

## Research Article

# Note on Relation between Double Laplace Transform and Double Differential Transform

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Double differential transform method has been employed to compute double Laplace transform. To illustrate the method, four examples of different forms have been prepared.

## 1. Introduction

The concept of the DTM was first proposed by Zhou [1], who solved linear and nonlinear problems in electrical circuit problems. Chen and Ho [2] developed this method for partial differential equations and applied it to the system of differential equations. During the recent years, many authors have used this method for solving various types of equations. For example, this method has been used for differential algebraic equations [3], partial differential equations [4, 5], fractional differential equations [6], Volterra integral equations [7], and difference equations [8]. The main goal of this paper is to extend the study of single Laplace transform by using differential transform (see [9]) and to compute double Laplace transform by means of double differential transform method. As we know, the standard derivation of Laplace transforms inherits an improper integration which may, in certain cases, not be analytically tractable. However, in contrast, the proposed straightforward approach merely requires easy differentiations and algebraic operations. Three examples are proposed.

The one-dimensional differential transform of the function  $u(x)$  is defined by the following formula:

$$U(k) = \frac{1}{k!} \left[ \frac{d^k u(x)}{dx^k} \right]_{x=0}, \quad (1)$$

where  $u(x)$  and  $U(k)$  are the original and transform functions, respectively. The inverse differential transform of  $U(k)$  is specified as follows:

$$u(x) = \sum_{k=0}^{\infty} [U(k) x^k]. \quad (2)$$

Consider an analytical function  $u(x, t)$  of two variables; then this function can be represented as a series in  $(x_0, t_0) \in D$  using differential transform

$$U(m, n) = \frac{1}{m!n!} \left[ \frac{\partial^{m+n} u(x, t)}{\partial x^m \partial t^n} \right]_{x=x_0, t=t_0} \quad (3)$$

and inverse double differential transform

$$u(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left[ \frac{\partial^{m+n} u(x, t)}{\partial x^m \partial t^n} \right]_{x=0, t=0}. \quad (4)$$

From the definition of double Laplace transform, we can write

$$\begin{aligned} L_x L_t [f(x, t)] &= F(p, s) \\ &= \left[ \int_{x_0}^x e^{-px} \left[ \int_{t_0}^t e^{-st} f(x, t) dt \right]_{t_0=0}^{t=\infty} dx \right]_{x_0=0}^{x=\infty}, \end{aligned} \quad (5)$$

where  $x, t > 0$  and  $p, s$  is a complex value.

On using double direct and inverse differential transform with respect to  $x$  and  $t$  for both sides of the previous,

$$\begin{aligned}
 F(p, s) &= DT_p^{-1} DT_s^{-1} \\
 &\times \left\{ DT_x DT_t \right. \\
 &\quad \times \left\{ \left[ \int_{x_0}^x e^{-px} \left[ \int_{t_0}^t e^{-st} f(x, t) dt \right]_{t_0=0}^{t=\infty} dx \right]_{x_0=0}^{x=\infty} \right\} \Bigg\}, \\
 F(p, s) &= \left[ \left[ DT_p^{-1} DT_s^{-1} \right. \right. \\
 &\quad \times \left\{ DT_x \right. \\
 &\quad \left. \left. \int_{x_0}^x e^{-px} \left\{ DT_t \int_{t_0}^t e^{-st} f(x, t) dt \right\} dx \right\} \right]_{t_0=0}^{t=\infty} \right]_{x_0=0}^{x=\infty}. \quad (6)
 \end{aligned}$$

**Lemma 1.** Let  $m$  and  $n$  be a finite positive integers such that  $m, n \geq 1$  and  $s, p > 0, s, p \in \mathbb{R}^2$ ; then

$$\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^m r^n}{k! r!} (-st)^k (-px)^r \right] = 0. \quad (7)$$

