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Research Article

Sequential Derivatives of Nonlinear q-Difference Equations with Three-Point q-Integral Boundary Conditions

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This paper studies sufficient conditions for the existence of solutions to the problem of sequential derivatives of nonlinear q-difference equations with three-point q-integral boundary conditions. Our results are concerned with several quantum numbers of derivatives and integrals. By using Banach's contraction mapping, Krasnoselskii's fixed-point theorem, and Leray-Schauder degree theory, some new existence results are obtained. Two examples illustrate our results.

1. Introduction

The study of q-calculus or quantum calculus was initiated by the pioneer works of Jackson [1], Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5], and so forth. Since then, in the last few decades, this subject has evolved into a multidisciplinary research area with many applications; for example, see [6–14]. For some recent works, we refer the reader to [15–21] and references therein. However, the theory of boundary value problems for nonlinear q-difference equations is still in the beginning stages and it needs to be explored further.

In [22], Ahmad investigated the existence of solutions for a nonlinear boundary value problem of third-order q-difference equation:

$$\begin{gathered} D_q^3 u(t) = f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = 0, \qquad D_q u(0) = 0, \qquad u(1) = 0. \end{gathered} \tag{1}$$

Using Leray-Schauder degree theory and standard fixed-point theorems, some existence results were obtained. Moreover, he showed that if $q \to 1$, then his results corresponded to the classical results. Ahmad et al. [23] studied a boundary value problem of a nonlinear second-order q-difference equation with nonseparated boundary conditions

$$\begin{gathered} D_q^2 u(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T). \end{gathered} \tag{2}$$

They proved the existence and uniqueness theorems of the problem (2) using the Leray-Schauder nonlinear alternative and some standard fixed-point theorems. For some very recent results on nonlocal boundary value problems of nonlinear q-difference equations and inclusions, see [24–26].

In this paper, we discuss the existence of solutions for the following nonlinear q-difference equation with three-point integral boundary condition:

$$\begin{gathered} D_q \left(D_p + \lambda\right) x(t) = f(t, x(t)), \quad t \in [0, T], \\ x(0) = 0, \qquad \beta \int_0^\eta x(s)\, d_r s = x(T), \end{gathered} \tag{3}$$

where $0 < p, q, r < 1$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $\beta \neq T(1 + r)/\eta^2$, $\eta \in (0, T)$ is a fixed point, and λ is a given constant.

The aim of this paper is to prove some existence and uniqueness results for the boundary value problem (3). Our results are based on Banach's contraction mapping, Krasnoselskii's fixed-point theorem, and Leray-Schauder degree theory. Since the problem (3) has different values of the quantum numbers of the q-derivative and the q-integral, the existence results of such problem are also new.

2. Preliminaries

Let us recall some basic concepts of quantum calculus [15].

For $0 < q < 1$, we define the q-derivative of a real-valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t\to 0} D_q f(t). \quad (4)$$

The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \quad (5)$$

The q-integral of a function f defined on the interval $[0, T]$ is given by

$$\int_a^t f(s)\, d_q s := \sum_{n=0}^{\infty} (1-q) q^n \times [t f(tq^n) - a f(q^n a)], \quad t \in [0, T], \quad (6)$$

and for $a = 0$, we denote

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n), \quad (7)$$

provided the series converges. If $a \in [0, T]$ and f is defined on the interval $[0, T]$, then

$$\int_a^b f(s)\, d_q s = \int_0^b f(s)\, d_q s - \int_0^a f(s)\, d_q s. \quad (8)$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}. \quad (9)$$

Observe that

$$D_q I_q f(t) = f(t), \quad (10)$$

and if f is continuous at $t = 0$, then

$$I_q D_q f(t) = f(t) - f(0). \quad (11)$$

In q-calculus, the product rule and integration by parts formula are

$$D_q (gh)(t) = (D_q g(t)) h(t) + g(qt) D_q h(t),$$
$$\int_0^t f(s) D_q g(s)\, d_q s = [f(s) g(s)]_0^t - \int_0^t D_q f(s) g(qs)\, d_q s. \quad (12)$$

In the limit $q \to 1$, the q-calculus corresponds to the classical calculus. The above results are also true for quantum numbers p, r such that $0 < p < 1$ and $0 < r < 1$.

