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Research Article

Positive Almost Periodic Solutions for a Discrete Competitive System Subject to Feedback Controls

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This paper concerns a discrete competitive system subject to feedback controls. By using Lyapunov function and some preliminary lemmas, the existence and uniformly asymptotic stability of unique positive almost periodic solution of the system are investigated. Numerical simulations suggest the feasibility of our theoretical results.

1. Introduction

Many real world phenomena are studied through discrete mathematical models involving difference equations which are more suitable than the continuous ones when the populations have nonoverlapping generations. On the other hand, discrete models can also provide efficient computational models of continuous models for numerical stimulations; therefore, the studies of dynamic systems governed by difference equations have attracted more attention from scholars. Many good results concerned with discrete systems are deliberated (see [1–7] in detail).

Recently, in [1] we consider the following discrete two-species competitive almost system

$$x_1(n+1)=x_1(n)\exp\left[r_1(n)-a_1(n)x_1(n)-\frac{c_2(n)x_2(n)}{1+x_2(n)}\right],$$
$$x_2(n+1)=x_2(n)\exp\left[r_2(n)-a_2(n)x_2(n)-\frac{c_1(n)x_1(n)}{1+x_1(n)}\right],$$
$$n=0,1,2,\ldots. \tag{1}$$

Here $x_i(n)$ stand for the densities of species x_i at the nth generation, $r_i(n)$ represent the natural growth rates of species x_i at the nth generation, $a_i(n)$ are the intraspecific effects of the nth generation of species x_i on own population, and $c_i(n)$ measure the interspecific effects of the nth generation of species x_i on species x_j $(i,j=1,2;\ i\neq j)$. The coefficients $\{r_i(n)\}$, $\{a_i(n)\}$, and $\{c_i(n)\}$ are bounded positive almost periodic sequences. We established a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (1) (see [1]).

Note that ecosystems in the real world are often disturbed by outside continuous forces. In the language of control, we call the disturbance functions control variables and they can be regarded as feedback controls. For more discussions on this direction, we can refer to [8–14] and the references cited therein. Motivated by the above ideas we can establish the discrete two-species competitive almost system with feedback controls

$$\begin{aligned}x_1(n+1)&=x_1(n)\exp\left[r_1(n)-a_1(n)x_1(n)-\frac{c_2(n)x_2(n)}{1+x_2(n)}\right.\\&\qquad\left.-e_1(n)u_1(n)\vphantom{\frac{c_2}{1}}\right],\\x_2(n+1)&=x_2(n)\exp\left[r_2(n)-a_2(n)x_2(n)-\frac{c_1(n)x_1(n)}{1+x_1(n)}\right.\\&\qquad\left.-e_2(n)u_2(n)\vphantom{\frac{c_1}{1}}\right],\end{aligned}$$

$$\begin{aligned}
&\Delta u_1(n) = -b_1(n)\,u_1(n) + d_1(n)\,x_1(n),\\
&\Delta u_2(n) = -b_2(n)\,u_2(n) + d_2(n)\,x_2(n),\\
&\qquad\qquad n = 0, 1, 2, \ldots,
\end{aligned} \tag{2}$$

where u_1 and u_2 represent control variables and $\Delta u_1(n) = u_1(n+1) - u_1(n), \Delta u_2(n) = u_2(n+1) - u_2(n)$ are the forward difference operators. $\{b_i(n)\}$, $\{d_i(n)\}$ and $\{e_i(n)\}$ are bounded positive almost periodic sequences, where $0 < b_i(n) < 1$, $i = 1, 2$. To belong to the direction of [1], in this contribution, we continue to discuss the effect of feedback controls and establish a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (2).

The rest of this paper is organized as follows. In the next section, we introduce some notations, definitions, and lemmas which are available for our main results. Sufficient conditions for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (2) are established in Section 3. In Section 4, we carry out numerical simulations to substantiate our analytical results.

2. Preliminaries

In this section, we give some notations, definitions, and lemmas which will be useful for the later sections.