*Proof.* By using mathematical induction, letting  $m, n = 1$ , we have

$$\begin{aligned}
 &\left( \sum_{k=1}^{\infty} \frac{k}{k!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r}{r!} (-px)^r \right) \\
 &= \left( \sum_{k=1}^{\infty} \frac{1}{(k-1)!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{1}{(r-1)!} (-px)^r \right) \\
 &= \left( \sum_{k=0}^{\infty} \frac{1}{k!} (-st)^{k+1} \right) \left( \sum_{r=0}^{\infty} \frac{1}{r!} (-px)^{r+1} \right) \\
 &= \left( (-st) \sum_{k=0}^{\infty} \frac{1}{k!} (-st)^k \right) \left( (-px) \sum_{r=0}^{\infty} \frac{1}{r!} (-px)^r \right) \\
 &= (-st) e^{-st} (-px) e^{-px}.
 \end{aligned} \quad (8)$$

Then  $\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} [(kr/k!r!)(-st)^k(-px)^r] = \lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st)e^{-st}(-px)e^{-px} = 0$ ; since  $(\sum_{k=1}^{\infty} (k/k!)(-st)^k)(\sum_{r=1}^{\infty} (r/r!)(-px)^r) = (-st)e^{-st}(-px)e^{-px}$  we conclude that

$$\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st)(-px) \left( \sum_{k=1}^{\infty} \frac{k}{k!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r}{r!} (-px)^r \right) = 0, \quad (9)$$

for  $m, n = 2$ ,

$$\begin{aligned}
 &\left( \sum_{k=1}^{\infty} \frac{k^2}{k!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r^2}{r!} (-px)^r \right) \\
 &= \left( \sum_{k=1}^{\infty} \frac{k}{(k-1)!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r}{(r-1)!} (-px)^r \right) \\
 &= \left( \sum_{k=0}^{\infty} \frac{k+1}{k!} (-st)^{k+1} \right) \left( \sum_{r=0}^{\infty} \frac{r+1}{r!} (-px)^{r+1} \right) \\
 &= (-st) \left( \sum_{k=0}^{\infty} \frac{k}{k!} (-st)^k + \sum_{k=0}^{\infty} \frac{(-st)^k}{k!} \right) \\
 &\quad \times (-px) \left( \sum_{r=0}^{\infty} \frac{r}{r!} (-px)^r + \sum_{r=0}^{\infty} \frac{(-px)^r}{r!} \right) \\
 &= (-st) [(-st)e^{-st} + e^{-st}] \\
 &\quad \times (-px) [(-px)e^{-px} + e^{-px}].
 \end{aligned} \quad (10)$$

Consequently

$$\begin{aligned}
 &\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \left( \sum_{k=1}^{\infty} \frac{k^2}{k!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r^2}{r!} (-px)^r \right) \\
 &= \lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st) [(-st)e^{-st} + e^{-st}] \\
 &\quad \times (-px) [(-px)e^{-px} + e^{-px}] = 0.
 \end{aligned} \quad (11)$$

Similarly

$$\begin{aligned}
 &\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st)(-px) \left( \sum_{k=1}^{\infty} \frac{k^2}{k!} (-st)^k \right) \left( \sum_{r=1}^{\infty} \frac{r^2}{r!} (-px)^r \right) \\
 &= \lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st)^2 [(-st)e^{-st} + e^{-st}] \\
 &\quad \times (-px)^2 [(-px)e^{-px} + e^{-px}] = 0.
 \end{aligned} \quad (12)$$

Assume that, for  $m, n = 1$  to  $i, j$ , it holds that

$$\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^m r^n}{k! r!} (-st)^k (-px)^r \right] = 0, \quad (13)$$

and also

$$\begin{aligned}
 &\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} (-st)(-px) \left( \sum_{k=1}^{\infty} \frac{k^m}{k!} (-st)^k \right) \\
 &\quad \times \left( \sum_{r=1}^{\infty} \frac{r^n}{r!} (-px)^r \right) = 0.
 \end{aligned} \quad (14)$$

Now, we are going to prove that

$$\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^{m+1} r^{n+1}}{k! r!} (-st)^k (-px)^r \right] = 0, \quad (15)$$