Lemma 1. *Let $T(1+r) \neq \beta\eta^2$, $0 < p, q, r < 1$, and let λ be a constant. Then for any $h \in C[0, T]$, the boundary value problem*

$$D_q (D_p + \lambda) x(t) = h(t), \quad t \in [0, T], \quad (13)$$

$$x(0) = 0, \qquad \beta \int_0^\eta x(s)\, d_r s = x(T), \quad 0 < \eta < T, \quad (14)$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = {} & \int_0^t \int_0^s h(u)\, d_q u\, d_p s - \lambda \int_0^t x(s)\, d_p s \\ & + \frac{\beta(1+r)t}{T(1+r) - \beta\eta^2} \\ & \times \int_0^\eta \int_0^v \left(\int_0^s h(u)\, d_q u - \lambda x(s) \right) d_p s\, d_r v \\ & + \frac{\lambda(1+r)t}{T(1+r) - \beta\eta^2} \int_0^T x(s)\, d_p s \\ & - \frac{(1+r)t}{T(1+r) - \beta\eta^2} \int_0^T \int_0^s h(u)\, d_q u\, d_p s. \end{aligned} \quad (15)$$

Proof. For $t \in [0, T]$, q-integrating (13) from 0 to t, we obtain

$$(D_p + \lambda) x(t) = \int_0^t h(s)\, d_q s + c_1. \quad (16)$$

Equation (16) can be written as

$$D_p x(t) = \int_0^t h(s)\, d_q s - \lambda x(t) + c_1. \quad (17)$$

For $t \in [0, T]$, p-integrating (17) from 0 to t, we have

$$\begin{aligned} x(t) = {} & \int_0^t \int_0^s h(u)\, d_q u\, d_p s \\ & - \lambda \int_0^t x(s)\, d_p s + c_1 t + c_2. \end{aligned} \quad (18)$$

From the first condition of (14), it follows that $c_2 = 0$. For $t \in [0, T]$, r-integrating equation (18) from 0 to t, we get

$$\begin{aligned} \int_0^t x(v)\, d_r v = {} & \int_0^t \int_0^v \int_0^s h(u)\, d_q u\, d_p s\, d_r v \\ & - \lambda \int_0^t \int_0^v x(s)\, d_p s\, d_r v + c_1 \frac{t^2}{1+r}. \end{aligned} \quad (19)$$

The second boundary condition (14) implies that

$$\begin{aligned}\beta \int_0^\eta x(v)\, d_r v &= \beta \int_0^\eta \int_0^v \int_0^s h(u)\, d_q u d_p s d_r v \\ &\quad - \beta\lambda \int_0^\eta \int_0^v x(s)\, d_p s d_r v + c_1 \frac{\beta\eta^2}{1+r} \\ &= \beta \int_0^\eta \int_0^v \left(\int_0^s h(u)\, d_q u - \lambda x(s) \right) d_p s d_r v + c_1 \frac{\beta\eta^2}{1+r} \\ &= \int_0^T \int_0^s h(u)\, d_q u d_p s - \lambda \int_0^T x(s)\, d_p s + c_1 T.\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}c_1 &= \frac{\beta(1+r)}{T(1+r)-\beta\eta^2} \int_0^\eta \int_0^v \left(\int_0^s h(u)\, d_q u - \lambda x(s) \right) d_p s d_r v \\ &\quad + \frac{\lambda(1+r)}{T(1+r)-\beta\eta^2} \int_0^T x(s)\, d_p s \\ &\quad - \frac{1+r}{T(1+r)-\beta\eta^2} \int_0^T \int_0^s h(u)\, d_q u d_p s.\end{aligned} \tag{21}$$

Substituting the values of c_1 and c_2 in (18), we obtain (15). This completes the proof. □

For the forthcoming analysis, let $\mathscr{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$.

