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Research Article

Stability of a Logarithmic Functional Equation in Distributions on a Restricted Domain

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Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $\epsilon \in \mathbb{R}_+$, and $f, g, h : \mathbb{R}^+ \to \mathbb{C}$. As classical and L^∞ versions of the Hyers-Ulam stability of the logarithmic type functional equation in a restricted domain, we consider the following inequalities: $|f(x+y) - g(xy) - h((1/x) + (1/y))| \le \epsilon$, and $\| f(x+y) - g(xy) - h((1/x) + (1/y))\|_{L^\infty(\Gamma_d)} \le \epsilon$ in the sectors $\Gamma_d = \{(x,y) : x > 0, y > 0, (y/x) > d\}$. As consequences of the results, we obtain asymptotic behaviors of the previous inequalities. We also consider its distributional version $\|u \circ S - v \circ \Pi - w \circ R\|_{\Gamma_d} \le \epsilon$, where $u, v, w \in \mathscr{D}'(\mathbb{R}^+)$, $S(x,y) = x + y$, $\Pi(x,y) = xy$, $R(x,y) = 1/x + 1/y$, $x, y \in \mathbb{R}^+$, and the inequality $\| \cdot \|_{\Gamma_d} \le \epsilon$ means that $|\langle \cdot, \varphi\rangle| \le \epsilon \| \varphi\|_{L^1}$ for all test functions $\varphi \in C_c^\infty(\Gamma_d)$.

1. Introduction

The Hyers-Ulam stability problem of functional equations was originated in 1940 when Ulam proposed a question concerning the approximate homomorphisms from a group to a metric group (see [1]). A partial answer was given by Hyers et al. [2, 3] under the assumption that the target space of the involved mappings is a Banach space. It is possible to prove stability results similar to Hyers for functions that do not have bounded Cauchy difference. In 1950, Aoki [4] first proved such a result for additive functions. Bourgin [5, 6] and Aoki [4] studied the Ulam problem from 1949 to 1951. The area rested there for a while until 1978 when Rassias [7] published a generalized version of Hyers' result on linear mappings, where the Cauchy difference was allowed to be unbounded. Rassias' work provided an impetus for the study on the stability of functional equations (see [2, 7–31]).

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, and $\mathbb{C}$ the set of complex numbers. The subset, for fixed real number $d > 0$,

$$\Gamma_d = \left\{(x,y) : x > 0, y > 0, \frac{y}{x} > d\right\} \tag{1}$$

of the plane, $\mathbb{R}^2$, will be referred to as a sector. A function $f : \mathbb{R}_+ \to \mathbb{C}$ is said to be logarithmic if and only if it satisfies the logarithmic functional equation:

$$f(xy) - f(x) - f(y) = 0, \quad \forall x, y \in \mathbb{R}_+, \tag{2}$$

for all $x, y \in \mathbb{R}_+$. There are several variants of logarithmic functional equations (see [14–16]). It was shown by Heuvers and Kannappan [16] that the logarithmic functional equation is equivalent to the following functional equation:

$$f(x+y) - f(xy) - f\left(\frac{1}{x} + \frac{1}{y}\right) = 0, \quad \forall x, y \in \mathbb{R}_+. \tag{3}$$

They have also studied the following pexiderized version of (3):

$$f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) = 0, \quad \forall x, y \in \mathbb{R}_+. \tag{4}$$

The general solution of the functional equation (4) has the form (see [16])

$$\begin{aligned} f(x) &= L(x) + c_1 + c_2, \\ g(x) &= L(x) + c_1, \\ h(x) &= L(x) + c_2, \end{aligned} \tag{5}$$

where $L : \mathbb{R}^+ \to \mathbb{C}$ is a logarithmic function and c_1, c_2 are arbitrary constants.

In this paper, we study Hyers-Ulam stability of the functional equation (4). In Section 2, we treat the Hyers-Ulam stability of the functional equation (4) in the classical sense and present its asymptotic behavior. In Section 3, we consider the stability of (4) in L^∞-sense and its asymptotic behavior. Finally, in Section 4 we present the stability of (4) in Schwartz distributions.

2. Stability of (4) in Classical Sense and Its Asymptotic Behavior

In this section, we consider the classical Hyers-Ulam stability of the functional equation (4) on the sector Γ_d and then study its asymptotic behavior.

The following theorem is a direct consequence of the Hyers' result [3] (see also result of Forti [32]).

Theorem 1. *Let ϵ be a nonnegative real number. Suppose that $f : \mathbb{R}^+ \to \mathbb{C}$ satisfies*

$$\left|f(xy) - f(x) - f(y)\right| \le \epsilon \tag{6}$$

for all $x, y \in \mathbb{R}_+$. Then there exists a unique logarithmic function $L : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\left|f(x) - L(x)\right| \le \epsilon, \quad \forall x \in \mathbb{R}_+. \tag{7}$$

Next, we establish the Hyers-Ulam stability of the functional equation (4) on the restricted domain Γ_d.

Theorem 2. *Suppose that $\epsilon \ge 0$, $d > 0$, and f, g, h satisfy the functional inequality*

$$\left|f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right)\right| \le \epsilon \tag{8}$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique logarithmic function $L : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\begin{aligned} &\left|f(x) - L(x) - f(1)\right| \le 4\epsilon, \\ &\left|g(x) - L(x) - g(1)\right| \le 4\epsilon, \\ &\left|h(x) - L(x) - h(1)\right| \le 4\epsilon \end{aligned} \tag{9}$$

for all $x \in \mathbb{R}_+$.