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^{m+1} r^{n+1}}{k! r!} (-st)^k (-px)^r \right] \\ &= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^m r^n}{(k-1)! (r-1)!} (-st)^k (-px)^r \right] \\ &= (-st) (-px) \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \left[ \frac{(k+1)^m (r+1)^n}{k! r!} (-st)^k (-px)^r \right]. \end{aligned} \quad (16)$$

By using the definition of polynomial, we have

$$\begin{aligned} & (-st) \sum_{k=0}^{\infty} \left[ \frac{(k+1)^m}{k!} (-st)^k \right] (-px) \sum_{r=0}^{\infty} \left[ \frac{(r+1)^n}{r!} (-px)^r \right] \\ &= (-st) \sum_{k=0}^{\infty} \left[ \frac{\left(1 + \binom{m}{1} k + \binom{m}{2} k^2 + \cdots + k^m\right)}{k!} (-st)^k \right] \\ & \quad \times (-px) \sum_{r=0}^{\infty} \left[ \frac{\left(1 + \binom{n}{1} r + \binom{n}{2} r^2 + \cdots + r^n\right)}{r!} (-px)^r \right] \\ &= \left( (-st) \sum_{k=0}^{\infty} \frac{(-st)^k}{k!} + \binom{m}{1} (-st) \sum_{k=0}^{\infty} \frac{k(-st)^k}{k!} \right. \\ & \quad + \binom{m}{2} (-st) \sum_{k=0}^{\infty} \frac{k^2(-st)^k}{k!} \\ & \quad + \cdots + (-st) \sum_{k=0}^{\infty} \frac{k^m(-st)^k}{k!} \Big) \\ & \quad \times \left( (-px) \sum_{r=0}^{\infty} \frac{(-px)^r}{r!} + \binom{n}{1} (-px) \sum_{r=0}^{\infty} \frac{r(-px)^r}{r!} \right. \\ & \quad + \binom{n}{2} (-px) \sum_{r=0}^{\infty} \frac{r^2(-px)^r}{r!} \\ & \quad + \cdots + (-px) \sum_{r=0}^{\infty} \frac{r^n(-px)^r}{r!} \Big). \end{aligned} \quad (17)$$

Thus

$$\lim_{\substack{x \rightarrow \infty \\ t \rightarrow \infty}} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \left[ \frac{k^{m+1} r^{n+1}}{k! r!} (-st)^k (-px)^r \right] = 0. \quad (18) \quad \square$$

## 2. Relation between Double Laplace and Differential Transforms

In this section, we compute the double Laplace transform by means of double differential transform by proposing some examples as follows.

*Example 2.* Double Laplace transform of function  $e^{ax+bt}$ ; consider

$$\begin{aligned} & L_x L_t \{e^{ax+bt}\} \\ &= \left[ DT_p^{-1} DT_s^{-1} \right. \\ & \quad \times \left\{ DT_x DT_t \left\{ \int_{y_0}^y e^{ax-px} dx \int_{\beta_0}^{\beta} e^{bt-st} dt \right\} \right\} \Big]_{y_0=0, \beta_0=0}^{y=\infty, \beta=\infty} \\ &= \left[ DT_p^{-1} \left\{ \frac{(-p+a)^{r-1}}{r(r-1)!} \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \left[ DT_s^{-1} \left\{ \frac{(-s+b)^{k-1}}{k(k-1)!} \right\} \right]_{\beta_0=0}^{\beta=\infty} \\ &= \left[ \sum_{r=0}^{\infty} \left\{ \frac{(-p+a)^{r-1} y^r}{r(r-1)!} \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \left[ \sum_{k=0}^{\infty} \left\{ \frac{(-s+b)^{k-1} \beta^k}{k(k-1)!} \right\} \right]_{\beta_0=0}^{\beta=\infty} \\ &= \frac{1}{(a-p)} \left[ \sum_{r=0}^{\infty} \left\{ \frac{(-p+a)^r y^r}{r!} \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \frac{1}{(b-s)} \left[ \sum_{k=0}^{\infty} \left\{ \frac{(-s+b)^k \beta^k}{k!} \right\} \right]_{\beta_0=0}^{\beta=\infty} \\ &= \frac{1}{(a-p)} \left[ e^{(a-p)y} \right]_{y_0=0}^{y=\infty} \frac{1}{(b-s)} \left[ e^{(b-s)\beta} \right]_{\beta_0=0}^{\beta=\infty} \\ &= \frac{1}{(p-a)(s-b)}, \quad p > a, \quad s > b. \end{aligned} \quad (19)$$

In the next example, we apply double differential transform to compute double Laplace transform as follows.

