

Research Article

Ranks of a Constrained Hermitian Matrix Expression with Applications

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We establish the formulas of the maximal and minimal ranks of the quaternion Hermitian matrix expression $C_4 - A_4XA_4^*$ where X is a Hermitian solution to quaternion matrix equations $A_1X = C_1$, $XB_1 = C_2$, and $A_3XA_3^* = C_3$. As applications, we give a new necessary and sufficient condition for the existence of Hermitian solution to the system of matrix equations $A_1X = C_1$, $XB_1 = C_2$, $A_3XA_3^* = C_3$, and $A_4XA_4^* = C_4$, which was investigated by Wang and Wu, 2010, by rank equalities. In addition, extremal ranks of the generalized Hermitian Schur complement $C_4 - A_4\tilde{A}_3A_4^*$ with respect to a Hermitian g-inverse \tilde{A}_3 of A_3 , which is a common solution to quaternion matrix equations $A_1X = C_1$ and $XB_1 = C_2$, are also considered.

1. Introduction

Throughout this paper, we denote the real number field by \mathbb{R} , the complex number field by \mathbb{C} , the set of all $m \times n$ matrices over the quaternion algebra

$$\begin{aligned} \mathbb{H} &= \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 \\ &= k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\} \end{aligned} \quad (1)$$

by $\mathbb{H}^{m \times n}$, the identity matrix with the appropriate size by I , the column right space, the row left space of a matrix A over \mathbb{H} by $\mathcal{R}(A)$, $\mathcal{N}(A)$, respectively, the dimension of $\mathcal{R}(A)$ by $\dim \mathcal{R}(A)$, a Hermitian g-inverse of a matrix A by $X = A^\dagger$ which satisfies $AA^\dagger A = A$ and $X = X^*$, and the Moore-Penrose inverse of matrix A over \mathbb{H} by A^+ which satisfies four Penrose equations $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^* = AA^+$, and $(A^+A)^* = A^+A$. In this case A^+ is unique and $(A^+)^* = (A^*)^+$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by A . Clearly, R_A and L_A are idempotent, Hermitian and $R_A = L_{A^*}$. By [1], for a quaternion matrix A , $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$. $\dim \mathcal{R}(A)$ is called the rank of a quaternion matrix A and denoted by $r(A)$.

Mitra [2] investigated the system of matrix equations

$$A_1X = C_1, \quad XB_1 = C_2. \quad (2)$$

Khatri and Mitra [3] gave necessary and sufficient conditions for the existence of the common Hermitian solution to (2) and presented an explicit expression for the general Hermitian solution to (2) by generalized inverses. Using the singular value decomposition (SVD), Yuan [4] investigated the general symmetric solution of (2) over the real number field \mathbb{R} . By the SVD, Dai and Lancaster [5] considered the symmetric solution of equation

$$AXA^* = C \quad (3)$$

over \mathbb{R} , which was motivated and illustrated with an inverse problem of vibration theory. Groß [6], Tian and Liu [7] gave the solvability conditions for Hermitian solution and its expressions of (3) over \mathbb{C} in terms of generalized inverses, respectively. Liu, Tian and Takane [8] investigated ranks of Hermitian and skew-Hermitian solutions to the matrix equation (3). By using the generalized SVD, Chang and Wang [9] examined the symmetric solution to the matrix equations

$$A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4 \quad (4)$$

over \mathbb{R} . Note that all the matrix equations mentioned above are special cases of

$$\begin{aligned} A_1X &= C_1, & XB_1 &= C_2, \\ A_3XA_3^* &= C_3, & A_4XA_4^* &= C_4. \end{aligned} \quad (5)$$

Wang and Wu [10] gave some necessary and sufficient conditions for the existence of the common Hermitian solution to (5) for operators between Hilbert C^* -modules by generalized inverses and range inclusion of matrices. In view of the complicated computations of the generalized inverses of matrices, we naturally hope to establish a more practical, necessary, and sufficient condition for system (5) over quaternion algebra to have Hermitian solution by rank equalities.