$\mathbb{R}, \mathbb{R}^+, \mathbb{Z}$, and $\mathbb{Z}^+$ denote the sets of real numbers, nonnegative real numbers, integers, and nonnegative integers, respectively. $\mathbb{R}^4$ and $\mathbb{R}^k$ denote the cone of 4-dimensional and k-dimensional real Euclidean space, respectively. For an almost periodic sequence $\{g(n)\}$ defined on $\mathbb{Z}^+$, the notations below will be used

$$g^U = \sup_{n\in\mathbb{Z}^+} \{g(n)\}, \qquad g^L = \inf_{n\in\mathbb{Z}^+} \{g(n)\}. \tag{3}$$

Definition 1 (see [4]). A sequence $x : \mathbb{Z} \to \mathbb{R}^k$ is called an almost periodic sequence if the ε-translation set of x,

$$E\{\varepsilon, x\} := \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \varepsilon,\ \forall n \in \mathbb{Z}\}, \tag{4}$$

is a relatively dense set in $\mathbb{Z}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each discrete interval of length $l(\varepsilon)$ contains a $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \varepsilon, \qquad \forall n \in \mathbb{Z}. \tag{5}$$

τ is called the ε-translation number of $x(n)$.

Definition 2 (see [4]). Let $f : \mathbb{Z} \times \mathbb{D} \to \mathbb{R}^k$, where $\mathbb{D}$ is an open set in $\mathbb{R}^k$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in \mathbb{D}$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and any compact set $\mathbb{S}$ in $\mathbb{D}$, there exists a positive integer $l(\varepsilon, \mathbb{S})$ such that any interval of length $l(\varepsilon, \mathbb{S})$ contains a integer τ for which

$$\left| f(n+\tau, x) - f(n, x) \right| < \varepsilon, \tag{6}$$

for all $n \in \mathbb{Z}$ and all $x \in \mathbb{S}$. τ is called the ε-translation number of $f(n, x)$.

Lemma 3 (see [4]). *$\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{t'_k\} \subset \mathbb{Z}$ there exists a subsequence $\{t_k\} \subset \{t'_k\}$ such that $x(n+t_k)$ converges uniformly on $n \in \mathbb{Z}$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Consider the following almost periodic difference system

$$x(n+1) = f(n, x(n)), \qquad n \in \mathbb{Z}^+, \tag{7}$$

where $f : \mathbb{Z}^+ \times \mathbb{S}_B \to \mathbb{R}^k, \mathbb{S}_B = \{x \in \mathbb{R}^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in n uniformly for $x \in \mathbb{S}_B$ and is continuous in x. The product system of (7) is the following system:

$$x(n+1) = f(n, x(n)), \qquad y(n+1) = f(n, y(n)), \tag{8}$$

and Zhang [5] established the following result.

Lemma 4 (see [5]). *Suppose that there exists a Lyapunov function $V(n, x, y)$ defined for $n \in \mathbb{Z}^+$, $\|x\| < B$, $\|y\| < B$ satisfying the following conditions:*

(i) *$a(\|x-y\|) \le V(n, x, y) \le b(\|x-y\|)$, where $a, b \in K$ with $K = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \alpha(0) = 0$ and α is increasing$\}$;*

(ii) *$|V(n, x_1, y_1) - V(n, x_2, y_2)| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|)$, where $L > 0$ is a constant;*

(iii) *$\Delta V_{(8)}(n, x, y) \le -\gamma V(n, x, y)$, where $0 < \gamma < 1$ is a constant and*

$$\Delta V_{(8)}(n, x, y) = V(n+1, f(n, x), f(n, y)) - V(n, x, y). \tag{9}$$

Moreover, if there exists a solution $\varphi(n)$ of system (7) such that $\|\varphi(n)\| \le B^* < B$ for $n \in \mathbb{Z}^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of system (7) which satisfies $\|p(n)\| \le B^*$. In particular, if $f(n, x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of system (7) of periodic ω.

3. Main Results

We first give the following two propositions which are useful for our main results.

Proposition 5. *Any positive solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ of system (2) satisfies*

$$\begin{aligned}
&\limsup_{n\to+\infty} x_1(n) \le x_1^*, \qquad \limsup_{n\to+\infty} x_2(n) \le x_2^*,\\
&\limsup_{n\to+\infty} u_1(n) \le u_1^*, \qquad \limsup_{n\to+\infty} u_2(n) \le u_2^*,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&x_1^* = \frac{\exp\left(r_1^U - 1\right)}{a_1^L}, \qquad x_2^* = \frac{\exp\left(r_2^U - 1\right)}{a_2^L},\\
&u_1^* = \frac{x_1^* d_1^U}{b_1^L}, \qquad u_2^* = \frac{x_2^* d_2^U}{b_2^L}.
\end{aligned} \tag{11}$$

Proof. We first prove that $\limsup_{n\to+\infty} x_1(n) \le x_1^*$; to do so, we consider *Cases* (1) and (2).