*Example 3.* If we consider the function  $x^m t^n$ , then double Laplace transform is given by

$$\begin{aligned} & L_x L_t \{x^m t^n; p, s\} \\ &= \left[ DT_p^{-1} DT_s^{-1} \right. \\ & \quad \times \left\{ DT_x DT_t \left\{ \int_{y_0}^y x^m e^{-px} dx \int_{\beta_0}^{\beta} t^n e^{-st} dt \right\} \right\} \Big]_{y_0=0, \beta_0=0}^{y=\infty, \beta=\infty} \end{aligned}$$

$$\begin{aligned}
&= \left[ \sum_{r=0}^{\infty} \left\{ DT_x \left\{ \int_{y_0}^y x^m e^{-px} dx \right\} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \left[ \sum_{k=0}^{\infty} \left\{ DT_t \left\{ \int_{\beta_0}^{\beta} t^n e^{-st} dt \right\} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty}.
\end{aligned} \tag{20}$$

From the properties of double differential transform, we have

$$\begin{aligned}
&L_x L_t \{x^m t^n\} \\
&= 0 + \left[ \sum_{r=0}^{\infty} \left\{ \sum_{l=0}^{r-1} \left\{ \frac{\delta(r-m-l-1)(-p)^l}{r} \right\} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \left[ \sum_{k=0}^{\infty} \left\{ \sum_{c=0}^{k-1} \left\{ \frac{\delta(k-n-c-1)(-s)^c}{k} \right\} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty}.
\end{aligned} \tag{21}$$

On using definition of Kronecker delta function forces, we have

$$\begin{aligned}
&L_x L_t \{x^m t^n\} \\
&= \left[ \sum_{r=0}^{\infty} \left\{ \frac{1}{r} \frac{(-p)^{r-1-m}}{(r-1-m)!} \right\} y^r \right]_{y_0=0}^{y=\infty} \\
&\quad \times \left[ \sum_{k=0}^{\infty} \left\{ \frac{1}{k} \frac{(-s)^{k-1-n}}{(k-1-n)!} \right\} \beta^k \right]_{\beta_0=0}^{\beta=\infty} \\
&= \left[ \frac{(-1)^{m+1}}{p^{m+1}} \right. \\
&\quad \times \sum_{r=1}^{\infty} \left\{ \frac{(r-1)(r-2) \cdots (r-m)(-p)^r}{r!} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \left[ \frac{(-1)^{n+1}}{s^{n+1}} \right. \\
&\quad \times \sum_{k=1}^{\infty} \left\{ \frac{(k-1)(k-2) \cdots (k-n)(-s)^k}{k!} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty}.
\end{aligned} \tag{22}$$

From the previous equation, we have

$$\begin{aligned}
&L_x L_t \{x^m t^n; p, s\} \\
&= \frac{(-1)^{m+1}}{p^{m+1}} \left( \left[ \sum_{r=1}^{\infty} \left\{ \frac{(-1)(-2) \cdots (-m)(-p)^r}{r!} y^r \right\} \right]_{y_0=0}^{y=\infty} \right. \\
&\quad \left. + \sum_{i=1}^m a_i \lim_{y \rightarrow \infty} \sum_{r=1}^{\infty} \left\{ \frac{r^i}{r!} y^r \right\} - 0 \right)
\end{aligned}$$

$$\begin{aligned}
&\times \frac{(-1)^{n+1}}{s^{n+1}} \left( \left[ \sum_{k=1}^{\infty} \left\{ \frac{(-1)(-2) \cdots (-n)(-s)^k}{k!} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty} \right. \\
&\quad \left. + \sum_{j=1}^n b_j \lim_{\beta \rightarrow \infty} \sum_{k=1}^{\infty} \left\{ \frac{k^j}{k!} \beta^k \right\} - 0 \right),
\end{aligned} \tag{23}$$