In the following, for the sake of convenience, we set

$$\begin{aligned}\Omega &= \frac{1}{1+p} \\ &\quad \times \left(T^2 + \frac{|\beta| T\eta^3 (1+r)}{|T(1+r)-\beta\eta^2| (1+r+r^2)} \right. \\ &\quad \left. + \frac{T^3(1+r)}{|T(1+r)-\beta\eta^2|} \right),\end{aligned} \tag{22}$$

$$\Phi = |\lambda| T + \frac{|\beta| |\lambda| T\eta^2}{|T(1+r)-\beta\eta^2|} + \frac{|\lambda| T^2 (1+r)}{|T(1+r)-\beta\eta^2|}. \tag{23}$$

3. Main Results

Now, we are in the position to establish the main results. We transform the boundary value problem (3) into a fixed-point problem. In view of Lemma 1, for $t \in [0,T]$, $x \in \mathscr{C}$, we define the operator $A : \mathscr{C} \to \mathscr{C}$ as

$$\begin{aligned}(Ax)(t) &= \int_0^t \int_0^s f(u, x(u))\, d_q u d_p s - \lambda \int_0^t x(s)\, d_p s \\ &\quad + \frac{\beta(1+r)t}{T(1+r)-\beta\eta^2} \\ &\quad \times \int_0^\eta \int_0^v \left(\int_0^s f(u, x(u))\, d_q u - \lambda x(s) \right) d_p s d_r v \\ &\quad + \frac{\lambda(1+r)t}{T(1+r)-\beta\eta^2} \int_0^T x(s)\, d_p s \\ &\quad - \frac{(1+r)t}{T(1+r)-\beta\eta^2} \int_0^T \int_0^s f(u, x(u))\, d_q u d_p s.\end{aligned} \tag{24}$$

Note that the problem (3) has solutions if and only if the operator equation $Ax = x$ has fixed points.

Our first result is based on Banach's fixed-point theorem.

Theorem 2. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying the conditions*

(H_1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all* $t \in [0,T]$, $x, y \in \mathbb{R}$;

(H_2) $\Lambda := (\Phi + L\Omega) < 1$,

where L is a Lipschitz constant, Ω and Φ are defined by (22) *and* (23), *respectively.*

Then, the boundary value problem (3) *has a unique solution.*

Proof. Assume that $\sup_{t\in[0,T]} |f(t,0)| = M_0$; we choose a constant

$$R \ge \frac{M_0 \Omega}{1-\Lambda}. \tag{25}$$

Now, we will show that $AB_R \subset B_R$, where $B_R = \{x \in \mathscr{C} : \|x\| \le R\}$. For any $x \in B_R$, we have

$$\begin{aligned}\|(Ax)\| &= \sup_{t\in[0,T]} \left| \int_0^t \int_0^s f(u, x(u))\, d_q u d_p s - \lambda \int_0^t x(s)\, d_p s \right. \\ &\quad + \frac{\beta(1+r)t}{T(1+r)-\beta\eta^2} \\ &\quad \times \int_0^\eta \int_0^v \left(\int_0^s f(u, x(u))\, d_q u - \lambda x(s) \right) d_p s d_r v \\ &\quad + \frac{\lambda(1+r)t}{T(1+r)-\beta\eta^2} \int_0^T x(s)\, d_p s \\ &\quad \left. - \frac{(1+r)t}{T(1+r)-\beta\eta^2} \int_0^T \int_0^s f(u, x(u))\, d_q u d_p s \right|\end{aligned}$$