As is known to us, solutions to matrix equations and ranks of solutions to matrix equations have been considered previously by many authors [10–34], and extremal ranks of matrix expressions can be used to characterize their rank invariance, nonsingularity, range inclusion, and solvability conditions of matrix equations. Tian and Cheng [35] investigated the maximal and minimal ranks of $A - BXC$ with respect to X with applications; Tian [36] gave the maximal and minimal ranks of $A_1 - B_1XC_1$ subject to a consistent matrix equation $B_2XC_2 = A_2$. Tian and Liu [7] established the solvability conditions for (4) to have a Hermitian solution over \mathbb{C} by the ranks of coefficient matrices. Wang and Jiang [20] derived extreme ranks of (skew)Hermitian solutions to a quaternion matrix equation $AXA^* + BYB^* = C$. Wang, Yu and Lin [31] derived the extremal ranks of $C_4 - A_4XB_4$ subject to a consistent system of matrix equations

$$A_1X = C_1, \quad XB_1 = C_2, \quad A_3XB_3 = C_3 \quad (6)$$

over \mathbb{H} and gave a new solvability condition to system

$$\begin{aligned} A_1X &= C_1, & XB_1 &= C_2, \\ A_3XB_3 &= C_3, & A_4XB_4 &= C_4. \end{aligned} \quad (7)$$

In matrix theory and its applications, there are many matrix expressions that have symmetric patterns or involve Hermitian (skew-Hermitian) matrices. For example,

$$\begin{aligned} A - BXB^*, & \quad A - BX \pm X^*B^*, \\ A - BXB^* - CYC^*, & \quad A - BXC \pm (BXC)^*, \end{aligned} \quad (8)$$

where $A = \pm A^*$, B , and C are given and X and Y are variable matrices. In recent papers [7, 8, 37, 38], Liu and Tian considered some maximization and minimization problems on the ranks of Hermitian matrix expressions (8).

Define a Hermitian matrix expression

$$f(X) = C_4 - A_4XA_4^*, \quad (9)$$

where $C_4 = C_4^*$; we have an observation that by investigating extremal ranks of (9), where X is a Hermitian solution to a system of matrix equations

$$A_1X = C_1, \quad XB_1 = C_2, \quad A_3XA_3^* = C_3. \quad (10)$$

A new necessary and sufficient condition for system (5) to have Hermitian solution can be given by rank equalities, which is more practical than one given by generalized inverses and range inclusion of matrices.

It is well known that Schur complement is one of the most important matrix expressions in matrix theory; there have been many results in the literature on Schur complements and their applications [39–41]. Tian [36, 42] has investigated the maximal and minimal ranks of Schur complements with applications.

Motivated by the work mentioned above, we in this paper investigate the extremal ranks of the quaternion Hermitian matrix expression (9) subject to the consistent system of quaternion matrix equations (10) and its applications. In Section 2, we derive the formulas of extremal ranks of (9) with respect to Hermitian solution of (10). As applications, in Section 3, we give a new, necessary, and sufficient condition for the existence of Hermitian solution to system (5) by rank equalities. In Section 4, we derive extremal ranks of generalized Hermitian Schur complement subject to (2). We also consider the rank invariance problem in Section 5.

2. Extremal Ranks of (9) Subject to System (10)

Corollary 8 in [10] over Hilbert C^* -modules can be changed into the following lemma over \mathbb{H} .

Lemma 1. *Let $A_1, C_1 \in \mathbb{H}^{m \times n}$, $B_1, C_2 \in \mathbb{H}^{n \times s}$, $A_3 \in \mathbb{H}^{r \times n}$, $C_3 \in \mathbb{H}^{r \times r}$ be given, and $F = B_1^*L_{A_1}$, $M = SL_F$, $S = A_3L_{A_1}$, $D = C_2^* - B_1^*A_1^\dagger C_1$, $J = A_1^\dagger C_1 + F^\dagger D$, $G = C_3 - A_3(J + L_{A_1}L_F^*J^*)A_3^*$; then the following statements are equivalent:*

(1) the system (10) have a Hermitian solution,

(2) $C_3 = C_3^*$,

$$A_1C_2 = C_1B_1, \quad A_1C_1^* = C_1A_1^*, \quad B_1^*C_2 = C_2^*B_1, \quad (11)$$

$$R_{A_1}C_1 = 0, \quad R_FD = 0, \quad R_MG = 0, \quad (12)$$

(3) $C_3 = C_3^*$; the equalities in (11) hold and

$$\begin{aligned} r[A_1 \ C_1] &= r(A_1), & r \begin{bmatrix} A_1 & C_1 \\ B_1^* & C_2^* \end{bmatrix} &= r \begin{bmatrix} A_1 \\ B_1^* \end{bmatrix}, \\ r \begin{bmatrix} A_1 & C_1A_3^* \\ B_1^* & C_2^*A_3^* \\ A_3 & C_3 \end{bmatrix} &= r \begin{bmatrix} A_1 \\ B_1^* \\ A_3 \end{bmatrix}. \end{aligned} \quad (13)$$