Case (1). Suppose that there exists an $n_0 \in \mathbb{Z}^+$ such that $x_1(n_0+1) \ge x_1(n_0)$; it follows from the first equation of system (2) that

$$r_1(n_0) - a_1(n_0)x_1(n_0) - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0) \ge 0, \tag{12}$$

which implies

$$x_1(n_0) \le \frac{r_1(n_0)}{a_1(n_0)} \le \frac{r_1^U}{a_1^L} \le \frac{\exp\left(r_1^U-1\right)}{a_1^L} = x_1^*. \tag{13}$$

Hence,

$$\begin{aligned} x_1(n_0+1) &= x_1(n_0)\exp\left[r_1(n_0) - a_1(n_0)x_1(n_0)\right. \\ &\qquad \left. - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0)\right] \\ &\le x_1(n_0)\exp\left[r_1^U - a_1^L x_1(n_0)\right] \\ &= \frac{r_1^U}{a_1^L}\cdot\frac{a_1^L}{r_1^U}x_1(n_0)\exp\left[r_1^U\left(1-\frac{a_1^L}{r_1^U}x_1(n_0)\right)\right] \\ &\le \frac{r_1^U}{a_1^L}\cdot\frac{\exp\left(r_1^U-1\right)}{r_1^U} = \frac{\exp\left(r_1^U-1\right)}{a_1^L} = x_1^*, \end{aligned} \tag{14}$$

where we use the fact that $\max_{x\in\mathbb{R}^+} x\exp[r(1-x)] = \exp(r-1)/r$ for $r>0$.

It is claimed that $x_1(n) \le x_1^*$ for all $n \ge n_0$. By way of contradiction, assume that there is a $q_0 > n_0$ such that $x_1(q_0) > x_1^*$, then $q_0 \ge n_0+2$. Set $\tilde{q}_0 = \min\{q_0 : q_0 \ge n_0+2, x_1(q_0) > x_1^*\}$; that is to say, $x_1(\tilde{q}_0) > x_1^*$ and $\tilde{q}_0 \ge n_0+2$; then $x_1(\tilde{q}_0) > x_1^* \ge x_1(\tilde{q}_0-1)$. It is easy to see that $x_1(\tilde{q}_0) \le x_1^*$ from the above argument, which is a contradiction. Therefore, $x_1(n) \le x_1^*$ for all $n \ge n_0$, then $\limsup_{n\to+\infty} x_1(n) \le x_1^*$. This proves the claim.

Case (2). Suppose that $x_1(n) > x_1(n+1)$ for all $n \in \mathbb{Z}^+$. In particular, $\lim_{n\to+\infty} x_1(n)$ exists, denoted by $\overline{x}_1$. We will prove that $\overline{x}_1 \le x_1^*$ by way of contradiction, if $\overline{x}_1 > x_1^*$, then by taking limit in the first equation in system (2) we have

$$\begin{aligned} &\lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n)\right] \\ &\quad = 0. \end{aligned} \tag{15}$$

Notice that $r_1^U/a_1^L \le \exp(r_1^U-1)/a_1^L = x_1^* < \overline{x}_1$, so we have

$$\begin{aligned} &\lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n)\right] \\ &\quad \le \lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n)\right] \\ &\quad \le r_1^U - a_1^L\overline{x}_1 < 0, \end{aligned} \tag{16}$$

which is a contradiction. This proves the claim. Hence, $\limsup_{n\to+\infty} x_1(n) \le x_1^*$.

Similar to $x_1(n)$, we can prove that $\limsup_{n\to+\infty} x_2(n) \le x_2^*$.

Next, we prove $\limsup_{n\to+\infty} u_1(t) \le u_1^*$. For any $\varepsilon > 0$, there exists a large enough integer $l_0 \in \mathbb{Z}^+$ such that $x_i(n) \le x_i^* + \varepsilon, i = 1,2$ for all $n \ge l_0$. We have from the third equation of system (2) that

$$\begin{aligned} u_1(n) &= \prod_{i=0}^{n-1}\left(1-b_1(i)\right)\left[u_1(0) + \sum_{i=0}^{n-1}\frac{d_1(i)x_1(i)}{\prod_{j=0}^{i}\left(1-b_1(j)\right)}\right] \\ &\le \left(1-b_1^L\right)^n\left(u_1(0)+\vartheta_1\right) + d_1^U\left(x_1^*+\varepsilon\right) \\ &\quad \times \sum_{i=l_0}^{n-1}\prod_{j=i+1}^{n-1}\left(1-b_1(j)\right) \\ &\le \left(1-b_1^L\right)^n\left(u_1(0)+\vartheta_1\right) + d_1^U\left(x_1^*+\varepsilon\right)\sum_{i=l_0}^{n-1}\left(1-b_1^L\right)^{n-i-1}, \end{aligned} \tag{17}$$