$$\leq \sup_{t\in[0,T]}\left\{\int_0^t\int_0^s\left(\left|f(u,x(u))-f(u,0)\right|+\left|f(u,0)\right|\right)d_qud_ps\right.$$
$$+|\lambda|\int_0^t|x(s)|\,d_ps$$
$$+\frac{|\beta|(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\times\left(\int_0^\eta\int_0^v\int_0^s\left(\left|f(u,x(u))-f(u,0)\right|+\left|f(u,0)\right|\right)\right.$$
$$\times d_qud_psd_rv$$
$$\left.+|\lambda|\int_0^\eta\int_0^v|x(s)|\,d_psd_rv\right)$$
$$+\frac{|\lambda|(1+r)t}{|T(1+r)-\beta\eta^2|}\int_0^T|x(s)|\,d_ps$$
$$+\frac{(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\times\int_0^T\int_0^s\left(\left|f(u,x(u))-f(u,0)\right|\right.$$
$$\left.\left.+\left|f(u,0)\right|\right)d_qud_ps\right\}$$
$$\leq \sup_{t\in[0,T]}\left\{\int_0^t\int_0^s\left(L|x(u)|+\left|f(u,0)\right|\right)d_qud_ps\right.$$
$$+|\lambda|\int_0^t|x(s)|\,d_ps+\frac{|\beta|(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\times\left(\int_0^\eta\int_0^v\int_0^s\left(L|x(u)|+\left|f(u,0)\right|\right)d_qud_psd_rv\right.$$
$$\left.+|\lambda|\int_0^\eta\int_0^v|x(s)|\,d_psd_rv\right)$$
$$+\frac{|\lambda|(1+r)t}{|T(1+r)-\beta\eta^2|}\int_0^T|x(s)|\,d_ps$$
$$+\frac{(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\left.\times\int_0^T\int_0^s\left(L|x(u)|+\left|f(u,0)\right|\right)d_qud_ps\right\}$$
$$\leq \sup_{t\in[0,T]}\left\{\left(L\|x\|+M_0\right)\int_0^t\int_0^s d_qud_ps\right.$$
$$+|\lambda|\,\|x\|\int_0^t d_ps+\frac{|\beta|(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\times\left(\left(L\|x\|+M_0\right)\int_0^\eta\int_0^v\int_0^s d_qud_psd_rv\right.$$
$$\left.+|\lambda|\,\|x\|\int_0^\eta\int_0^v d_psd_rv\right)$$
$$+\frac{|\lambda|\,\|x\|(1+r)t}{|T(1+r)-\beta\eta^2|}\int_0^T d_ps$$
$$\left.+\frac{(1+r)t}{|T(1+r)-\beta\eta^2|}\left(L\|x\|+M_0\right)\int_0^T\int_0^s d_qud_ps\right\}$$
$$=\sup_{t\in[0,T]}\left\{\left(L\|x\|+M_0\right)\frac{t^2}{1+p}+|\lambda|\,\|x\|\,t\right.$$
$$+\frac{|\beta|(1+r)t}{|T(1+r)-\beta\eta^2|}$$
$$\times\left(\left(L\|x\|+M_0\right)\frac{(1-r)\eta^3}{(1+p)(1-r^3)}+|\lambda|\,\|x\|\frac{\eta^2}{1+r}\right)$$
$$+\frac{|\lambda|\,\|x\|(1+r)t}{|T(1+r)-\beta\eta^2|}T$$
$$\left.+\frac{(1+r)t}{|T(1+r)-\beta\eta^2|}\left(L\|x\|+M_0\right)\frac{T^2}{1+p}\right\}$$
$$=\sup_{t\in[0,T]}\left\{\|x\|\left(|\lambda|t+\frac{|\beta||\lambda|\eta^2t}{|T(1+r)-\beta\eta^2|}+\frac{|\lambda|T(1+r)t}{|T(1+r)-\beta\eta^2|}\right)\right.$$
$$+\frac{\left(L\|x\|+M_0\right)}{1+p}$$
$$\times\left(t^2+\frac{|\beta|(1+r)\eta^3t}{|T(1+r)-\beta\eta^2|(1+r+r^2)}\right.$$
$$\left.\left.+\frac{T^2(1+r)t}{|T(1+r)-\beta\eta^2|}\right)\right\}$$
$$\leq R\left(|\lambda|T+\frac{|\beta||\lambda|T\eta^2}{|T(1+r)-\beta\eta^2|}+\frac{|\lambda|T^2(1+r)}{|T(1+r)-\beta\eta^2|}\right)$$
$$+\frac{(LR+M_0)}{1+p}$$
$$\times\left(T^2+\frac{|\beta|T\eta^3(1+r)}{|T(1+r)-\beta\eta^2|(1+r+r^2)}\right.$$
$$\left.+\frac{T^3(1+r)}{|T(1+r)-\beta\eta^2|}\right)$$
$$=R\Phi+(LR+M)\,\Omega\leq R. \tag{26}$$

Next, we will show that A is a contraction. For any $x, y \in \mathscr{C}$ and for each $t \in [0,T]$, we have

$$\|(Ax)-(Ay)\|$$
$$=\sup_{t\in[0,T]}\left|(Ax)(t)-(Ay)(t)\right|$$
$$=\sup_{t\in[0,T]}\left|\int_0^t\int_0^s\left(f(u,x(u))-f(u,y(u))\right)d_qud_ps\right.$$