In that case, the general Hermitian solution of (10) can be expressed as

$$\begin{aligned} X &= J + L_{A_1}L_FJ^* + L_{A_1}L_FM^\dagger G(M^\dagger)^*L_FL_{A_1} \\ &\quad + L_{A_1}L_FL_MVL_FL_{A_1} + L_{A_1}L_FV^*L_ML_FL_{A_1}, \end{aligned} \quad (14)$$

where V is Hermitian matrix over \mathbb{H} with compatible size.

Lemma 2 (see Lemma 2.4 in [24]). Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times k}$, $C \in \mathbb{H}^{l \times n}$, $D \in \mathbb{H}^{j \times k}$, and $E \in \mathbb{H}^{l \times i}$. Then the following rank equalities hold:

- (a) $r(CL_A) = r \begin{bmatrix} A \\ C \end{bmatrix} - r(A)$,
- (b) $r \begin{bmatrix} B & AL_C \\ 0 & C \end{bmatrix} = r \begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$,
- (c) $r \begin{bmatrix} C \\ R_B A \end{bmatrix} = r \begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$,
- (d) $r \begin{bmatrix} A & BL_D \\ R_E C & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E)$.

Lemma 2 plays an important role in simplifying ranks of various block matrices.

Liu and Tian [38] has given the following lemma over a field. The result can be generalized to \mathbb{H} .

Lemma 3. Let $A = \pm A^* \in \mathbb{H}^{m \times m}$, $B \in \mathbb{H}^{m \times n}$, and $C \in \mathbb{H}^{p \times m}$ be given; then

$$\begin{aligned} & \max_{X \in \mathbb{H}^{n \times p}} r [A - BXC \mp (BXC)^*] \\ &= \min \left\{ r \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, r \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \quad (15) \\ & \min_{X \in \mathbb{H}^{n \times p}} r [A - BXC \mp (BXC)^*] \\ &= 2r \begin{bmatrix} A & B & C^* \end{bmatrix} + \max \{s_1, s_2\}, \end{aligned}$$

where

$$\begin{aligned} s_1 &= r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \\ s_2 &= r \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - 2r \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}. \end{aligned} \quad (16)$$

If $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$,

$$\begin{aligned} \max_X r [A - BXC - (BXC)^*] &= \min \left\{ r \begin{bmatrix} A & C^* \\ B^* & 0 \end{bmatrix}, r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \\ \max_X r [A - BXC - (BXC)^*] &= \min \left\{ r \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}. \end{aligned} \quad (17)$$

Now we consider the extremal ranks of the matrix expression (9) subject to the consistent system (10).

Theorem 4. Let A_1, C_1, B_1, C_2, A_3 , and C_3 be defined as Lemma 1, $C_4 \in \mathbb{H}^{t \times t}$, and $A_4 \in \mathbb{H}^{t \times n}$. Then the extremal ranks of the quaternion matrix expression $f(X)$ defined as (9) subject to system (10) are the following:

$$\max r [f(X)] = \min \{a, b\}, \quad (18)$$

where

$$\begin{aligned} a &= r \begin{bmatrix} C_4 & A_4 \\ C_2^* A_4^* & B_1^* \\ C_1 A_4^* & A_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix}, \\ b &= r \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* A_3^* & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A_3^* & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} - 2r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix}, \end{aligned} \quad (19)$$

$$\begin{aligned} \min r [f(X)] &= 2r \begin{bmatrix} C_4 & A_4 \\ C_2^* A_4^* & B_1^* \\ C_1 A_4^* & A_1 \end{bmatrix} \\ &+ r \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* A_3^* & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A_3^* & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \\ &- 2r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix}. \end{aligned} \quad (20)$$