where  $a_i$  and  $b_j$  are constant coefficients of the polynomials generated by  $(r-1)(r-2) \cdots (r-m)$  and  $(k-1)(k-2) \cdots (k-n)$ , respectively. According to the lemma, the last summations of (23),  $\lim_{y \rightarrow \infty} \sum_{r=1}^{\infty} \{(r^i/r!)y^r\}$ , and  $\lim_{\beta \rightarrow \infty} \sum_{k=1}^{\infty} \{(k^j/k!)\beta^k\}$  are zeros, such that

$$\begin{aligned}
&L_x L_t \{x^m t^n\} \\
&= \frac{(-1)^{m+1}}{p^{m+1}} \left[ \sum_{r=1}^{\infty} \left\{ \frac{(-1)(-2) \cdots (-m)(-p)^r}{r!} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \frac{(-1)^{n+1}}{s^{n+1}} \left[ \sum_{k=1}^{\infty} \left\{ \frac{(-1)(-2) \cdots (-n)(-s)^k}{k!} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty} \\
&= \frac{(-1)^{m+1} m!}{p^{m+1}} \left[ \sum_{r=1}^{\infty} \left\{ \frac{(-p)^r}{r!} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \frac{(-1)^{n+1} n!}{s^{n+1}} \left[ \sum_{k=1}^{\infty} \left\{ \frac{(-s)^k}{k!} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty} \\
&= \frac{-m!}{p^{m+1}} \left[ \sum_{r=1}^{\infty} \left\{ \frac{(-p)^r}{r!} y^r \right\} \right]_{y_0=0}^{y=\infty} \\
&\quad \times \frac{-n!}{s^{n+1}} \left[ \sum_{k=1}^{\infty} \left\{ \frac{(-s)^k}{k!} \beta^k \right\} \right]_{\beta_0=0}^{\beta=\infty} \\
&= \frac{-m!}{p^{m+1}} [(e^{-py} - 1)]_{y_0=0}^{y=\infty} \\
&\quad \times \frac{-n!}{s^{n+1}} [(e^{-s\beta} - 1)]_{\beta_0=0}^{\beta=\infty} \\
&= \frac{m!n!}{p^{m+1} s^{n+1}}.
\end{aligned} \tag{24}$$

In the next example, we apply double integral transform as follows.

*Example 4.* If we consider the function  $\sin(ax) \sin(bt)$ , then double Laplace transform is given by