where $\vartheta_1 = \sum_{i=0}^{l_0-1}(d_1(i)x_1(i)/\prod_{j=0}^{i}(1-b_1(j)))$. Since $0 < b_1^L < 1$, we can find a positive number s such that $1-b_1^L = e^{-s}$, then by using Stolz's theorem, we obtain that

$$\lim_{n\to+\infty}\sum_{i=l_0}^{n-1}\left(1-b_1^L\right)^{n-i-1} = \lim_{n\to+\infty}\frac{\sum_{i=l_0}^{n-1}e^{s(i+1)}}{e^{sn}} = \frac{1}{1-e^{-s}} = \frac{1}{b_1^L}. \tag{18}$$

Thus $\limsup_{n\to+\infty} u_1(n) \le (x_1^*+\varepsilon)d_1^U/b_1^L$. Since ε is arbitrary, $\limsup_{n\to+\infty} u_1(n) \le u_1^*$ is valid. Analogously, we can prove $\limsup_{n\to+\infty} u_2(n) \le u_2^*$. Hence, the proof of Proposition 5 is complete. □

Proposition 6. *If the following inequalities*

$$r_1^L - c_2^U - e_1^U u_1^* > 0, \qquad r_2^L - c_1^U - e_2^U u_2^* > 0 \tag{19}$$

hold. Then any positive solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ of system (2) satisfies

$$\begin{aligned} &\liminf_{n\to+\infty} x_1(n) \ge x_{1*}, \qquad \liminf_{n\to+\infty} x_2(n) \ge x_{2*}, \\ &\liminf_{n\to+\infty} u_1(n) \ge u_{1*}, \qquad \liminf_{n\to+\infty} u_2(n) \ge u_{2*}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} x_{1*} &= \frac{r_1^L - c_2^U - e_1^U u_1^*}{a_1^U}\exp\left[r_1^L - a_1^U x_1^* - c_2^U - e_1^U u_1^*\right], \\ x_{2*} &= \frac{r_2^L - c_1^U - e_2^U u_2^*}{a_2^U}\exp\left[r_2^L - a_2^U x_2^* - c_1^U - e_2^U u_2^*\right], \\ u_{1*} &= \frac{x_{1*}d_1^L}{b_1^U}, \qquad u_{2*} = \frac{x_{2*}d_2^L}{b_2^U}. \end{aligned} \tag{21}$$

Proof. For any small enough $\varepsilon > 0$, which satisfies $r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon) > 0$, according to Proposition 5, there exists $n^* \in \mathbb{Z}^+$ such that

$$\begin{aligned} x_1(n) &\le x_1^* + \varepsilon, \qquad x_2(n) \le x_2^* + \varepsilon, \\ u_1(n) &\le u_1^* + \varepsilon, \qquad u_2(n) \le u_2^* + \varepsilon, \\ &\text{for } n \ge n^*. \end{aligned} \tag{22}$$

We first present *Cases* (1) and (2) to prove that $\liminf_{n\to+\infty} x_1(n) \ge x_{1*}$.

Case (1). Assuming that there exists a positive integer $n_0 \ge n^*$ such that $x_1(n_0+1) \le x_1(n_0)$, we have from the first equation of system (2) that

$$\begin{aligned} x_1(n_0+1) &= x_1(n_0)\exp\left[r_1(n_0) - a_1(n_0)x_1(n_0) - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0)\right] \\ &\ge x_1(n_0)\exp\left[r_1^L - a_1^U x_1(n_0) - c_2^U - e_1^U(u_1^* + \varepsilon)\right]. \end{aligned} \tag{23}$$

Therefore,

$$r_1^L - a_1^U x_1(n_0) - c_2^U - e_1^U(u_1^* + \varepsilon) \le 0, \tag{24}$$

which implies that

$$x_1(n_0) \ge \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U} > 0. \tag{25}$$

Then

$$\begin{aligned} x_1(n_0+1) &= x_1(n_0)\exp\left[r_1(n_0) - a_1(n_0)x_1(n_0) - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0)\right] \\ &\ge \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U} \\ &\quad \times \exp\left[r_1^L - a_1^U(x_1^* + \varepsilon) - c_2^U - e_1^U(u_1^* + \varepsilon)\right]. \end{aligned} \tag{26}$$