Proof. By Lemma 1, the general Hermitian solution of the system (10) can be expressed as

$$\begin{aligned} X &= J + L_{A_1} L_F J^* + L_{A_1} L_F M^\dagger G (M^\dagger)^* L_F L_{A_1} \\ &+ L_{A_1} L_F L_M V L_F L_{A_1} + L_{A_1} L_F V^* L_M L_F L_{A_1}, \end{aligned} \quad (21)$$

where V is Hermitian matrix over \mathbb{H} with appropriate size. Substituting (21) into (9) yields

$$\begin{aligned} f(X) &= C_4 - A_4 \left(J + L_{A_1} L_F J^* \right. \\ &\quad \left. + L_{A_1} L_F M^\dagger G (M^\dagger)^* L_F L_{A_1} \right) A_4^* \\ &- A_4 L_{A_1} L_F L_M V L_F L_{A_1} A_4^* \\ &- A_4 L_{A_1} L_F V^* L_M L_F L_{A_1} A_4^*. \end{aligned} \quad (22)$$

Put

$$C_4 - A_4(J + L_{A_1}L_FJ^* + L_{A_1}L_FM^\dagger G(M^\dagger)^* L_FL_{A_1})A_4^* = A,$$

$$J + L_{A_1}L_FJ^* + L_{A_1}L_FM^\dagger G(M^\dagger)^* L_FL_{A_1} = J',$$

$$A_4L_{A_1}L_FL_M = N,$$

$$L_FL_{A_1}A_4^* = P; \quad (23)$$

then

$$f(X) = A - NVP - (NVP)^*. \quad (24)$$

Note that $A = A^*$ and $\mathcal{R}(N) \subseteq \mathcal{R}(P^*)$. Thus, applying (17) to (24), we get the following:

$$\begin{aligned} \max r[f(X)] &= \max_V r(A - NVP - (NVP)^*) \\ &= \min \left\{ r[A \ P^*], r \begin{bmatrix} A & N \\ N^* & 0 \end{bmatrix} \right\}, \\ \min r[f(X)] &= \min_V r(A - NVP - (NVP)^*) \\ &= 2r[A \ P^*] + r \begin{bmatrix} A & N \\ N^* & 0 \end{bmatrix} - 2r \begin{bmatrix} A & N \\ P & 0 \end{bmatrix}. \end{aligned} \quad (25)$$

Now we simplify the ranks of block matrices in (25).

In view of Lemma 2, block Gaussian elimination, (11), (12), and (23), we have the following:

$$r(F) = r(B_1^*L_{A_1}) = r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} - r(A_1),$$