$$
\begin{aligned}
&\quad - \lambda \int_0^t (x(s) - y(s))\, d_p s \\
&\quad + \frac{\beta(1+r)t}{T(1+r) - \beta\eta^2} \\
&\quad \times \int_0^\eta \int_0^v \left(\int_0^s (f(u, x(u)) - f(u, y(u)))\, d_q u \right. \\
&\qquad\qquad \left. - \lambda (x(s) - y(s)) \right) d_p s d_r v \\
&\quad + \frac{\lambda(1+r)t}{T(1+r) - \beta\eta^2} \int_0^T (x(s) - y(s))\, d_q s \\
&\quad - \frac{(1+r)t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \int_0^T \int_0^s (f(u, x(u)) - f(u, y(u)))\, d_q u d_p s \Bigg| \\
&\leq \sup_{t \in [0,T]} \left\{ L\|x - y\| \int_0^t \int_0^s d_q u d_p s \right. \\
&\quad + |\lambda| \|x - y\| \int_0^t d_p s \\
&\quad + \frac{|\beta|(1+r)t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \left(L\|x - y\| \int_0^\eta \int_0^v \int_0^s d_q u d_p s d_r v \right. \\
&\qquad \left. + |\lambda| \|x - y\| \int_0^\eta \int_0^v d_p s d_r v \right) \\
&\quad + \frac{|\lambda|(1+r)t}{T(1+r) - \beta\eta^2} \|x - y\| \\
&\quad \times \int_0^T d_q s + \frac{(1+r)t}{|T(1+r) - \beta\eta^2|} \\
&\quad \left. \times L\|x - y\| \int_0^T \int_0^s d_q u d_p s \right\} \\
&= \sup_{t \in [0,T]} \left\{ L\|x - y\| \frac{t^2}{1+p} + |\lambda| \|x - y\| t \right. \\
&\quad + \frac{|\beta|(1+r)t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \left(L\|x - y\| \frac{(1-r)\eta^3}{(1+p)(1-r^3)} \right. \\
&\qquad \left. + |\lambda| \|x - y\| \frac{\eta^2}{1+r} \right) \\
&\quad + |\lambda| \|x - y\| \frac{(1+r)t}{|T(1+r) - \beta\eta^2|} T \\
&\quad \left. + \frac{(1+r)t}{|T(1+r) - \beta\eta^2|} L\|x - y\| \frac{T^2}{1+p} \right\} \\
&\leq \|x - y\| \left(|\lambda| T + \frac{|\beta||\lambda| T\eta^2}{|T(1+r) - \beta\eta^2|} \right. \\
&\qquad \left. + \frac{|\lambda| T^2(1+r)}{|T(1+r) - \beta\eta^2|} \right) \\
&\quad + \frac{L\|x - y\|}{1+p} \left(T^2 + \frac{|\beta|\eta^3 T(1+r)}{|T(1+r) - \beta\eta^2|(1+r+r^2)} \right. \\
&\qquad \left. + \frac{T^3(1+r)}{|T(1+r) - \beta\eta^2|} \right) \\
&= (\Phi + L\Omega)\|x - y\| \\
&\leq \Lambda \|x - y\|.
\end{aligned}
\tag{27}
$$

Since $\Lambda < 1$, A is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Our second result is based on the following Krasnoselskii's fixed-point theorem [27].

Theorem 3. *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that*

(i) *$Ax + By \in K$ whenever $x, y \in K$;*

(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping.*

Then, there exists $z \in K$ such that $z = Az + Bz$.

Theorem 4. *Assume that (H_1) and (H_2) hold with*

(H_3) $|f(t, x)| \leq \mu(t)$, for all $(t, x) \in [0, T] \times \mathbb{R}$, with $\mu \in L^1([0, T], \mathbb{R}^+)$.

If

$$
\begin{aligned}
&\frac{|\beta||\lambda| T\eta^2}{|T(1+r) - \beta\eta^2|} + \frac{|\lambda| T^2(1+r)}{|T(1+r) - \beta\eta^2|} \\
&\quad + \frac{1}{1+p} \left(\frac{|\beta|\eta^3 T(1+r)}{|T(1+r) - \beta\eta^2|(1+r+r^2)} \right. \\
&\qquad \left. + \frac{T^3(1+r)}{|T(1+r) - \beta\eta^2|} \right) < 1,
\end{aligned}
\tag{28}
$$

then the boundary value problem (3) *has at least one solution on* $[0, T]$.

Proof. Setting $\max_{t \in [0,T]} |\mu(t)| = \|\mu\|$ and choosing a constant

$$
R \geq \frac{\|\mu\| \Omega}{1 - \Phi},
\tag{29}
$$

where Ω and Φ are given by (22) and (23), respectively, we consider that $B_R = \{x \in \mathscr{C} : \|x\| \leq R\}$.