Proof. For given $t, s > 0$, choose a real number $u > 0$ such that

$$u \ge \max\left\{\frac{2}{\sqrt{ts^2}}, \frac{2}{\sqrt{ts}}, \frac{2}{\sqrt{s}}, \frac{2}{s}, 2\sqrt{\frac{d}{ts^2}}, 2\sqrt{\frac{d}{ts}}, 2\sqrt{\frac{d}{s}}, 2\sqrt{\frac{d}{s^2}}\right\}, \tag{10}$$

and let

$$\begin{aligned} x_1 &= \frac{tsu - \sqrt{t^2s^2u^2 - 4t}}{2}, & y_1 &= \frac{tsu + \sqrt{t^2s^2u^2 - 4t}}{2}, \\ x_2 &= \frac{tsu - \sqrt{t^2s^2u^2 - 4ts}}{2}, & y_2 &= \frac{tsu + \sqrt{t^2s^2u^2 - 4ts}}{2}, \\ x_3 &= \frac{su - \sqrt{s^2u^2 - 4s}}{2}, & y_3 &= \frac{su + \sqrt{s^2u^2 - 4s}}{2}, \\ x_4 &= \frac{su - \sqrt{s^2u^2 - 4}}{2}, & y_4 &= \frac{su + \sqrt{s^2u^2 - 4}}{2}. \end{aligned} \tag{11}$$

Then it is easy to check that $x_j, y_j > 0$, $y_j/x_j > d$ for all $j = 1, 2, 3, 4$. Replacing x, y by x_j, y_j in (8), respectively, for $j = 1, 2, 3, 4$ we have

$$\left|f(tsu) - g(t) - h(su)\right| \le \epsilon, \tag{12}$$

$$\left|f(tsu) - g(ts) - h(u)\right| \le \epsilon, \tag{13}$$

$$\left|f(su) - g(s) - h(u)\right| \le \epsilon, \tag{14}$$

$$\left|f(su) - g(1) - h(su)\right| \le \epsilon. \tag{15}$$

From (12)–(15), using the triangle inequality we have

$$\left|g(ts) - g(\mathrm{t}) - g(s) + g(1)\right| \le 4\epsilon \tag{16}$$

for all $t, s > 0$. Similarly, for given $t, s > 0$, choose $u > 0$ such that

$$u \ge \max\left\{\frac{4}{t^2s}, \frac{4}{t^2s^2}, \frac{4}{s^2}, \frac{4}{s}, \frac{4d}{t^2s}, \frac{4d}{t^2s^2}, \frac{4d}{s^2}, \frac{4d}{s}\right\} \tag{17}$$

and let

$$\begin{aligned} x_1 &= \frac{tsu - \sqrt{t^2s^2u^2 - 4su}}{2}, & y_1 &= \frac{tsu + \sqrt{t^2s^2u^2 - 4su}}{2}, \\ x_2 &= \frac{tsu - \sqrt{t^2s^2u^2 - 4u}}{2}, & y_2 &= \frac{tsu + \sqrt{t^2s^2u^2 - 4u}}{2}, \\ x_3 &= \frac{su - \sqrt{s^2u^2 - 4u}}{2}, & y_3 &= \frac{su + \sqrt{s^2u^2 - 4u}}{2}, \\ x_4 &= \frac{su - \sqrt{s^2u^2 - 4su}}{2}, & y_4 &= \frac{su + \sqrt{s^2u^2 - 4su}}{2}. \end{aligned} \tag{18}$$

Then it is easy to check that $x_j, y_j > 0$, $y_j/x_j > d$ for all $j = 1, 2, 3, 4$. Next, replacing x, y by x_j, y_j in (8), respectively, for $j = 1, 2, 3, 4$, we have

$$\begin{aligned} &\left|f(tsu) - h(t) - g(su)\right| \le \epsilon, \\ &\left|f(tsu) - h(ts) - g(u)\right| \le \epsilon, \\ &\left|f(su) - h(s) - g(u)\right| \le \epsilon, \\ &\left|f(su) - h(1) - g(su)\right| \le \epsilon. \end{aligned} \tag{19}$$

From (19), using the triangle inequality, we have

$$|h(ts) - h(t) - h(s) + h(1)| \le 4\epsilon \tag{20}$$

for all $t, s > 0$. Now we prove that

$$|f(ts) - f(t) - f(s) + f(1)| \le 4\epsilon \tag{21}$$

for all $t, s > 0$. For given $t, s > 0$, choose $u > 0$ such that

$$u \le \min\left\{\frac{t^2s^2}{4}, \frac{t^2s}{4}, \frac{s^2}{4}, \frac{s}{4}, \frac{t^2s^2}{4d}, \frac{t^2s}{4d}, \frac{s^2}{4d}, \frac{s}{4d}\right\} \tag{22}$$

and let

$$\begin{aligned} x_1 &= \frac{ts - \sqrt{t^2s^2 - 4u}}{2}, & y_1 &= \frac{ts + \sqrt{t^2s^2 - 4u}}{2}, \\ x_2 &= \frac{t - \sqrt{t^2 - 4u/s}}{2}, & y_2 &= \frac{t + \sqrt{t^2 - 4u/s}}{2}, \\ x_3 &= \frac{s - \sqrt{s^2 - 4u}}{2}, & y_3 &= \frac{s + \sqrt{s^2 - 4u}}{2}, \\ x_4 &= \frac{1 - \sqrt{1 - 4u/s}}{2}, & y_4 &= \frac{1 + \sqrt{1 - 4u/s}}{2}. \end{aligned} \tag{23}$$

Then $x_j, y_j > 0$, $y_j/x_j > d$ for all $j = 1, 2, 3, 4$. Replacing x, y by x_j, y_j in (8), respectively, for $j = 1, 2, 3, 4$, we have

$$\begin{aligned} &\left|f(ts) - g(u) - h\left(\frac{ts}{u}\right)\right| \le \epsilon, \\ &\left|f(t) - g\left(\frac{u}{s}\right) - h\left(\frac{ts}{u}\right)\right| \le \epsilon, \\ &\left|f(s) - g(u) - h\left(\frac{s}{u}\right)\right| \le \epsilon, \\ &\left|f(1) - g\left(\frac{u}{s}\right) - h\left(\frac{s}{u}\right)\right| \le \epsilon. \end{aligned} \tag{24}$$

From (24), using the triangle inequality we get (21).