$$\begin{aligned}
&L_x L_t \{\sin(ax) \sin(bt); p, s\} \\
&= \left[ DT_p^{-1} DT_s^{-1} \right. \\
&\quad \left. \times \left\{ DT_x DT_t \right\} \right]
\end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \int_{y_0}^y \int_{\beta_0}^{\beta} \sin(ax) \sin(bt) e^{-px} e^{-st} dt dx \right\} \Bigg|_{y_0=0, \beta_0=0}^{y=\infty, \beta=\infty}, \\
 L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \left[ DT_p^{-1} \right. \\
 & \quad \times \left\{ \sum_{l=0}^{r-1} \left\{ \frac{a^j}{j!} \sin\left(\frac{j\pi}{2}\right) \frac{(-p)^{r-j-1}/(r-j-1)!}{r} \right\} \right\} \Bigg|_{y_0=0}^{y=\infty} \\
 & \quad \times \left[ DT_s^{-1} \right. \\
 & \quad \times \left\{ \sum_{i=0}^{k-1} \left\{ \frac{b^i}{i!} \sin\left(\frac{i\pi}{2}\right) \frac{(-s)^{k-i-1}/(k-i-1)!}{k} \right\} \right\} \Bigg|_{\beta_0=0}^{\beta=\infty}, \\
 & L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \left[ \sum_{r=0}^{\infty} \left\{ \sum_{l=1,3,5}^{r-1} \left\{ \frac{a^j}{j!} \frac{(r-1)(r-2)\cdots(r-j)}{p^{j+1}} \right. \right. \right. \\
 & \quad \times (-1)^{(j-1)/2-j-1} \left. \left. \left\{ \frac{y^r (-p)^r}{r!} \right\} \right\} \right]_{y_0=0}^{y=\infty} \\
 & \quad \times \left[ \sum_{k=0}^{\infty} \left\{ \sum_{i=1,3,5}^{k-1} \left\{ \frac{b^i}{i!} \frac{(k-1)(k-2)\cdots(k-i)}{s^{i+1}} \right. \right. \right. \\
 & \quad \times (-1)^{(i-1)/2-i-1} \left. \left. \left\{ \frac{\beta^k (-s)^k}{k!} \right\} \right\} \right]_{\beta_0=0}^{\beta=\infty}, \\
 & L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \left[ \sum_{r=0}^{\infty} \left\{ \sum_{l=1,3,5}^{r-1} \left\{ \frac{a^j}{j!} \frac{(r-1)(r-2)\cdots(r-j)}{p^{j+1}} \right. \right. \right. \\
 & \quad \times (-1)^{(j+3)/2} \left. \left. \left\{ \frac{y^r (-p)^r}{r!} \right\} \right\} \right]_{y_0=0}^{y=\infty} \\
 & \quad \times \left[ \sum_{k=0}^{\infty} \left\{ \sum_{i=1,3,5}^{k-1} \left\{ \frac{b^i}{i!} \frac{(k-1)(k-2)\cdots(k-i)}{s^{i+1}} \right. \right. \right. \\
 & \quad \times (-1)^{(i+3)/2} \left. \left. \left\{ \frac{\beta^k (-s)^k}{k!} \right\} \right\} \right]_{\beta_0=0}^{\beta=\infty}. \tag{25}
 \end{aligned}$$

By calculating the summation inside the bracket we have

$$\begin{aligned}
 & L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \left[ \sum_{r=0}^{\infty} \left\{ \frac{a}{p^2} \frac{(r-1)(-py)^r}{r!} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{a^3 (r-1)(r-2)(r-3)(-py)^r}{p^4 r!} \\
 & + \frac{a^5 (r-1)(r-2)(r-3)(-py)^r}{p^6 r!} + \dots \Bigg]_{y_0=0}^{y=\infty} \\
 & \times \left[ \sum_{k=0}^{\infty} \left\{ \frac{b}{s^2} \frac{(k-1)(-s\beta)^k}{k!} \right. \right. \\
 & - \frac{b^3 (k-1)(k-2)(k-3)(-s\beta)^k}{s^4 k!} \\
 & + \frac{b^5 (k-1)(k-2)(k-3)(-s\beta)^k}{s^6 k!} + \dots \Bigg]_{\beta_0=0}^{\beta=\infty}. \tag{26}
 \end{aligned}$$

From the previous lemma, we know that, for  $m, n \geq 1$ ,  $\lim_{y \rightarrow \infty} \sum_{r=1}^{\infty} \{(r^m/r!)(-py)^r\} = 0$ , and  $\lim_{\beta \rightarrow \infty} \sum_{k=1}^{\infty} \{(k^m/k!)(-s\beta)^k\} = 0$ , we have the following form:

$$\begin{aligned}
 & L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \left[ \sum_{r=0}^{\infty} \left\{ -\frac{a}{p^2} \frac{(-py)^r}{r!} + \frac{a^3}{p^4} \frac{(-py)^r}{r!} \right. \right. \\
 & \quad \left. \left. - \frac{a^5}{p^6} \frac{(-py)^r}{r!} + \dots \right\} \right]_{y_0=0}^{y=\infty} \\
 & \quad \times \left[ \sum_{k=0}^{\infty} \left\{ -\frac{b}{s^2} \frac{(-s\beta)^k}{k!} + \frac{b^3}{s^4} \frac{(-s\beta)^k}{k!} \right. \right. \\
 & \quad \left. \left. - \frac{b^5}{s^6} \frac{(-s\beta)^k}{k!} + \dots \right\} \right]_{\beta_0=0}^{\beta=\infty} \tag{27} \\
 &= \left[ \left( -\frac{a}{p^2} + \frac{a^3}{p^4} - \frac{a^5}{p^6} + \dots \right) e^{-py} \right]_{y_0=0}^{y=\infty} \\
 & \quad \times \left[ \left( -\frac{b}{s^2} + \frac{b^3}{s^4} - \frac{b^5}{s^6} + \dots \right) e^{-s\beta} \right]_{\beta_0=0}^{\beta=\infty} \\
 &= \left( \frac{a}{p^2} - \frac{a^3}{p^4} + \frac{a^5}{p^6} + \dots \right) \\
 & \quad \times \left( \frac{b}{s^2} - \frac{b^3}{s^4} + \frac{b^5}{s^6} + \dots \right).
 \end{aligned}$$

From the definition of infinite geometric series and the summation of the previous terms, we have

$$\begin{aligned}
 & L_x L_t \{ \sin(ax) \sin(bt); p, s \} \\
 &= \lim_{m \rightarrow \infty} \frac{a}{p^2} \frac{(1 - (-a^2/p^2)^{m+1})}{1 - (-a^2/p^2)}
 \end{aligned}$$

$$\begin{aligned} & \times \lim_{n \rightarrow \infty} \frac{b \left(1 - (-b^2/s^2)^{n+1}\right)}{1 - (-b^2/s^2)} \\ & = \left( \frac{a/p^2}{1 + a^2/p^2} \right) \left( \frac{b/s^2}{1 + b^2/s^2} \right) = \frac{ab}{(p^2 + a^2)(s^2 + b^2)}. \end{aligned} \quad (28)$$

In the next example, we apply double differential transform to find double Laplace transform of the function  $(\sin(x)/x)(\sin(t)/t)$  as follows.

*Example 5.* The double Laplace transform of the function  $(\sin(x)/x)(\sin(t)/t)$ , as follows:

$$\begin{aligned} L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ = \left[ DT_p^{-1} DT_s^{-1} \left\{ DT_x DT_t \left\{ \int_{y_0}^y \int_{\beta_0}^{\beta} \frac{\sin(x)}{x} \frac{\sin(t)}{t} \right. \right. \right. \\ \left. \left. \left. \times e^{-px} e^{-st} dt dx \right\} \right\} \right]_{y_0=0, \beta_0=0}^{y=\infty, \beta=\infty}. \end{aligned} \quad (29)$$

By using sin series, we have

$$\begin{aligned} & \frac{\sin(x)}{x} \frac{\sin(t)}{t} \\ & = \left( \frac{\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}}{x} \right) \\ & \quad \times \left( \frac{\sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!}}{t} \right) \\ & = \left( \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i+1)!} \right) \left( \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j+1)!} \right). \end{aligned} \quad (30)$$

By applying double differential transform, we have

$$\begin{aligned} & DT_x DT_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t} \right\} \\ & = \left\{ \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} DT_x \{x^{2i}\} \right\} \\ & \quad \times \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} DT_t \{t^{2j}\} \right\}. \end{aligned} \quad (31)$$

On using double inverse differential transform, we have

$$\begin{aligned} & L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ & = \left[ DT_p^{-1} \left\{ \frac{(-1)^{m/2}}{(m+1)!} \frac{(-p)^{k-m-1}}{k} \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \left[ DT_s^{-1} \left\{ \frac{(-1)^{n/2}}{(n+1)!} \frac{(-s)^{\beta-n-1}}{\beta} \right\} \right]_{\beta_0=0}^{\beta=\infty} \\ & = \left[ \sum_{k=0}^{\infty} \left\{ \sum_{m=0,2,4}^{k-1} \left\{ \frac{(-1)^{m/2} (k-1)(k-2) \cdots (k-m)}{(m+1)!(-p)^{m+1}} \right. \right. \right. \\ & \quad \left. \left. \left. \times (-1)^{m+1} \right\} \frac{(-px)^k}{k!} \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \left[ \sum_{\beta=0}^{\infty} \left\{ \sum_{n=0,2,4}^{\beta-1} \left\{ \frac{(-1)^{n/2} (\beta-1)(\beta-2) \cdots (\beta-n)}{(n+1)!(-s)^{n+1}} \right. \right. \right. \\ & \quad \left. \left. \left. \times (-1)^{n+1} \right\} \frac{(-st)^{\beta}}{\beta!} \right\} \right]_{\beta_0=0}^{\beta=\infty}, \end{aligned} \quad (32)$$