In view of Lemma 1, we define the operators F_1 and F_2 on the set B_R as

$$\begin{aligned}(F_1x)(t) &= \int_0^t\int_0^s f(u,x(u))\,d_qu\,d_ps - \lambda\int_0^t x(s)\,d_ps,\\ (F_2x)(t) &= \frac{\beta(1+r)t}{T(1+r)-\beta\eta^2}\\ &\quad\times\int_0^\eta\int_0^\nu\left(\int_0^s f(u,x(u))\,d_qu-\lambda x(s)\right)d_ps\,d_r\nu\\ &\quad+\frac{\lambda(1+r)t}{T(1+r)-\beta\eta^2}\int_0^T x(s)\,d_ps\\ &\quad-\frac{(1+r)t}{T(1+r)-\beta\eta^2}\int_0^T\int_0^s f(u,x(u))\,d_qu\,d_ps,\end{aligned} \tag{30}$$

for $x, y \in B_R$. By computing directly, we have

$$\begin{aligned}&\|(F_1x)+(F_2y)\|\\ &\leq \|\mu\|\int_0^t\int_0^s d_qu\,d_ps + |\lambda|\,\|x\|\int_0^t d_ps\\ &\quad+\frac{|\beta|(1+r)t}{|T(1+r)-\beta\eta^2|}\\ &\quad\times\left(\|\mu\|\int_0^\eta\int_0^\nu\int_0^s d_qu\,d_ps\,d_r\nu+|\lambda|\,\|y\|\int_0^\eta\int_0^\nu d_ps\,d_r\nu\right)\\ &\quad+\frac{|\lambda|(1+r)t}{|T(1+r)-\beta\eta^2|}\|y\|\int_0^T d_ps\\ &\quad+\frac{(1+r)t}{|T(1+r)-\beta\eta^2|}\|\mu\|\int_0^T\int_0^s d_qu\,d_ps,\\ &\leq R\Phi+\|\mu\|\,\Omega\leq R.\end{aligned} \tag{31}$$

Therefore, $(F_1x) + (F_2y) \in B_R$. The condition (28) implies that F_2 is a contraction mapping. Next, we will show that F_1 is compact and continuous. Continuity of f coupled with the assumption (H_3) implies that the operator F_1 is continuous and uniformly bounded on B_R. We define $\sup_{(t,x)\in[0,T]\times B_R}|f(t,x)| = f_{\max} < \infty$. For $t_1, t_2 \in [0,T]$ with $t_2 < t_1$ and $x \in B_R$, we have

$$\begin{aligned}&|F_1x(t_1)-F_2x(t_2)|\\ &\leq \sup_{(t,x)\in[0,T]\times B_R}\left|\int_0^{t_1}\int_0^s f(u,x(u))\,d_qu\,d_ps-\lambda\int_0^{t_1}x(s)\,d_ps\right.\\ &\qquad-\int_0^{t_2}\int_0^s f(u,x(u))\,d_qu\,d_ps\\ &\qquad\left.+\lambda\int_0^{t_2}x(s)\,d_ps\right|\\ &= \sup_{(t,x)\in[0,T]\times B_R}\left|\int_{t_2}^{t_1}\int_0^s f(u,x(u))\,d_qu\,d_ps-\lambda\int_{t_2}^{t_1}x(s)\,d_ps\right|\\ &\leq f_{\max}\frac{|t_1^2-t_2^2|}{1+p}+|\lambda|(t_1-t_2)R.\end{aligned} \tag{32}$$

Actually, as $t_1 - t_2 \to 0$, the right-hand side of the above inequality tends to zero. So F_1 is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, F_1 is compact on B_R. Therefore, all the assumptions of Theorem 5 are satisfied and the conclusion of Theorem 5 implies that the three-point integral boundary value problem (3) has at least one solution on $[0,T]$. This completes the proof. □

As the third result, we prove the existence of solutions of (3) by using Leray-Schauder degree theory.