Now by Theorem 1, there exist $L_j : \mathbb{R}^+ \to \mathbb{C}$ for $j = 1, 2, 3$ satisfying the logarithmic functional equation

$$L_j(ts) = L_j(t) + L_j(s), \quad j = 1, 2, 3, \tag{25}$$

for which

$$|f(t) - L_1(t) - f(1)| \le 4\epsilon, \tag{26}$$

$$|g(t) - L_2(t) - g(1)| \le 4\epsilon, \tag{27}$$

$$|h(t) - L_3(t) - h(1)| \le 4\epsilon. \tag{28}$$

Now we show that $L_1 = L_2 = L_3$. Putting $s = u = 1$ and $t = u = 1$ in (12) separately, we have

$$|f(t) - g(t) - h(1)| \le \epsilon \quad \text{for } t \ge \max\{4, 4d\}, \tag{29}$$

$$|f(s) - h(s) - g(1)| \le \epsilon \quad \text{for } s \ge \max\left\{2, 2\sqrt{d}\right\}. \tag{30}$$

From (26), (27), and (29), using the triangle inequality we have

$$\begin{aligned} &|L_1(t) - L_2(t)| \\ &\quad \le 9\epsilon + |f(1) - g(1) - h(1)| \\ &\quad := M \quad \text{for } t \ge \max\{4, 4d\}. \end{aligned} \tag{31}$$

Let $t > 1$. Then we can choose a positive integer n_0 such that $t^n \ge \max\{4, 4d\}$ for all integers $n \ge n_0$. In view of (25), and (31) we have

$$|L_1(t) - L_2(t)| = \frac{1}{n}|L_1(t^n) - L_2(t^n)| \le \frac{M}{n} \tag{32}$$

for all integer $n \ge n_0$. Letting $n \to \infty$ in (32), we have $L_1(t) = L_2(t)$ for all $t > 1$. For $0 < t < 1$, we have $L_1(t) = -L_1(1/t) = -L_2(1/t) = L_2(t)$. Thus, we have $L_1(t) = L_2(t)$ for all $t > 0$. Similarly, using (26), (28), and (30) we can show that $L_1 = L_3$. The uniqueness of the logarithmic function L is obvious. This completes the proof of the theorem. □

Letting $g = h = f$ in Theorem 2 and using the inequalities (12)–(14) together with the triangle inequality, we obtain

$$|f(ts) - f(t) - f(s)| \le 3\epsilon \tag{33}$$

for all $t, s > 0$. Thus, by Theorem 1 we have the following theorem.

Theorem 3. *Let $d > 0$. Suppose that $f : \mathbb{R}^+ \to \mathbb{C}$ satisfies the functional inequality*

$$\left|f(x + y) - f(xy) - f\left(\frac{1}{x} + \frac{1}{y}\right)\right| \le \epsilon \tag{34}$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique logarithmic function $L : \mathbb{R}^+ \to \mathbb{C}$ such that

$$|f(x) - L(x)| \le 3\epsilon, \quad \forall x \in \mathbb{R}^+. \tag{35}$$

Now we prove the following asymptotic result concerning (8).

Theorem 4. *Suppose that $f, g, h : \mathbb{R}^+ \to \mathbb{C}$ satisfy the asymptotic condition*

$$\left|f(x + y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right)\right| \longrightarrow 0 \tag{36}$$

as $(y/x) \to \infty$. Then there exists a logarithmic function $L : \mathbb{R}^+ \to \mathbb{C}$ and $c_1, c_2 \in \mathbb{C}$ such that

$$\begin{aligned} f(x) &= L(x) + c_1 + c_2, \\ g(x) &= L(x) + c_1, \\ h(x) &= L(x) + c_2 \end{aligned} \tag{37}$$

for all $x > 0$.

Proof. By the condition (36), for any positive integer n, there exists $d_n > 0$ such that

$$\left| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right| \leq \frac{1}{n} \tag{38}$$

for all $x, y > 0$ with $(y/x) > d_n$. By Theorem 1, there exists a logarithmic function $L_n : \mathbb{R}^+ \to \mathbb{C}$ such that

$$|f(x) - L_n(x) - f(1)| \leq \frac{4}{n}, \tag{39}$$

$$|g(x) - L_n(x) - g(1)| \leq \frac{4}{n}, \tag{40}$$

$$|h(x) - L_n(x) - h(1)| \leq \frac{4}{n} \tag{41}$$

for all $x > 0$. Replacing n by m in (39) and using the triangle inequality, we have

$$|L_n(x) - L_m(x)| \leq \frac{4}{n} + \frac{4}{m} \leq 8 \tag{42}$$

for all $x > 0$. Thus, we obtain

$$|L_n(x) - L_m(x)| = \frac{1}{k} \left| L_n(x^k) - L_m(x^k) \right| \leq \frac{8}{k} \tag{43}$$

for all $x > 0$ and $k \in \mathbb{N}$. Letting $k \to \infty$ in (43), we have $L_n(x) = L_m(x) := L(x)$ for all $x > 0$. Finally, letting $n \to \infty$ in (39), (40), and (41), we have

$$\begin{aligned} f(x) &= L(x) + f(1), \\ g(x) &= L(x) + g(1), \\ h(x) &= L(x) + h(1) \end{aligned} \tag{44}$$

for all $x > 0$. Finally, substituting (44) in (36) we get $f(1) = g(1) + h(1)$. Letting $c_1 = g(1)$ and $c_2 = h(1)$ we obtain the asserted result. □