Hence $x_1(n_0+1) \ge x_{1\varepsilon}$, where

$$x_{1\varepsilon} = \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U}\exp\left[r_1^L - a_1^U(x_1^* + \varepsilon) - c_2^U - e_1^U(u_1^* + \varepsilon)\right]. \tag{27}$$

We claim that $x_1(n) \ge x_{1\varepsilon}$ for $n \ge n_0$. By way of contradiction, assume that there exists a $q_0 \ge n_0$ such that $x_1(q_0) < x_{1\varepsilon}$, then $q_0 \ge n_0 + 2$. Let $\overline{q}_0 = \min\{q_0 : q_0 \ge n_0+2, x_1(q_0) < x_{1\varepsilon}\}$, that is, $x_1(\overline{q}_0) < x_{1\varepsilon}$ and $\overline{q}_0 \ge n_0+2$, then $x_1(\overline{q}_0) < x_{1\varepsilon} \le x_1(\overline{q}_0 - 1)$, and the above argument produces that $x_1(\overline{q}_0) \ge x_{1\varepsilon}$, which is a contradiction. Thus, $x_1(n) \ge x_{1\varepsilon}$ for all $n \ge n_0$; since ε can be sufficiently small, it obtains that $\liminf_{n\to+\infty} x_1(n) \ge x_{1*}$. This proves the claim.

Case (2). We assume that $x_1(n+1) > x_1(n)$ for all $n \in \mathbb{Z}^+$. Then $\lim_{n\to+\infty} x_1(n)$ exists, denoted by $\underline{x}_1$. We claim that

$$\underline{x}_1 \ge \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U}. \tag{28}$$

By way of contradiction, assume that $\underline{x}_1 < (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U$. Taking limit in the first equation in system (2) yields

$$\lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n)\right] = 0. \tag{29}$$

However,

$$\begin{aligned} &\lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n)\right] \\ &\quad \ge \lim_{n\to+\infty}\left[r_1(n) - a_1(n)x_1(n) - c_2(n) - e_1(n)u_1(n)\right] \\ &\quad \ge r_1^L - a_1^U\underline{x}_1 - c_2^U - e_1^U(u_1^* + \varepsilon) > 0, \end{aligned} \tag{30}$$

which is a contradiction. It implies that $\underline{x}_1 \ge (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U$. By the fact that $\min_{x\in\mathbb{R}^+}\{[\exp(x-1)]/x\} = 1$, we obtain that $x_1^* = \exp(r_1^U - 1)/a_1^L \ge r_1^U/a_1^L \ge r_1^L/a_1^U$, which means $x_1^* + \varepsilon > r_1^L/a_1^U$. From (27), we know that $\underline{x}_1 \ge (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U \ge x_{1\varepsilon}$. Therefore, $\liminf_{n\to+\infty} x_1(n) \ge x_{1\varepsilon}$.

Since ε can be sufficiently small, we have

$$\liminf_{n\to+\infty} x_1(n) \ge x_{1*} = \frac{r_1^L - c_2^U - e_1^U u_1^*}{a_1^U}\exp\left[r_1^L - a_1^U x_1^* - c_2^U - e_1^U u_1^*\right]. \tag{31}$$

By a similar argument, we can prove that

$$\liminf_{n\to+\infty} x_2(n) \ge x_{2*} = \frac{r_2^L - c_1^U - e_2^U u_2^*}{a_2^U}\exp\left[r_2^L - a_2^U x_2^* - c_1^U - e_2^U u_2^*\right]. \tag{32}$$

Now we prove that $\liminf_{n\to+\infty} u_1(n) \ge u_{1*}$. For any small enough $\varepsilon > 0$, there exists a positive integer l_0 such that $x_i(n) \ge x_{i*} - \varepsilon > 0$ for $n \ge l_0$.

By the third equation of system (2), we obtain that

$$\begin{aligned} u_1(n) &= \prod_{i=0}^{n-1}(1-b_1(i))\left[u_1(0)+\sum_{i=0}^{n-1}\frac{d_1(i)x_1(i)}{\prod_{j=0}^{i}(1-b_1(j))}\right] \\ &\geq \left(1-b_1^U\right)^n\left(u_1(0)+\varrho_1\right)+d_1^L\left(x_{1*}-\varepsilon\right) \\ &\quad\times\sum_{i=l_0}^{n-1}\prod_{j=i+1}^{n-1}(1-b_1(j)) \\ &\geq \left(1-b_1^U\right)^n\left(u_1(0)+\varrho_1\right) \\ &\quad+d_1^L\left(x_{1*}-\varepsilon\right)\sum_{i=l_0}^{n-1}\left(1-b_1^U\right)^{n-i-1}, \end{aligned} \tag{33}$$