Theorem 5. *Let $f : [0,T]\times\mathbb{R}\to\mathbb{R}$. Assume that there exist constants $0 \leq \kappa < (1-\Phi)\Omega^{-1}$, where Ω and Φ are given by* (22) *and* (23)*, respectively, and $M > 0$ such that $|f(t,x)| \leq \kappa|x|+M$ for all $t \in [0,T]$, $x \in \mathscr{C}$. Then, the boundary value problem* (3) *has at least one solution.*

Proof. Let us define an operator $A : \mathscr{C} \to \mathscr{C}$ as in (24). We will prove that there exists at least one solution $x \in \mathscr{C}$ of the fixed-point equation

$$x = Ax. \tag{33}$$

We define a ball $B_R \subset \mathscr{C}$, with a constant radius $R > 0$, given by

$$B_R = \left\{x\in\mathscr{C} : \max_{t\in[0,T]}|x(t)| < R\right\}. \tag{34}$$

Then, it is sufficient to show that $A : \overline{B}_R \to \mathscr{C}$ satisfies

$$x \neq \theta Ax,\quad \forall x\in\partial B_R,\ \forall\theta\in[0,1]. \tag{35}$$

Now, we set

$$H(\theta,x)=\theta Ax,\qquad x\in\mathscr{C},\ \theta\in[0,1]. \tag{36}$$

Then, by the Arzelá-Ascoli theorem, we get that $h_\theta(x) = x-H(\theta,x) = x-\theta Ax$ is completely continuous. If (35) holds, then the following Leray-Schauder degrees are well defined. From the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\theta,B_R,0)&=\deg(I-\theta A,B_R,0)\\ &=\deg(h,B_R,0)\\ &=\deg(h_0,B_R,0)\\ &=\deg(I,B_R,0)=1\neq0,\quad 0\in B_R,\end{aligned} \tag{37}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x-Ax = 0$ for at least one

$x \in B_R$. Let us assume that $x = \theta Ax$ for some $\theta \in [0,1]$. Then, for all $t \in [0,T]$, we obtain

$$\begin{aligned}
|x(t)| &= |\theta (Ax)(t)| \\
&\leq \int_0^t \int_0^s \left| f(u, x(u)) \right| d_q u d_p s \\
&\quad + |\lambda| \int_0^t |x(s)| d_p s \\
&\quad + \frac{|\beta| (1+r) t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \int_0^\eta \int_0^v \left(\int_0^s \left| f(u, x(u)) \right| d_q u + |\lambda x(s)| \right) d_p s d_r v \\
&\quad + \frac{|\lambda| (1+r) t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \int_0^T |x(s)| d_p s + \frac{(1+r) t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \int_0^T \int_0^s \left| f(u, x(u)) \right| d_q u d_p s \\
&\leq (\kappa |x| + M) \int_0^t \int_0^s d_q u d_p s + |\lambda| |x| \int_0^t d_p s \\
&\quad + \frac{|\beta| (1+r) t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \left((\kappa |x| + M) \int_0^\eta \int_0^v \int_0^s d_q u d_p s d_r v \right. \\
&\qquad \left. + |\lambda| |x| \int_0^\eta \int_0^v d_p s d_r v \right) \\
&\quad + \frac{|\lambda| |x| (1+r) t}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \int_0^T d_p s + \frac{(1+r) t}{|T(1+r) - \beta\eta^2|} (\kappa |x| + M) \\
&\quad \times \int_0^T \int_0^s d_q u d_p s \\
&\leq (\kappa |x| + M) \frac{T^2}{1+p} + |\lambda| |x| T \\
&\quad + \frac{|\beta| (1+r) T}{|T(1+r) - \beta\eta^2|} \\
&\quad \times \left((\kappa |x| + M) \frac{(1-r)\eta^3}{(1+p)(1-r^3)} + |\lambda| |x| \frac{\eta^2}{1+r} \right) \\
&\quad + \frac{|\lambda| |x| (1+r) T^2}{|T(1+r) - \beta\eta^2|} \\
&\quad + (\kappa |x| + M) \frac{(1+r) T^3}{|T(1+r) - \beta\eta^2| (1+p)} \\
&= |x| \left(|\lambda| T + \frac{|\beta| |\lambda| T\eta^2}{|T(1+r) - \beta\eta^2|} + \frac{|\lambda| T^2 (1+r)}{|T(1+r) - \beta\eta^2|} \right) \\
&\quad + \frac{(\kappa |x| + M)}{1+p} \left(T^2 + \frac{|\beta| T\eta^3 (1+r)}{|T(1+r) - \beta\eta^2| (1+r+r^2)} \right. \\
&\qquad \left. + \frac{T^3 (1+r)}{|T(1+r) - \beta\eta^2|} \right) \\
&= |x| \Phi + (\kappa |x| + M) \Omega.
\end{aligned} \tag{38}$$