3. Stability of (4) in L^∞-Sense and Its Asymptotic Behavior

In this section, we consider the Hyers-Ulam stability of the functional equation (4) in L^∞-sense on the sector Γ_d and then examine its asymptotic behavior. Consider the functional inequality

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right\|_{L^\infty(\Gamma_d)} \leq \epsilon, \tag{45}$$

where $\Gamma_d = \{(x, y) : x > 0, y > 0, (y/x) > d\}$ and $d > 1$ is fixed, where $\|\cdot\|_{L^\infty(\Gamma_d)}$ denotes the essential supremum norm of $D(x, y) = f(x+y) - g(xy) - h((1/x) + (1/y))$ on the set Γ_d. We employ the function δ on $\mathbb{R}$ defined by

$$\delta(x) = \begin{cases} q e^{-(1-x^2)^{-1}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \tag{46}$$

where

$$q = \left(\int_{-1}^{1} e^{-(1-x^2)^{-1}} dx \right)^{-1}. \tag{47}$$

It is easy to see that $\delta(x)$ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Let f be a locally integrable function and $\delta_t(x) := t^{-1}\delta(x/t)$, $t > 0$. Then for each $t > 0$,

$$f * \delta_t(x) = \int_{-\infty}^{\infty} f(y)\, \delta_t(x-y)\, dy \tag{48}$$

is a smooth function of $x \in \mathbb{R}$ and $f * \delta_t(x) \to f(x)$ for almost every $x \in \mathbb{R}$ as $t \to 0^+$.

Now we prove the Hyers-Ulam stability of the functional equation (4) in L^∞-sense on the sector Γ_d.

Theorem 5. *Let f, g, h be locally integrable functions satisfying (45). Then there exist constants $c_1, c_2, c_3, a \in \mathbb{C}$ such that*

$$\begin{aligned} \| f(x) - c_1 - a \ln x \|_{L^\infty(R^+)} &\leq 4\epsilon, \\ \| g(x) - c_2 - a \ln x \|_{L^\infty(R^+)} &\leq 4\epsilon, \\ \| h(x) - c_3 - a \ln x \|_{L^\infty(R^+)} &\leq 4\epsilon. \end{aligned} \tag{49}$$

Proof. We will use the diffeomorphism

$$J(x, y) = \left(\ln xy, \ln \frac{x+y}{xy} \right). \tag{50}$$

Let $u = \ln xy$, $v = \ln((x+y)/xy)$ and $y/x = t > 1$. Then, we have

$$\begin{aligned} u + 2v &= \ln xy + 2 \ln \frac{x+y}{xy} \\ &= \ln \left(2 + \frac{x}{y} + \frac{y}{x} \right) \\ &= \ln \left(2 + t + \frac{1}{t} \right). \end{aligned} \tag{51}$$

Thus, we have $J(\Gamma_d) := U_d = \{(u, v) : u + 2v > \ln(2 + d + 1/d)\}$. Consequently, (45) is converted to

$$\| f(e^{u+v}) - g(e^u) - h(e^v) \|_{L^\infty(U_d)} \leq \epsilon. \tag{52}$$

Now, let

$$F(u) = f(e^u), \qquad G(u) = g(e^u), \qquad H(u) = h(e^u). \tag{53}$$

Then, we have

$$\| F(u+v) - G(u) - H(v) \|_{L^\infty(U_d)} \leq \epsilon. \tag{54}$$

For each $x, y \in \mathbb{R}$ and $t, s > 0$, we have

$$\begin{aligned}\iint_{-\infty}^{\infty} F(u+v)\,\delta_t(x-u)\,\delta_s(y-v)\,du\,dv &= \int_{-\infty}^{\infty} F(u)\left(\int_{-\infty}^{\infty}\delta_t(x-u+v)\,\delta_s(y-v)\,dv\right)du \\ &= \int_{-\infty}^{\infty} F(u)\left(\int_{-\infty}^{\infty}\delta_t(v)\,\delta_s(x+y-u-v)\,dv\right)du \\ &= \int_{-\infty}^{\infty} F(u)\,(\delta_t*\delta_s)(x+y-u)\,du \\ &= F*\delta_t*\delta_s(x+y).\end{aligned} \tag{55}$$

We also have

$$\begin{aligned}\iint_{-\infty}^{\infty} G(u)\,\delta_t(x-u)\,\delta_s(y-v)\,du\,dv &= \int_{-\infty}^{\infty} G(u)\,\delta_t(x-u)\left(\int_{-\infty}^{\infty}\delta_s(y-v)\,dv\right)du \\ &= \int_{-\infty}^{\infty} G(u)\,\delta_t(x-u)\,du \\ &= G*\delta_t(x).\end{aligned} \tag{56}$$

Similarly, we have

$$\iint_{-\infty}^{\infty} H(v)\,\delta_t(x-u)\,\delta_s(y-v)\,du\,dv = H*\delta_s(y). \tag{57}$$

On the other hand, let $x+2y > 3+\ln(2+d+1/d)$ and $0 < t < 1$, $0 < s < 1$. Then, we have

$$\begin{aligned}&\operatorname{supp}(\delta_t(x-u)\,\delta_s(y-v)) \\ &\quad = \{(u,v) : x-t \le u \le x+t,\ y-s \le v \le y+s\} \subset U_d.\end{aligned} \tag{58}$$