where $\varrho_1=\sum_{i=0}^{l_0-1}(d_1(i)x_1(i)/\prod_{j=0}^{i}(1-b_1(j)))$. Since $0<b_1^U<1$, we can find a positive number t such that $1-b_1^U=e^{-t}$, then by Stolz's theorem, we have

$$\lim_{n\to+\infty}\sum_{i=l_0}^{n-1}\left(1-b_1^U\right)^{n-i-1}=\lim_{n\to+\infty}\frac{\sum_{i=l_0}^{n-1}e^{t(i+1)}}{e^{tn}}=\frac{1}{1-e^{-t}}=\frac{1}{b_1^U}. \tag{34}$$

Thus $\liminf_{n\to+\infty}u_1(n)\geq(x_{1*}-\varepsilon)d_1^L/b_1^U$, by the arbitrary of ε, $\liminf_{n\to+\infty}u_1(n)\geq u_{1*}$ is valid. The conclusion about $u_2(n)$ can be obtained in a similar way. Thus the proof of Proposition 6 is complete. □

Denote

$$\begin{aligned} \Omega=\{&(x_1,x_2,u_1,u_2)\mid x_{i*}\leq x_i(n)\leq x_i^*, \\ &u_{i*}\leq u_i(n)\leq u_i^*\ (i=1,2)\}. \end{aligned} \tag{35}$$

From the proofs of Propositions 5 and 6, we know that the set Ω is an invariant set of system (2) under the assumptions in (19).

Theorem 7. *If the assumptions in* (19) *are satisfied, then* $\Omega\neq\phi$.

Proof. We have from system (2) that

$$\begin{aligned} x_i(n)&=x_i(0)\exp\sum_{m=0}^{n-1}\left[r_i(m)-a_i(m)x_i(m)\right. \\ &\qquad\left.-\frac{c_j(m)x_j(m)}{1+x_j(m)}-e_i(m)u_i(m)\right], \\ u_i(n)&=u_i(0)-\sum_{m=0}^{n-1}\left[b_i(m)u_i(m)-d_i(m)x_i(m)\right], \end{aligned} \tag{36}$$

for $i,j=1,2,i\neq j$. Based on Propositions 5 and 6, any solution $(x_1(n),x_2(n),u_1(n),u_2(n))$ of system (2) satisfies (10) and (20). Thus, for any $\varepsilon>0$, there exists p_0 large enough such that

$$\begin{gathered} x_{i*}-\varepsilon\leq x_i(n)\leq x_i^*+\varepsilon, \\ u_{i*}-\varepsilon\leq u_i(n)\leq u_i^*+\varepsilon, \\ \forall n\geq p_0,\quad i=1,2. \end{gathered} \tag{37}$$

Setting to $\{t_k\}$ be any positive integer valued sequence such that $t_k\to+\infty$ as $k\to+\infty$, it is easy to show that there exists a subsequence of $\{t_k\}$ still denoted by $\{t_k\}$, such that $x_i(n+t_k)\to x_i^*(n), u_i(n+t_k)\to u_i^*(n)$ uniformly in n on any finite subset D of $\mathbb{Z}^+$ as $k\to+\infty$, where $D=\{\alpha_1,\alpha_2,\ldots,\alpha_l\}$, $\alpha_h\in\mathbb{Z}^+$ $(h=1,2,\ldots,l)$ and l is a finite number.

In reality, for any finite subset $D\subset\mathbb{Z}^+$, $t_k+\alpha_h>p_0, h=1,2,\ldots,l$, when k is large enough. Thus, $x_{i*}-\varepsilon\leq x_i(n+t_k)\leq x_i^*+\varepsilon, u_{i*}-\varepsilon\leq u_i(n+t_k)\leq u_i^*+\varepsilon$, which mean that $\{x_i(n+t_k)\}, \{u_i(n+t_k)\}$ are uniformly bounded for k large enough.

