(x)$.

Proof. By the frame inequality (1), for any $t \in \mathbb{R}^d$, we have

$$\sum_{\delta \in \Delta} \left| \widehat{\chi_\Omega}(t - \delta) \right|^2$$

$$= \sum_{\delta \in \Delta} \left| m(\Omega) \left\langle e_t(x), e_\delta(x) \right\rangle_\Omega \right|^2 \quad (6)$$

$$\leq B(m(\Omega))^2 \left\| e_t(x) \right\|_\Omega^2 = B(m(\Omega))^2.$$

Similarly, we get $A(m(\Omega))^2 \leq \sum_{\delta \in \Delta} |\widehat{\chi_\Omega}(t - \delta)|^2$. $\square$

Remark 3. In the case for $A = B$, if $\mathscr{E}(\Delta)$ is a tight frame for $L^2(\Omega)$ with the frame bound B, then

$$\sum_{\delta \in \Delta} \left| \widehat{\chi_\Omega}(t - \delta) \right|^2 = B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \quad (7)$$

If $\mathscr{E}(\Delta)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B, then

$$\sum_{\delta \in \Delta} \left| \widehat{\chi_\Omega}(t - \delta) \right|^2 \leq B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \quad (8)$$

Moreover, $\mathscr{E}(\Delta)$ is an orthonormal basis for $L^2(\Omega)$ if and only if

$$\sum_{\delta \in \Delta} \left| \widehat{\chi_\Omega}(t - \delta) \right|^2 = (m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \quad (9)$$

Since $|\widehat{\chi_\Omega}(u)| = |\widehat{\chi_\Omega}(-u)|$, for all $u \in \mathbb{R}^d$, if we substitute "$-$" for "$+$" in (4), (7), (8), and (9), all the above results also hold.

In the remainder of this paper, we assume that $\Theta \subset \mathbb{R}^d$ is a discrete subset and Λ and Γ are two finite subsets of $\mathbb{R}^d$ such that $\Theta + \Lambda$ and $\Theta + \Gamma$ are two direct sums.

Theorem 4. *If $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, and $(D, \Theta + \Gamma)$ is a tiling pair, then*

$$\frac{\#\Lambda}{B\#\Gamma} \leq m(\Omega) m(D) \leq \frac{\#\Lambda}{A\#\Gamma}, \quad (10)$$

where $\#$ denotes the cardinality of some set.

Proof. Let $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ and $\Gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_q\}$ with $\#\Lambda = p$ and $\#\Gamma = q$. Since $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, it follows from Lemma 2 that

$$A(m(\Omega))^2 \leq \sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega}(t + \theta + \lambda_i) \right|^2 \leq B(m(\Omega))^2, \quad (11)$$

$$\forall t \in \mathbb{R}^d.$$

Note that $(D, \Theta + \Gamma)$ is a tiling pair, from the Plancherel's formula on $L^2(\mathbb{R}^d)$, we have the following:

$$m(\Omega) = \left\| \chi_\Omega \right\|_{\mathbb{R}^d}^2 = \left\| \widehat{\chi_\Omega} \right\|_{\mathbb{R}^d}^2 = \int_{\mathbb{R}^d} \left| \widehat{\chi_\Omega}(t) \right|^2 dt$$

$$= \frac{1}{p} \int_{\mathbb{R}^d} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega}(t + \lambda_i) \right|^2 dt$$

$$= \frac{1}{p} \int_{\bigcup_{\theta \in \Theta, 1 \leq j \leq q} (D + \theta + \gamma_j)} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega}(t + \lambda_i) \right|^2 dt$$

$$= \frac{1}{p} \sum_{j=1}^{q} \sum_{\theta \in \Theta} \int_{D} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega}(t + \theta + \lambda_i + \gamma_j) \right|^2 dt \quad (12)$$

$$= \frac{1}{p} \sum_{j=1}^{q} \int_{D} \sum_{\theta \in \Theta} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega}(t + \theta + \lambda_i + \gamma_j) \right|^2 dt$$

$$= \frac{1}{p} \sum_{j=1}^{q} \int_{D} \sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega}(t + \theta + \lambda_i + \gamma_j) \right|^2 dt$$

$$\leq \frac{q}{p} B(m(\Omega))^2 m(D).$$

The bottom third equality holds by Lebesgue dominated convergence theorem and the last inequality follows from (11). Thus, $p/qB \leq m(\Omega)m(D)$. Similarly, we get $m(\Omega)m(D) \leq p/qA$. Hence, the proof is completed. □

Since an orthonormal basis is also a tight frame with frame bounds $A = B = 1$, we get the following corollary.