Let $d' = \ln(2+d+1/d)$. Then it follows from (54)~(58) that

$$\begin{aligned}&\left|F*\delta_t*\delta_s(x+y) - G*\delta_t(x) - H*\delta_s(y)\right| \\ &\quad = \left|\iint_{-\infty}^{\infty}(F(u+v)-G(u)-H(v)) \times \delta_t(x-u)\,\delta_s(y-v)\,du\,dv\right| \\ &\quad = \left|\int_{-\infty}^{\infty}\int_{d'-2v}^{\infty}(F(u+v)-G(u)-H(v)) \times \delta_t(x-u)\,\delta_s(y-v)\,du\,dv\right| \\ &\quad = \int_{-\infty}^{\infty}\int_{d'-2v}^{\infty}|F(u+v)-G(u)-H(v)| \times \left|\delta_t(x-u)\,\delta_s(y-v)\right|du\,dv \\ &\quad \le \epsilon\iint_{-\infty}^{\infty}\left|\delta_t(x-u)\,\delta_s(y-v)\right|du\,dv = \epsilon.\end{aligned} \tag{59}$$

Thus, we have the functional inequality

$$\left|F*\delta_t*\delta_s(x+y) - G*\delta_t(x) - H*\delta_s(y)\right| \le \epsilon \tag{60}$$

for all $x + 2y > d_1 := 3 + \ln(2 + d + 1/d)$ and $0 < t < 1$, $0 < s < 1$. From now on, we assume that $0 < t < 1$, $0 < s < 1$. From (60), we have

$$\left|F*\delta_t*\delta_s(x+y+z) - G*\delta_t(x+y) - H*\delta_s(z)\right| \le \epsilon \tag{61}$$

for $x + y + 2z > d_1$,

$$\left|F*\delta_t*\delta_s(x+y+z) - G*\delta_t(x) - H*\delta_s(y+z)\right| \le \epsilon \tag{62}$$

for $x + 2y + 2z > d_1$,

$$\left|F*\delta_t*\delta_s(y+z) - G*\delta_t(y) - H*\delta_s(z)\right| \le \epsilon \tag{63}$$

for $y + 2z > d_1$,

$$\left|F*\delta_t*\delta_s(y+z) - G*\delta_t(0) - H*\delta_s(y+z)\right| \le \epsilon \tag{64}$$

for $2y + 2z > d_1$.

For given $x, y \in \mathbb{R}$, choose $z > (1/2)(d_1 + |x| + 2|y|)$. Then, using the triangle inequality with (61)~(64), we have

$$\left|G*\delta_t(x+y) - G*\delta_t(x) - G*\delta_t(y) + G*\delta_t(0)\right| \le 4\epsilon \tag{65}$$

for all $x, y \in \mathbb{R}$. Replacing (x,t) by (y,s), (y,s) by (x,t) in (60) and changing the roles of G and H, we have

$$\left|H*\delta_t(x+y) - H*\delta_t(x) - H*\delta_t(y) + H*\delta_t(0)\right| \le 4\epsilon \tag{66}$$

for all $x, y \in \mathbb{R}$. Now we prove that

$$\left|F*\delta_t(x+y) - F*\delta_t(x) - F*\delta_t(y) + F*\delta_t(0)\right| \le 4\epsilon \tag{67}$$

for all $x, y \in \mathbb{R}$. From (60), we have

$$\begin{aligned}&\left|F*\delta_t*\delta_s(x+y) - G*\delta_t(z) - H*\delta_s(x+y-z)\right| \le \epsilon, \\ &\left|F*\delta_t*\delta_s(x) - G*\delta_t(z-y) - H*\delta_s(x+y-z)\right| \le \epsilon, \\ &\left|F*\delta_t*\delta_s(y) - G*\delta_t(z) - H*\delta_s(y-z)\right| \le \epsilon, \\ &\left|F*\delta_t*\delta_s(0) - G*\delta_t(z-y) - H*\delta_s(y-z)\right| \le \epsilon,\end{aligned} \tag{68}$$

for all x, y, z such that $2x + 2y - z > d_1$, $2x + y - z > d_1$, $2y - z > d_1$, and $y - z > d_1$. For given $x, y \in \mathbb{R}$, choose $z \le -d_1 - 2|x| - 2|y|$. Using the triangle inequality with (68), we have

$$\begin{aligned}&\left|F*\delta_t*\delta_s(x+y) - F*\delta_t*\delta_s(x) - F*\delta_t*\delta_s(y)\right. \\ &\quad \left.+F*\delta_t*\delta_s(0)\right| \le 4\epsilon.\end{aligned} \tag{69}$$

Letting $s \to 0^+$ in (69), we get (67).

Applying Hyers' stability theorem from [3] for (65), (66), and (67), we obtain that for each $0 < t < 1$ there exist functions $A_j(\cdot, t)$, $j = 1, 2, 3$, satisfying

$$A_j(x+y,t) = A_j(x,t) + A_j(y,t), \quad x, y \in \mathbb{R}, \tag{70}$$

for which

$$\left|F * \delta_t(x) - A_1(x,t) - F * \delta_t(0)\right| \le 4\epsilon, \tag{71}$$

$$\left|G * \delta_t(x) - A_2(x,t) - G * \delta_t(0)\right| \le 4\epsilon, \tag{72}$$

$$\left|H * \delta_t(x) - A_3(x,t) - H * \delta_t(0)\right| \le 4\epsilon, \tag{73}$$

for all $x \in \mathbb{R}$.