Now, for $\alpha_1\in D$, we can select a subsequence $\{t_k^{(1)}\}$ of $\{t_k\}$ such that $\{x_i(\alpha_1+t_k^{(1)})\}, \{u_i(\alpha_1+t_k^{(1)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Similarly, for $\alpha_2\in D$, we can also select a subsequence $\{t_k^{(2)}\}$ of $\{t_k^{(1)}\}$ such that $\{x_i(\alpha_2+t_k^{(2)})\}, \{u_i(\alpha_2+t_k^{(2)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Repeating the above process, for $\alpha_l\in D$, we choose a subsequence $\{t_k^{(l)}\}$ of $\{t_k^{(l-1)}\}$ such that $\{x_i(\alpha_l+t_k^{(l)})\}, \{u_i(\alpha_l+t_k^{(l)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Then we choose the sequence $\{t_k^{(l)}\}$ which is a subsequence of $\{t_k\}$ still denoted by $\{t_k\}$; for all $n\in D$, we have $x_i(n+t_k)\to x_i^*(n), u_i(n+t_k)\to u_i^*(n)$ uniformly in $n\in D$ as $k\to+\infty$. Therefore, the conclusion is true by the arbitrary of D.

Consider the almost periodicity of $\{r_i(n)\}, \{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}$, and $\{e_i(n)\}, i=1,2$, for the above sequence $\{t_k\}$, $t_k\to+\infty$ as $k\to+\infty$, there exists a subsequence still denoted by $\{t_k\}$ such that

$$\begin{aligned} r_i(n+t_k)&\longrightarrow r_i(n), & a_i(n+t_k)&\longrightarrow a_i(n), \\ b_i(n+t_k)&\longrightarrow b_i(n), & c_i(n+t_k)&\longrightarrow c_i(n), \\ d_i(n+t_k)&\longrightarrow d_i(n), & e_i(n+t_k)&\longrightarrow e_i(n), \end{aligned} \tag{38}$$

as $k\to+\infty$ uniformly on $\mathbb{Z}^+$.

For any $\beta\in\mathbb{Z}^+$, we can presume that $t_k+\beta\geq p_0$ for k large enough. Let $n\in\mathbb{Z}^+$, by an inductive argument of system (2) from $t_k+\beta$ to $n+t_k+\beta$ which results in

$$\begin{aligned} x_i(n+t_k+\beta)&=x_i(t_k+\beta) \\ &\quad\times\exp\sum_{m=t_k+\beta}^{n+t_k+\beta-1}\left[r_i(m)-a_i(m)x_i(m)\right. \\ &\qquad-\frac{c_j(m)x_j(m)}{1+x_j(m)} \\ &\qquad\left.-e_i(m)u_i(m)\right], \end{aligned}$$

$$u_i(n+t_k+\beta) = u_i(t_k+\beta) - \sum_{m=t_k+\beta}^{n+t_k+\beta-1} [b_i(m) u_i(m) - d_i(m) x_i(m)], \tag{39}$$

for $i, j = 1, 2$, $i \neq j$, so it derives that

$$\begin{aligned} x_i(n+t_k+\beta) &= x_i(t_k+\beta) \exp \sum_{m=\beta}^{n+\beta-1} \Bigg[r_i(m+t_k) - a_i(m+t_k) \times x_i(m+t_k) - \frac{c_j(m+t_k) x_j(m+t_k)}{1 + x_j(m+t_k)} - e_i(m+t_k) u_i(m+t_k) \Bigg], \\ u_i(n+t_k+\beta) &= u_i(t_k+\beta) - \sum_{m=\beta}^{n+\beta-1} [b_i(m+t_k) u_i(m+t_k) - d_i(m+t_k) x_i(m+t_k)]. \end{aligned} \tag{40}$$

Letting $k \to +\infty$, one has

$$\begin{aligned} x_i^*(n+\beta) &= x_i^*(\beta) \exp \sum_{m=\beta}^{n+\beta-1} \Bigg[r_i(m) - a_i(m) x_i^*(m) - \frac{c_j(m) x_j^*(m)}{1 + x_j^*(m)} - e_i(m) u_i^*(m) \Bigg], \\ u_i^*(n+\beta) &= u_i^*(\beta) - \sum_{m=\beta}^{n+\beta-1} [b_i(m) u_i^*(m) - d_i(m) x_i^*(m)]. \end{aligned} \tag{41}$$

By the arbitrary of β, we can easily see that $(x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n))$ is a solution of system (2) on $\mathbb{Z}^+$, and

$$\begin{gathered} 0 < x_{i*} - \varepsilon \leq x_i^*(n) \leq x_i^* + \varepsilon, \\ 0 < u_{i*} - \varepsilon \leq u_i^*(n) \leq u_i^* + \varepsilon, \\ \forall n \in \mathbb{Z}^+, \quad i = 1, 2. \end{gathered} \tag{42}$$

Since ε is an arbitrarily small positive number, we obtain that

$$\begin{gathered} 0 < x_{i*} \leq x_i^*(n) \leq x_i^*, \\ 0 < u_{i*} \leq u_i^*(n) \leq u_i^*, \\ \forall n \in \mathbb{Z}^+, \quad i = 1, 2. \end{gathered} \tag{43}$$

So $\Omega \neq \phi$. This completes the proof. □

The following theorem concerns the existence and uniformly asymptotical stability of unique positive almost periodic solution of system (2).