$$r(M) = r(SL_F) = r \begin{bmatrix} S \\ F \end{bmatrix} - r(F)$$

$$= r \begin{bmatrix} A_3L_{A_1} \\ B_1^*L_{A_1} \end{bmatrix} - r(F)$$

$$= r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} - r(A_1) - r(F),$$

$$\begin{aligned} r[A \ P^*] &= r[C_4 - A_4JA_4^* \ P^*] \\ &= r \begin{bmatrix} C_4 - A_4JA_4^* & A_4L_{A_1} \\ 0 & F \end{bmatrix} - r(F) \\ &= r \begin{bmatrix} C_4 - A_4JA_4^* & A_4 \\ 0 & B_1^* \\ 0 & A_1 \end{bmatrix} - r(F) - r(A_1) \\ &= r \begin{bmatrix} C_4 & A_4 \\ C_2^*A_4^* & B_1^* \\ C_1A_4^* & A_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} r \begin{bmatrix} A & N \\ N^* & 0 \end{bmatrix} &= r \begin{bmatrix} C_4 - A_4J'A_4^* & A_4L_{A_1}L_FL_M \\ R_{M^*}R_{F^*}R_{A_1^*}A_4^* & 0 \end{bmatrix} \\ &= r \begin{bmatrix} C_4 - A_4J'A_4^* & A_4 & 0 & 0 & 0 \\ A_4^* & 0 & A_3^* & B_1 & A_1^* \\ 0 & A_3 & 0 & 0 & 0 \\ 0 & B_1^* & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - 2r(M) - 2r(F) - 2r(A_1) \\ &= r \begin{bmatrix} C_4 & A_4 & 0 & 0 & 0 \\ A_4^* & 0 & A_3^* & B_1 & A_1^* \\ A_3J'A_4^* & A_3 & 0 & 0 & 0 \\ B_1^*J'A_4^* & B_1^* & 0 & 0 & 0 \\ A_1J'A_4^* & A_1 & 0 & 0 & 0 \end{bmatrix} - 2r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} \\ &= r \begin{bmatrix} C_4 & A_4 & 0 & 0 & 0 \\ A_4^* & 0 & A_3^* & B_1 & A_1^* \\ 0 & A_3 & -C_3 & -A_3C_2 & -A_3C_1^* \\ 0 & B_1^* & -C_2^*A_3^* & -C_2^*B_1 & -C_2^*A_1^* \\ 0 & A_1 & -C_1A_3^* & -C_1B_1 & -C_1A_1^* \end{bmatrix} \\ &\quad - 2r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} \\ &= r \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3C_2 & -A_3C_1^* \\ B_1^* & 0 & -C_2^*A_3^* & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1A_3^* & -C_1B_1 & -C_1A_1^* \end{bmatrix} \\ &\quad - 2r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix}, \\ r \begin{bmatrix} A & N \\ P & 0 \end{bmatrix} &= r \begin{bmatrix} C_4 - A_4J'A_4^* & A_4L_{A_1}L_FL_M \\ R_{F^*}R_{A_1^*}A_4^* & 0 \end{bmatrix} \\ &= r \begin{bmatrix} C_4 & A_4 & 0 & 0 \\ A_4^* & 0 & B_1 & A_1^* \\ 0 & A_3 & -A_3C_2 & -A_3C_1^* \\ 0 & B_1^* & -C_2^*B_1 & -C_2^*A_1^* \\ 0 & A_1 & -C_1B_1 & -C_1A_1^* \end{bmatrix} \\ &\quad - r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 &= r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3C_2 & -A_3C_1^* \\ B_1^* & 0 & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1B_1 & -C_1A_1^* \end{bmatrix} \\
 &\quad - r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix}.
 \end{aligned} \tag{26}$$

Substituting (26) into (25) yields (18) and (20). \square

In Theorem 4, letting C_4 vanish and A_4 be I with appropriate size, respectively, we have the following.

Corollary 5. Assume that $A_1, C_1 \in \mathbb{H}^{m \times n}$, $B_1, C_2 \in \mathbb{H}^{n \times s}$, $A_3 \in \mathbb{H}^{r \times n}$, and $C_3 \in \mathbb{H}^{r \times r}$ are given; then the maximal and minimal ranks of the Hermitian solution X to the system (10) can be expressed as

$$\max r(X) = \min \{a, b\}, \tag{27}$$

where

$$\begin{aligned}
 a &= n + r \begin{bmatrix} C_2^* \\ C_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix}, \\
 b &= 2n + r \begin{bmatrix} C_3 & A_3C_2 & A_3C_1^* \\ C_2^*A_3 & C_2^*B_1 & C_2^*A_1^* \\ C_1A_3^* & C_1B_1 & C_1A_1^* \end{bmatrix} - 2r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix}, \\
 \min r(X) &= 2r \begin{bmatrix} C_2^* \\ C_1 \end{bmatrix} \\
 &\quad + r \begin{bmatrix} C_3 & A_3C_2 & A_3C_1^* \\ C_2^*A_3 & C_2^*B_1 & C_2^*A_1^* \\ C_1A_3^* & C_1B_1 & C_1A_1^* \end{bmatrix} \\
 &\quad - 2r \begin{bmatrix} A_3C_2 & A_3C_1^* \\ C_2^*B_1 & C_2^*A_1^* \\ C_1B_1 & C_1A_1^* \end{bmatrix}.
 \end{aligned} \tag{28}$$

In Theorem 4, assuming that A_1, B_1, C_1 , and C_2 vanish, we have the following.

Corollary 6. Suppose that the matrix equation $A_3XA_3^* = C_3$ is consistent; then the extremal ranks of the quaternion matrix expression $f(X)$ defined as (9) subject to $A_3XA_3^* = C_3$ are the following:

$$\begin{aligned}
 &\max r[f(X)] \\
 &= \min \left\{ r[C_4 \ A_4], r \begin{bmatrix} 0 & A_4^* & A_3^* \\ A_4 & C_4 & 0 \\ A_3 & 0 & -C_3 \end{bmatrix} - 2r(A_3) \right\}, \\
 &\min r[f(X)] = 2r[C_4 \ A_4] \\
 &\quad + r \begin{bmatrix} 0 & A_4^* & A_3^* \\ A_4 & C_4 & 0 \\ A_3 & 0 & -C_3 \end{bmatrix} - 2r \begin{bmatrix} 0 & A_4^* \\ A_4 & C_4 \\ A_3 & 0 \end{bmatrix}.
 \end{aligned} \tag{29}$$

3. A Practical Solvability Condition for Hermitian Solution to System (5)

In this section, we use Theorem 4 to give a necessary and sufficient condition for the existence of Hermitian solution to system (5) by rank equalities.