Now we prove that $A_1 = A_2 = A_3$. From (60), using the triangle inequality we have

$$\left|G * \delta_t(x)\right| \le \epsilon + \left|F * \delta_t * \delta_s(x+y)\right| + \left|H * \delta_s(y)\right| \tag{74}$$

for all $x + 2y > d_1$. Since $F * \delta_t * \delta_s(x) \to F * \delta_s(x)$ as $t \to 0^+$, in view of (74) it is easy to see that

$$\widetilde{G}(x) := \limsup_{t \to 0^+} G * \delta_t(x) \tag{75}$$

exists for all $x \in \mathbb{R}$. Similarly, we can show that

$$\widetilde{H}(x) := \limsup_{s \to 0^+} H * \delta_s(x) \tag{76}$$

exists for all $x \in \mathbb{R}$. Putting $y = 0$ in (60) and letting $s \to 0^+$ so that $H * \delta_s(0) \to \widetilde{H}(0)$ we have

$$\left|F * \delta_t(x) - G * \delta_t(x) - \widetilde{H}(0)\right| \le \epsilon \tag{77}$$

for all $x > d_1$. Similarly, we have

$$\left|F * \delta_t(x) - H * \delta_t(x) - \widetilde{G}(0)\right| \le \epsilon \tag{78}$$

for all $x > (d_1/2)$. Using (71), (72), (77), and the triangle inequality, we have

$$\begin{aligned}\left|A_1(x,t) - A_2(x,t)\right| &\le 9\epsilon + \left|F * \delta_t(0) - G * \delta_t(0) - \widetilde{H}(0)\right| \\ &:= M(t)\end{aligned} \tag{79}$$

for all $x > d_1$. From (71) and (80), we have

$$\begin{aligned}\left|A_1(x,t) - A_2(x,t)\right| &= \frac{1}{|k|}\left|A_1(kx,t) - A_2(kx,t)\right| \\ &\le \frac{1}{|k|} M(t)\end{aligned} \tag{80}$$

for all $x \in \mathbb{R}$, $x \neq 0$, and all integers k with $kx > d_1$. Letting $k \to \infty$ if $x > 0$ and letting $k \to -\infty$ if $x < 0$ in (80), we have $A_1(x,t) = A_2(x,t)$ for $x \neq 0$, which implies $A_1 = A_2$ since $A_1(0,t) = A_2(0,t) = 0$. Similarly, using (71), (73), and (78) we obtain that $A_1 = A_3$.

Finally, we prove that A_1 is independent of t. Fixing $x \in \mathbb{R}$ and letting $t \to 0^+$ so that $G * \delta_t(x) \to \widetilde{G}(x)$ in (60), we have

$$\left|F * \delta_s(x+y) - \widetilde{G}(x) - H * \delta_s(y)\right| \le \epsilon \tag{81}$$

for all $x + 2y > d_1$. From (81), using the same substitutions as in (61)~(64) we have

$$\left|\widetilde{G}(x+y) - \widetilde{G}(x) - \widetilde{G}(y) + \widetilde{G}(0)\right| \le 4\epsilon. \tag{82}$$

By Hyers' stability theorem [3], there exists a unique function A satisfying the Cauchy functional equation

$$A(x+y) - A(x) - A(y) = 0 \tag{83}$$

for which

$$\left|\widetilde{G}(x) - A(x) - \widetilde{G}(0)\right| \le 4\epsilon. \tag{84}$$

Now we show that $A_1(x,t) = A(x)$ for all $x \in \mathbb{R}$ and $0 < t < 1$. Putting $y = 0$ in (81), we have

$$\left|F * \delta_s(x) - \widetilde{G}(x) - H * \delta_s(0)\right| \le \epsilon \tag{85}$$

for all $x > d_1$. From (71), (84), and (85), using the triangle inequality we have

$$\left|A_1(x,t) - A(x)\right| \le 9\epsilon + \left|F * \delta_t(0) - H * \delta_t(0) - \widetilde{G}(0)\right| \tag{86}$$

for all $x > d_1$. From (86), using the method of proving $A_1 = A_2$ we can show that $A_1(x,t) = A(x)$ for all $x \in \mathbb{R}$ and $0 < t < 1$. Thus, we have $A_1 = A_2 = A_3 := A$.

Letting $t \to 0^+$ in (72) so that $G * \delta_t(0) \to \widetilde{G}(0)$, we have

$$\left\|G(x) - A(x) - \widetilde{G}(0)\right\|_{L^\infty} \le 4\epsilon. \tag{87}$$

Similarly, letting $t \to 0^+$ in (73) so that $H * \delta_t(0) \to \widetilde{H}(0)$, we have

$$\left\|H(x) - A(x) - \widetilde{H}(0)\right\|_{L^\infty} \le 4\epsilon. \tag{88}$$

Now we prove the inequality

$$\left\|F(x) - A(x) - \widetilde{F}(0)\right\|_{L^\infty} \le 4\epsilon. \tag{89}$$

For given $x \in \mathbb{R}$, choosing z such that $x + z > d_1$ replacing x by $x - z$ and y by z in (81), and using the triangle inequality, we have

$$\left|F * \delta_s(x)\right| \le \epsilon + \left|\widetilde{G}(x-z) + H * \delta_s(z)\right|. \tag{90}$$

From (90), it is easy to see that

$$\widetilde{F}(x) := \limsup_{s \to 0^+} F * \delta_s(x) \tag{91}$$

exists for all $x \in \mathbb{R}$. Letting $t \to 0^+$ in (71) so that $F * \delta_t(0) \to \widetilde{F}(0)$, we get (89). Replacing x by $\ln x$ in (87), (88), and (89), we have

$$\begin{aligned}&\left\|f(x) - A(\ln x) - \widetilde{F}(0)\right\|_{L^\infty(\mathbb{R}^+)} \le 4\epsilon, \\ &\left\|g(x) - A(\ln x) - \widetilde{G}(0)\right\|_{L^\infty(\mathbb{R}^+)} \le 4\epsilon, \\ &\left\|h(x) - A(\ln x) - \widetilde{H}(0)\right\|_{L^\infty(\mathbb{R}^+)} \le 4\epsilon.\end{aligned} \tag{92}$$