Corollary 5. *If $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$, and $(D, \Theta + \Gamma)$ is a tiling pair, then $m(\Omega)m(D) = \#\Lambda/\#\Gamma$.*

Lemma 6. *Let $\Theta + \Lambda$ be such that the set of exponentials $\mathscr{E}(\Theta + \Lambda)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B. If $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$ with $m(\Omega)m(D) \leq \#\Lambda/B\#\Gamma$, then*

$$\sum_{\lambda \in \Lambda} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda \right) \right|^2 = B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{13}$$

Proof. Keep the assumptions on Λ and Γ in the above proof. Since $\mathscr{E}(\Theta + \Lambda)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B, it follows from Remark 3 that

$$\sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i \right) \right|^2 \leq B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{14}$$

Since $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$ and $m(\Omega)m(D) \leq \#\Lambda/B\#\Gamma = p/qB$, for any $y \in \mathbb{R}^d$, then we have

$$
\begin{aligned}
m(\Omega) &= \int_{\mathbb{R}^d} \left| \widehat{\chi_\Omega}(t) \right|^2 dt \\
&= \frac{1}{p} \int_{\mathbb{R}^d} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \lambda_i \right) \right|^2 dt \\
&= \frac{1}{p} \int_{\cup_{\theta \in \Theta, 1 \leq j \leq q}(D+y+\theta+\gamma_j)} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \lambda_i \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \sum_{\theta \in \Theta} \int_{D+y} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \int_{D+y} \sum_{\theta \in \Theta} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \int_{D+y} \sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&\leq \frac{q}{p} B(m(\Omega))^2 m(D) \leq m(\Omega),
\end{aligned}
\tag{15}
$$

which yields

$$\sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i \right) \right|^2 = B(m(\Omega))^2 \tag{16}$$

for almost every t in $D + y$. Since y is arbitrary, (16) holds for almost every t in $\mathbb{R}^d$. By the continuity of the function on the left side of (16), we see that (16) holds for every t in $\mathbb{R}^d$. □

Theorem 7. *If $\mathscr{E}(\Theta + \Lambda)$ is orthogonal in $L^2(\Omega)$ and $(D, \Theta + \Gamma)$ is a tiling pair with $m(\Omega)m(D) = \#\Lambda/\#\Gamma$, then $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$.*

Proof. The proof is straightforward by the above lemma. □

Theorem 8. *Suppose that $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$ with $\#\Lambda/A\#\Gamma \leq m(\Omega)m(D)$. If $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with the frame bounds A, B, then $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$.*

Proof. Since $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with the frame bounds A, B, then (11) holds. If $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$, then it follows from (11) and $\#\Lambda/A\#\Gamma \leq m(\Omega)m(D)$ that

$$
\begin{aligned}
m(\Omega) &= \int_{\mathbb{R}^d} \left| \widehat{\chi_\Omega}(t) \right|^2 dt \\
&= \frac{1}{p} \int_{\mathbb{R}^d} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \lambda_i \right) \right|^2 dt \\
&\geq \frac{1}{p} \int_{\cup_{\theta \in \Theta, 1 \leq j \leq q}(D+\theta+\gamma_j)} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \lambda_i \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \sum_{\theta \in \Theta} \int_{D} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \int_{D} \sum_{\theta \in \Theta} \sum_{i=1}^{p} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&= \frac{1}{p} \sum_{j=1}^{q} \int_{D} \sum_{i=1}^{p} \sum_{\theta \in \Theta} \left| \widehat{\chi_\Omega} \left(t + \theta + \lambda_i + \gamma_j \right) \right|^2 dt \\
&\geq \frac{q}{p} A(m(\Omega))^2 m(D) \geq m(\Omega).
\end{aligned}
\tag{17}
$$

Thus, $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$. □

It is clear that the above theorem yields the following corollary.

Corollary 9. *Suppose that $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$ with $\#\Lambda/\#\Gamma = m(\Omega)m(D)$. If $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$, then $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$.*

Combining Theorem 7 with Corollary 9, we obtain a more general form of the theorem in [11] and Theorem 1.

Theorem 10. *Suppose that $\mathscr{E}(\Theta + \Lambda)$ is orthogonal in $L^2(\Omega)$, and $(D, \Theta + \Gamma)$ is a packing pair with $m(\Omega)m(D) = \#\Lambda/\#\Gamma$. Then, $(\Omega, \Theta + \Lambda)$ is a spectral pair if and only if $(D, \Theta + \Gamma)$ is a tiling pair.*

Example 11. Let p, q be two positive integers. Take the following:

$$\Omega = [0, 1], \qquad D = \left[0, \frac{p}{q}\right], \qquad \Theta = \{pk : k \in \mathbb{Z}\},$$

$$\Lambda = \{0, 1, \ldots, p-1\}, \qquad \Gamma = \left\{0, \frac{p}{q}, \frac{2p}{q}, \ldots, \frac{(q-1)p}{q}\right\}.$$

$$(18)$$

We see that $m(\Omega)m(D) = \#\Lambda/\#\Gamma$, $(\Omega, \Theta+\Lambda)$ is a spectral pair and $(D, \Theta + \Gamma)$ is a tiling pair.

Acknowledgment

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