Theorem 7. Let $A_1, C_1 \in \mathbb{H}^{m \times n}$, $B_1, C_2 \in \mathbb{H}^{n \times s}$, $A_3 \in \mathbb{H}^{r \times n}$, $C_3 \in \mathbb{H}^{r \times r}$, $A_4 \in \mathbb{H}^{t \times n}$, and $C_4 \in \mathbb{H}^{t \times t}$ be given; then the system (5) have Hermitian solution if and only if $C_3 = C_3^*$, (11), (13) hold, and the following equalities are all satisfied:

$$r[A_4 \ C_4] = r(A_4), \tag{30}$$

$$r \begin{bmatrix} C_4 & A_4 \\ C_2^*A_4^* & B_1^* \\ C_1A_4^* & A_1 \end{bmatrix} = r \begin{bmatrix} A_4 \\ B_1^* \\ A_1 \end{bmatrix}, \tag{31}$$

$$r \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3C_2 & -A_3C_1^* \\ B_1^* & 0 & -C_2^*A_3^* & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1A_3^* & -C_1B_1 & -C_1A_1^* \end{bmatrix} = 2r \begin{bmatrix} A_4 \\ A_3 \\ B_1^* \\ A_1 \end{bmatrix}. \tag{32}$$

Proof. It is obvious that the system (5) have Hermitian solution if and only if the system (10) have Hermitian solution and

$$\min r[f(X)] = 0, \tag{33}$$

where $f(X)$ is defined as (9) subject to system (10). Let X_0 be a Hermitian solution to the system (5); then X_0 is a Hermitian solution to system (10) and X_0 satisfies $A_4X_0A_4^* = C_4$. Hence, Lemma 1 yields $C_3 = C_3^*$, (11), (13), and (30). It follows

from

$$\begin{aligned} & \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ A_3 X_0 & 0 & I & 0 & 0 \\ B_1^* X_0 & 0 & 0 & I & 0 \\ A_1 X_0 & 0 & 0 & 0 & I \end{bmatrix} \\ & \times \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* A_3 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A_3^* & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \\ & \times \begin{bmatrix} I & -X_0 A_4^* & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & 0 & 0 & 0 & 0 \\ A_3 & 0 & 0 & 0 & 0 \\ B_1^* & 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{34}$$

that (32) holds. Similarly, we can obtain (31).

Conversely, assume that $C_3 = C_3^*$, (11), (13) hold; then by Lemma 1, system (10) have Hermitian solution. By (20), (31)-(32), and

$$r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \geq r \begin{bmatrix} A_4 \\ A_3 \\ B_1^* \\ A_1 \end{bmatrix} + r \begin{bmatrix} A_4 \\ B_1^* \\ A_1 \end{bmatrix} \tag{35}$$

we can get

$$\min r [f(X)] \leq 0. \tag{36}$$

However,

$$\min r [f(X)] \geq 0. \tag{37}$$

Hence (33) holds, implying that the system (5) have Hermitian solution. \square

By Theorem 7, we can also get the following.

Corollary 8. Suppose that $A_3, C_3, A_4,$ and C_4 are those in Theorem 7; then the quaternion matrix equations $A_3 X A_3^* = C_3$ and $A_4 X A_4^* = C_4$ have common Hermitian solution if and only if (30) hold and the following equalities are satisfied:

$$\begin{aligned} & r [A_3 \ C_3] = r (A_3), \\ & r \begin{bmatrix} 0 & A_4^* & A_3^* \\ A_4 & C_4 & 0 \\ A_3 & 0 & -C_3 \end{bmatrix} = 2r \begin{bmatrix} A_3 \\ A_4 \end{bmatrix}. \end{aligned} \tag{38}$$

Corollary 9. Suppose that $A_1, C_1 \in \mathbb{H}^{m \times n}, B_1, C_2 \in \mathbb{H}^{n \times s},$ and $A, B \in \mathbb{H}^{n \times n}$ are Hermitian. Then A and B have a common