Finally, we show that the solution A of the Cauchy equation (83) has the form $A(x) = cx$ for some $c \in \mathbb{C}$. Since $\widetilde{G}$ is the supremum limit of a collection of continuous functions $G * \delta_t$, $0 < t < 1$, $\widetilde{G}$ is a Lebesgue measurable function. Also, as we see in the proof of Hyers-Ulam stability theorem (see [3]), the function A is given by

$$A(x) = \lim_{n\to\infty} 2^{-n}\widetilde{G}(2^n x). \tag{93}$$

Thus, A is a Lebesgue measurable function since it is the limit of a sequence of Lebesgue measurable functions. It is well known that every Lebesgue measurable solution A of the Cauchy functional equation (83) has the form $A(x) = ax$ for some $a \in \mathbb{C}$. Letting $c_1 = \widetilde{F}(0)$, $c_2 = \widetilde{G}(0)$, $c_3 = \widetilde{H}(0)$ we get the asserted result. □

Now we discuss an asymptotic behavior of the inequality (45).

Theorem 6. *Let $f, g, h : \mathbb{R}^+ \to \mathbb{C}$, $j = 1, 2, 3$, be locally integrable functions satisfying*

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right)\right\|_{L^\infty(\Gamma_d)} \longrightarrow 0 \tag{94}$$

as $d \to \infty$. Then there exist constants $a, c_1, c_2, c_3 \in \mathbb{C}$ such that

$$\begin{aligned}\|f(x) - c_1 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0,\\ \|g(x) - c_2 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0,\\ \|h(x) - c_3 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0.\end{aligned} \tag{95}$$

Proof. By the condition (94), for any positive integer n there exists $d_n > 1$ such that

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right)\right\|_{L^\infty(\Gamma_d)} \le \frac{1}{n} \tag{96}$$

for all $x, y > 0$ with $(y/x) > d_n$. Now by Theorem 5, there exist constants $a, c_1, c_2, c_3 \in \mathbb{C}$ (which are independent of n) such that

$$\begin{aligned}\|f(x) - c_1 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &\le \frac{4}{n},\\ \|g(x) - c_2 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &\le \frac{4}{n},\\ \|h(x) - c_3 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &\le \frac{4}{n}.\end{aligned} \tag{97}$$

Letting $n \to \infty$ in (97), we obtain the asserted result. □

As a direct consequence of the previous result we have found the solution of functional equation (4) in the L^∞-sense.

Corollary 7. *Let $f, g, h : \mathbb{R}^+ \to \mathbb{C}$ be locally integrable functions satisfying*

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right)\right\|_{L^\infty(\Gamma_d)} = 0 \tag{98}$$

for all $x, y \in \mathbb{R}_+$. Then there exist $a, c_1, c_2, c_3 \in \mathbb{C}$ such that

$$\begin{aligned}\|f(x) - c_1 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0,\\ \|g(x) - c_2 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0,\\ \|h(x) - c_3 - a\ln x\|_{L^\infty(\mathbb{R}^+)} &= 0.\end{aligned} \tag{99}$$

Finally, we discuss the locally integrable solution $f, g, h : \mathbb{R}^+ \to \mathbb{C}$ of the functional equation (c.f. [16])

$$f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) = 0 \tag{100}$$

for all $(x, y) \in \Gamma_d$. The following result is a direct consequence of Theorem 2. However, we introduce an alternative proof using Corollary 7. The following method of proof will be useful when we know only regular solution in L^∞-sense.

Corollary 8. *Every locally integrable solution $f, g, h : \mathbb{R}^+ \to \mathbb{C}$ of the functional equation (100) has the form*

$$f(x) = c_1 + c_2 + a\ln x, \tag{101}$$

$$g(x) = c_1 + a\ln x, \tag{102}$$

$$h(x) = c_2 + a\ln x \tag{103}$$

for some constants $a, c_1, c_2 \in \mathbb{C}$.

Proof. It follows from Corollary 7 that (101), (102), and (103) hold in almost everywhere sense; that is, there exists a subset $\Omega \subset \mathbb{R}^+$ with Lebesgue measure $m(\Omega^c) = 0$ such that (101), (102), and (103) hold for all $x \in \Omega$. For given $x > 0$, let $p, q : (0, x) \to \mathbb{R}$ by $p(t) = (1/t) + (1/(x-t))$, $q(t) = t(x-t)$. Since $m[(p^{-1}(\Omega) \cap q^{-1}(\Omega))^c] = m[p^{-1}(\Omega^c) \cup q^{-1}(\Omega^c)] = 0$, we can choose $y \in p^{-1}(\Omega) \cap q^{-1}(\Omega)$. Let $z = x - y$. Then $y + z = x$ and $yz, (1/y) + (1/z) \in \Omega$. Thus, we can write

$$\begin{aligned}f(x) &= g(yz) + h\left(\frac{1}{y} + \frac{1}{z}\right)\\ &= c_1 + a\ln(yz) + c_2 + a\ln\left(\frac{1}{y} + \frac{1}{z}\right)\\ &= c_1 + c_2 + a\ln(y+z) = c_1 + c_2 + a\ln x,\end{aligned} \tag{104}$$

which gives (101). For given $x > 0$, let $p : \mathbb{R}^+ \to \mathbb{R}$ by $p(t) = (1/t) + (t/x)$. Then, we have $p^{-1}(\Omega) \neq \emptyset$. Choose $y \in p^{-1}(\Omega)$ and let $z = (x/y)$. Then $yz = x$, $(1/y) + (1/z) \in \Omega$. Thus, using (101) we can write

$$\begin{aligned}g(x) &= f(y+z) - h\left(\frac{1}{y} + \frac{1}{z}\right)\\ &= c_1 + c_2 + a\ln(y+z) - c_2 - a\ln\left(\frac{1}{y} + \frac{1}{z}\right)\\ &= c_1 + a\ln(yz) = c_1 + a\ln x,\end{aligned} \tag{105}$$

which gives (102). Finally, (103) follows from (100), (101), and (102). This completes the proof of the corollary. □

4. Stability of (4) in Schwartz Distributions

Let Ω be an open subset of $\mathbb{R}^n$. We briefly introduce the space $\mathcal{D}'(\Omega)$ of distributions. We denote $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = (\partial/\partial x_j)$, $j = 1, 2, \ldots, n$.