$$\begin{aligned} & L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ & = \left[ \sum_{k=0}^{\infty} \left\{ \left( -\frac{1}{p} + \frac{(k-1)(k-2)}{3!p^3} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{(k-1)(k-2)(k-3)(k-4)}{5!p^5} \right. \right. \right. \\ & \quad \left. \left. \left. + ((k-1)(k-2)(k-3)(k-1) \right. \right. \right. \\ & \quad \left. \left. \left. \times (k-5)(k-6)) \times (7!p^7)^{-1} + \cdots \right) \right\} \right]_{y_0=0}^{y=\infty} \\ & \quad \times \left[ \sum_{\beta=0}^{\infty} \left\{ \left( -\frac{1}{s} + \frac{(\beta-1)(\beta-2)}{3!s^3} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}{5!s^5} \right. \right. \right. \\ & \quad \left. \left. \left. + ((\beta-1)(\beta-2)(\beta-3) \right. \right. \right. \end{aligned}$$

$$\begin{aligned} & \times (\beta - 1)(\beta - 5)(\beta - 6)) \times (7!s^7)^{-1} + \dots \Big) \\ & \times \frac{(-st)^\beta}{\beta!} \Bigg\} \Bigg]_{\beta_0=0}^{\beta=\infty}. \end{aligned} \quad (33)$$

According to the previous lemma, we have

$$\begin{aligned} & L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ &= \left[ \sum_{k=0}^{\infty} \left\{ \left( -\frac{1}{p} + \frac{2!}{3!p^3} - \frac{4!}{5!p^5} + \frac{6!}{7!p^7} + \dots \right) \right. \right. \\ & \quad \times \left. \left. \frac{(-px)^k}{k!} \right\} \right]_{y_0=0}^{y=\infty} \\ & \times \left[ \sum_{\beta=0}^{\infty} \left\{ \left( -\frac{1}{s} + \frac{2!}{3!s^3} - \frac{4!}{5!s^5} + \frac{6!}{7!s^7} + \dots \right) \right. \right. \\ & \quad \times \left. \left. \frac{(-st)^\beta}{\beta!} \right\} \right]_{\beta_0=0}^{\beta=\infty}, \end{aligned} \quad (34)$$

$$\begin{aligned} & L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ &= \left( -\frac{1}{p} + \frac{2!}{3!p^3} - \frac{4!}{5!p^5} + \frac{6!}{7!p^7} + \dots \right) [e^{-px}]_{y_0=0}^{y=\infty} \\ & \times \left( -\frac{1}{s} + \frac{2!}{3!s^3} - \frac{4!}{5!s^5} + \frac{6!}{7!s^7} + \dots \right) [e^{-st}]_{\beta_0=0}^{\beta=\infty}, \\ & L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} \\ &= \left( \frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \frac{1}{7p^7} + \dots \right) \\ & \times \left( \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \dots \right). \end{aligned} \quad (35)$$

By using the definition of

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right), \quad (36)$$

we have

$$\begin{aligned} & \arctan\left(\frac{1}{p}\right) \arctan\left(\frac{1}{s}\right) \\ &= \left( \frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \frac{1}{7p^7} + \dots \right) \\ & \times \left( \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \dots \right). \end{aligned} \quad (37)$$

So that

$$L_x L_t \left\{ \frac{\sin(x)}{x} \frac{\sin(t)}{t}; p, s \right\} = \arctan\left(\frac{1}{p}\right) \arctan\left(\frac{1}{s}\right). \quad (38)$$

Also, we can use the same idea to compute double Laplace transform for convolution function, single or double.

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