Hermitian g -inverse which is a solution to the system (2) if and only if (11) holds and the following equalities are all satisfied:

$$r \begin{bmatrix} A_1 & C_1 A \\ B_1^* & C_2^* A \\ A & A \end{bmatrix} = r \begin{bmatrix} A_1 \\ B_1^* \\ A \end{bmatrix}, \tag{39}$$

$$r \begin{bmatrix} A_1 & C_1 B \\ B_1^* & C_2^* B \\ B & B \end{bmatrix} = r \begin{bmatrix} A_1 \\ B_1^* \\ B \end{bmatrix},$$

$$r \begin{bmatrix} 0 & B & A & B_1 & A_1^* \\ B & B & 0 & 0 & 0 \\ A & 0 & -A & -A C_2 & -A C_1^* \\ B_1^* & 0 & -C_2^* A & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} = 2r \begin{bmatrix} B \\ A \\ B_1^* \\ A_1 \end{bmatrix}. \tag{40}$$

4. Extremal Ranks of Schur Complement Subject to (2)

As is well known, for a given block matrix

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \tag{41}$$

where A and D are Hermitian quaternion matrices with appropriate sizes, then the Hermitian Schur complement of A in M is defined as

$$S_A = D - B^* A^{\sim} B, \tag{42}$$

where A^{\sim} is a Hermitian g -inverse of A , that is, $A^{\sim} \in \{X \mid AXA = A, X = X^*\}$.

Now we use Theorem 4 to establish the extremal ranks of S_A given by (42) with respect to A^{\sim} which is a solution to system (2).

Theorem 10. Suppose $A_1, C_1 \in \mathbb{H}^{m \times n}, B_1, C_2 \in \mathbb{H}^{n \times s}, D \in \mathbb{H}^{t \times t}, B \in \mathbb{H}^{n \times t},$ and $A \in \mathbb{H}^{n \times n}$ are given and system (2) is consistent; then the extreme ranks of S_A given by (42) with respect to A^{\sim} which is a solution of (2) are the following:

$$\max_{\substack{A_1 A^{\sim} = C_1 \\ A^{\sim} B_1 = C_2}} r (S_A) = \min \{a, b\}, \tag{43}$$

where

$$\begin{aligned}
 a &= r \begin{bmatrix} D & B^* \\ C_2^* B & B_1^* \\ C_1 B & A_1 \end{bmatrix} - r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix}, \\
 b &= r \begin{bmatrix} 0 & B & A & B_1 & A_1^* \\ B^* & D & 0 & 0 & 0 \\ A & 0 & -A & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^* A & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} - 2r \begin{bmatrix} A \\ B_1^* \\ A_1 \end{bmatrix}, \\
 \min_{\substack{A_1 A^- = C_1 \\ A^- B_1 = C_2}} r(S_A) &= 2r \begin{bmatrix} D & B^* \\ C_2^* B & B_1^* \\ C_1 B & A_1 \end{bmatrix} \\
 &+ r \begin{bmatrix} 0 & B & A & B_1 & A_1^* \\ B^* & D & 0 & 0 & 0 \\ A & 0 & -A & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^* A & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \\
 &- 2r \begin{bmatrix} 0 & B & B_1 & A_1^* \\ B^* & D & 0 & 0 \\ A & 0 & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix}. \tag{44}
 \end{aligned}$$

Proof. It is obvious that

$$\begin{aligned}
 &\max_{A_1 A^- = C_1, A^- B_1 = C_2} r(D - B^* A^- B) \\
 &= \max_{A_1 X = C_1, X B_1 = C_2, A X A = A} r(D - B^* X B), \\
 &\min_{A_1 A^- = C_1, A^- B_1 = C_2} r(D - B^* A^- B) \\
 &= \min_{A_1 X = C_1, X B_1 = C_2, A X A = A} r(D - B^* X B). \tag{45}
 \end{aligned}$$

Thus in Theorem 4 and its proof, letting $A_3 = A_3^* = C_3 = A$, $A_4 = B^*$, and $C_4 = D$, we can easily get the proof. \square

In Theorem 10, let A_1, C_1, B_1 , and C_2 vanish. Then we can easily get the following.