Definition 9. Let $C_c^\infty(\Omega)$ be the set of all infinitely differentiable functions on Ω with compact supports. A distribution u is a linear form on $C_c^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ for which

$$\left|\langle u, \varphi\rangle\right| \le C \sum_{|\alpha|\le k} \sup \left|\partial^\alpha \varphi\right| \tag{106}$$

holds for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.

Let Ω_j be open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$, with $n_1 \ge n_2$.

Definition 10. Let $u_j \in \mathcal{D}'(\Omega_j)$ and let $\lambda : \Omega_1 \to \Omega_2$ be a smooth function such that for each $x \in \Omega_1$ the derivative $\lambda'(x)$ is surjective; that is, the Jacobian matrix $\nabla\lambda$ of λ has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. We call $\lambda^* u$ the pullback of u by λ and it is usually denoted by $u \circ \lambda$.

If λ is a diffeomorphism (a bijection with λ, λ^{-1} smooth functions) the pullback $u \circ \lambda$ can be written as

$$\langle u \circ \lambda, \varphi\rangle = \left\langle u, \left(\varphi \circ \lambda^{-1}\right)(x) \left|\nabla \lambda^{-1}(x)\right|\right\rangle. \tag{107}$$

For more details of distributions we refer the reader to [29, 33].

In this section, we consider the Hyers-Ulam stability of the functional equation of (4) in Schwartz distributions, that is, the functional inequality

$$\|u \circ S - v \circ \Pi - w \circ R\|_{\Gamma_d} \le \epsilon, \tag{108}$$

where $u, v, w \in \mathcal{D}'(\mathbb{R}^+)$, $\Pi : \mathbb{R}^2 \to \mathbb{R}$, and $R : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$S(x, y) = x + y, \qquad \Pi(x, y) = xy, \qquad R(x, y) = \frac{1}{x} + \frac{1}{y} \tag{109}$$

and the inequality $\|\cdot\|_{\Gamma_d} \le \epsilon$ in (108) means that $|\langle \cdot, \varphi\rangle| \le \epsilon\|\varphi\|_{L^1}$ for all test functions $\varphi \in C_c^\infty(\Gamma_d)$. For each $t > 0$, $u * \delta_t(x) = \langle u_y, \delta_t(x - y)\rangle$ is a smooth function of $x \in \mathbb{R}^n$ and $u * \delta_t(x) \to u$ as $t \to 0^+$ in the sense that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} (u * \delta_t)(x)\varphi(x)\, dx = \langle u, \varphi\rangle \tag{110}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Theorem 11. *Let $u, v, w \in \mathcal{D}'(\mathbb{R}^+)$ satisfy* (108). *Then there exist constants $a, c_1, c_2, c_3 \in \mathbb{C}$ such that*

$$\begin{aligned} \|u - c_1 - a\ln x\| &\le 4\epsilon, \\ \|v - c_2 - a\ln x\| &\le 4\epsilon, \\ \|w - c_3 - a\ln x\| &\le 4\epsilon. \end{aligned} \tag{111}$$

Proof. The idea of the following proof is essentially the same as that of Theorem 5, only with different terminologies. For the reader we give a sketch of proof. Let u_d and $J : \Gamma_d \to U_d$ be the set and mapping in the proof of Theorem 5, respectively. Then, $J^{-1} : U_d \to \Gamma_d$ is given by

$$J^{-1}(x, y) = \left(\frac{e^{x+y} + \sqrt{e^{2x+2y} - 4e^x}}{2}, \frac{e^{x+y} - \sqrt{e^{2x+2y} - 4e^x}}{2}\right). \tag{112}$$

Taking pullback by J^{-1} in (108), we have

$$\|u \circ E \circ S - v \circ E \circ P_1 - w \circ E \circ P_2\|_{U_d} \le \epsilon, \tag{113}$$

where $E : \mathbb{R} \to \mathbb{R}$, $S, P_1, P_2 : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\begin{aligned} E(x) &= e^x, \qquad S(x, y) = x + y, \\ P_1(x, y) &= x, \qquad P_2(x, y) = y. \end{aligned} \tag{114}$$

Thus, instead of (54) we have the inequality

$$\|\widetilde{u} \circ S - \widetilde{v} \circ P_1 - \widetilde{w} \circ P_2\|_{U_d} \le \epsilon, \tag{115}$$

where $\widetilde{u} = u \circ E$, $\widetilde{v} = v \circ E$, and $\widetilde{w} = w \circ E$. Using the same approach as in the proof of Theorem 5, we have

$$\begin{aligned} \|\widetilde{u} - cx - c_1\| &\le 4\epsilon, \\ \|\widetilde{v} - cx - c_2\| &\le 4\epsilon, \\ \|\widetilde{w} - cx - c_3\| &\le 4\epsilon \end{aligned} \tag{116}$$

for some $c \in \mathbb{C}$. Taking pullback by $E^{-1}(x) = \ln x$ in (116), we have

$$\begin{aligned} \|u - a\ln x - c_1\| &\le 4\epsilon, \\ \|v - a\ln x - c_2\| &\le 4\epsilon, \\ \|w - a\ln x - c_3\| &\le 4\epsilon, \end{aligned} \tag{117}$$

for some constants $a, c_1, c_2, c_3 \in \mathbb{C}$. This completes the proof of the theorem. □

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