Taking norm $\sup_{t\in[0,T]} |x(t)| = \|x\|$ and solving for $\|x\|$, this yields

$$\|x\| \leq \frac{M\Omega}{1 - (\Phi + \kappa\Omega)}. \tag{39}$$

Let $R = M\Omega/(1-(\Phi+\kappa\Omega))+1$, then (35) holds. This completes the proof. □

4. Examples

In this section, we give two examples to illustrate our results.

Example 6. Consider the following nonlinear q-difference equation with boundary value problem:

$$\begin{gathered}
D_{1/2} \left(D_{1/3} - \frac{2}{7} \right) x(t) = \frac{1}{(t+2)^2} \cdot \frac{|x|}{|x|+1}, \quad t \in [0,1], \\
x(0) = 0, \qquad x(1) + \frac{2}{3} \int_0^{3/4} x(s) d_{1/4} s = 0.
\end{gathered} \tag{40}$$

Set $q = 1/2$, $p = 1/3$, $r = 1/4$, $T = 1$, $\lambda = -2/7$, $\eta = 3/4$, $\beta = -2/3$, and $f(t,x) = (1/(t+2)^2)(\|x\|/(1+\|x\|))$. Since $|f(t,x) - f(t,y)| \leq (1/4)\|x - y\|$, then, (H_1) and (H_2) are satisfied with $T(1+r) - \beta\eta^2 = 13/8 \neq 0$,

$$\begin{aligned}
\Omega &= \frac{1}{1+p} \left(T^2 + \frac{|\beta| T\eta^3 (1+r)}{|T(1+r) - \beta\eta^2| (1+r+r^2)} \right. \\
&\qquad \left. + \frac{T^3 (1+r)}{|T(1+r) - \beta\eta^2|} \right) = \frac{132}{91}, \\
\Phi &= |\lambda| T + \frac{|\beta| |\lambda| T\eta^2}{|T(1+r) - \beta\eta^2|} \\
&\quad + \frac{|\lambda| T^2 (1+r)}{|T(1+r) - \beta\eta^2|} = \frac{4}{7},
\end{aligned} \tag{41}$$

$L = 1/4$. Hence, $\Lambda =: \Phi + L\Omega = 85/91 < 1$. Therefore, by Theorem 2, the boundary value problem (40) has a unique solution on $[0,1]$.

Example 7. Consider the following nonlinear q-difference equation with boundary value problem:

$$D_{2/3}\left(D_{4/5}+\frac{1}{9}\right)x(t)=\frac{\sin(5\pi x)}{25\pi}+\frac{|x|}{|x|+1},\quad t\in\left[0,\frac{3}{2}\right],$$
$$x(0)=0,\qquad x\left(\frac{3}{2}\right)=\frac{1}{4}\int_0^1 x(s)\,d_{1/2}s. \tag{42}$$

Set $q=2/3$, $p=4/5$, $r=1/2$, $T=3/2$, $\lambda=1/9$, $\eta=1$, and $\beta=1/4$. Here, $|f(t,x)|=|\sin(5\pi x)/25\pi+|x|/(1+|x|)|\le(|x|/5)+1$. So, $M=1$, $T(1+r)-\beta\eta^2=2\neq0$, and

$$\begin{aligned}\Omega&=\frac{1}{1+p}\\&\quad\times\left(T^2+\frac{|\beta|T\eta^3(1+r)}{|T(1+r)-\beta\eta^2|(1+r+r^2)}\right.\\&\qquad\left.+\frac{T^3(1+r)}{|T(1+r)-\beta\eta^2|}\right)=\frac{615}{224},\\\Phi&=|\lambda|T+\frac{|\beta||\lambda|T\eta^2}{|T(1+r)-\beta\eta^2|}\\&\quad+\frac{|\lambda|T^2(1+r)}{|T(1+r)-\beta\eta^2|}=\frac{3}{8},\\\kappa&=\frac{1}{5}<(1-\Phi)\Omega^{-1}=\frac{28}{123}.\end{aligned}\tag{43}$$

Hence, by Theorem 5, the boundary value problem (42) has at least one solution on $[0,3/2]$.

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