Corollary 11. *The extreme ranks of S_A given by (42) with respect to A^- are the following:*

$$\begin{aligned}
 \max_{A^-} (S_A) &= \min \left\{ r \begin{bmatrix} D & B^* \end{bmatrix}, r \begin{bmatrix} 0 & B & A \\ B^* & D & 0 \\ A & 0 & -A \end{bmatrix} - 2r(A) \right\}, \\
 \min_{A^-} (S_A) &= 2r \begin{bmatrix} D & B^* \end{bmatrix} + r \begin{bmatrix} 0 & B & A \\ B^* & D & 0 \\ A & 0 & -A \end{bmatrix} - 2r \begin{bmatrix} 0 & B \\ B^* & D \\ A & 0 \end{bmatrix}. \tag{46}
 \end{aligned}$$

5. The Rank Invariance of (9)

As another application of Theorem 4, we in this section consider the rank invariance of the matrix expression (9) with respect to the Hermitian solution of system (10).

Theorem 12. *Suppose that (10) have Hermitian solution; then the rank of $f(X)$ defined by (9) with respect to the Hermitian solution of (10) is invariant if and only if*

$$\begin{aligned}
 &r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \\
 &= r \begin{bmatrix} C_4 & A_4 \\ C_2^* A_4^* & B_1^* \\ C_1 A_4^* & A_1 \end{bmatrix} + r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix}, \\
 &r \begin{bmatrix} 0 & A_4^* & A_3^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 & 0 \\ A_3 & 0 & -C_3 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* A_3^* & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 A_3^* & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} + r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} \tag{47} \\
 &= r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} + r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix},
 \end{aligned}$$

or

$$\begin{aligned}
 &r \begin{bmatrix} 0 & A_4^* & B_1 & A_1^* \\ A_4 & C_4 & 0 & 0 \\ A_3 & 0 & -A_3 C_2 & -A_3 C_1^* \\ B_1^* & 0 & -C_2^* B_1 & -C_2^* A_1^* \\ A_1 & 0 & -C_1 B_1 & -C_1 A_1^* \end{bmatrix} \\
 &= r \begin{bmatrix} C_4 & A_4 \\ C_2^* A_4^* & B_1^* \\ C_1 A_4^* & A_1 \end{bmatrix} + r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix}. \tag{48}
 \end{aligned}$$

Proof. It is obvious that the rank of $f(X)$ with respect to Hermitian solution of system (10) is invariant if and only if

$$\max r[f(X)] - \min r[f(X)] = 0. \tag{49}$$

By (49), Theorem 4, and simplifications, we can get (47) and (48). \square

Corollary 13. *The rank of S_A defined by (42) with respect to A^* which is a solution to system (2) is invariant if and only if*

$$\begin{aligned}
 r \begin{bmatrix} 0 & B & B_1 & A_1^* \\ B^* & D & 0 & 0 \\ A & 0 & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1B_1 & -C_1A_1^* \end{bmatrix} &= r \begin{bmatrix} D & B^* \\ C_2^*B & B_1^* \\ C_1B & A_1 \end{bmatrix} + r \begin{bmatrix} A \\ B_1^* \\ A_1 \end{bmatrix}, \\
 r \begin{bmatrix} 0 & B & A & B_1 & A_1^* \\ B^* & D & 0 & 0 & 0 \\ A & 0 & -A & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^*A & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1A & -C_1B_1 & -C_1A_1^* \end{bmatrix} &+ r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} \\
 = r \begin{bmatrix} 0 & B & B_1 & A_1^* \\ B^* & D & 0 & 0 \\ A & 0 & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1B_1 & -C_1A_1^* \end{bmatrix} &+ r \begin{bmatrix} A \\ B_1^* \\ A_1 \end{bmatrix}, \tag{50}
 \end{aligned}$$

or

$$\begin{aligned}
 r \begin{bmatrix} 0 & B & B_1 & A_1^* \\ B^* & D & 0 & 0 \\ A & 0 & -AC_2 & -AC_1^* \\ B_1^* & 0 & -C_2^*B_1 & -C_2^*A_1^* \\ A_1 & 0 & -C_1B_1 & -C_1A_1^* \end{bmatrix} & \\
 = r \begin{bmatrix} D & B^* \\ C_2^*B & B_1^* \\ C_1B & A_1 \end{bmatrix} &+ r \begin{bmatrix} A \\ B_1^* \\ A_1 \end{bmatrix}. \tag{51}
 \end{aligned}$$

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