Theorem 8. *Suppose the inequalities in* (19) *are satisfied; furthermore,* $0 < \gamma < 1$, *where* $\gamma = \min\{s_{ij}, s_{ij}^*\}$ *and*

$$\begin{aligned} s_{ij} &= 2a_i^L x_{i*} - \left(a_i^U\right)^2 \left(x_i^*\right)^2 - \left(d_i^U\right)^2 \left(x_i^*\right)^2 - \frac{a_i^U c_j^U x_i^* x_j^*}{\left(1 + x_{j*}\right)^2} - \frac{c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} \\ &\quad - a_i^U e_i^U x_i^* - e_i^U - d_i^U \left(1 - b_i^L\right) x_i^* - \frac{a_j^U c_i^U x_i^* x_j^*}{\left(1 + x_{i*}\right)^2} \\ &\quad - \frac{c_i^U x_i^*}{\left(1 + x_{i*}\right)^2} - \frac{e_j^U c_i^U x_i^*}{\left(1 + x_{i*}\right)^2} - \frac{\left(c_i^U\right)^2 \left(x_i^*\right)^2}{\left(1 + x_{i*}\right)^4}, \\ s_{ij}^* &= 2b_i^L - a_i^U e_i^U x_i^* - d_i^U \left(1 - b_i^L\right) x_i^* - e_i^U - \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} - \left(e_i^U\right)^2 - \left(b_i^U\right)^2, \end{aligned} \tag{44}$$

$i, j = 1, 2, i \neq j$, then there exists a unique uniformly asymptotically stable positive almost periodic solution of system (2) *which is bounded by Ω for all $n \in \mathbb{Z}^+$.*

Proof. We first make the change of variables

$$y_1(n) = \ln x_1(n), \qquad y_2(n) = \ln x_2(n), \tag{45}$$

then system (2) can be reformulated as

$$\begin{gathered} y_i(n+1) = y_i(n) + r_i(n) - a_i(n) e^{y_i(n)} - \frac{c_j(n) e^{y_j(n)}}{1 + e^{y_j(n)}} - e_i(n) u_i(n), \\ \Delta u_i(n) = -b_i(n) u_i(n) + d_i(n) e^{y_i(n)}, \\ i = 1, 2, \quad i \neq j. \end{gathered} \tag{46}$$

By Theorem 7, it is easy to see that there exists a bounded solution $(y_1(n), y_2(n), u_1(n), u_2(n))$ of system (46) satisfying

$$\begin{gathered} \ln x_{i*} \leq y_i(n) \leq \ln x_i^*, \qquad u_{i*} \leq u_i(n) \leq u_i^*, \\ i = 1, 2, \quad n \in \mathbb{Z}^+. \end{gathered} \tag{47}$$

Then $|y_i(n)| \le A_i, |u_i(n)| \le B_i$, where $A_i = \max\{|\ln x_{i*}|, |\ln x_i^*|\}, B_i = \max\{|u_{i*}|, |u_i^*|\}$, and $i = 1, 2$. Let $\|(y_1(n), y_2(n), u_1(n), u_2(n))\| = \sum_{i=1}^{2}\{|y_i(n)| + |u_i(n)|\}$, where $(y_1(n), y_2(n), u_1(n), u_2(n)) \in \mathbb{R}^4$.

The following associate product system of system (46) can be expressed as

$$\begin{aligned} y_i(n+1) &= y_i(n) + r_i(n) - a_i(n)e^{y_i(n)} \\ &\quad - \frac{c_j(n)e^{y_j(n)}}{1+e^{y_j(n)}} - e_i(n)u_i(n), \\ \Delta u_i(n) &= -b_i(n)u_i(n) + d_i(n)e^{y_i(n)}, \\ z_i(n+1) &= z_i(n) + r_i(n) - a_i(n)e^{z_i(n)} \\ &\quad - \frac{c_j(n)e^{z_j(n)}}{1+e^{z_j(n)}} - e_i(n)v_i(n), \\ \Delta v_i(n) &= -b_i(n)v_i(n) + d_i(n)e^{z_i(n)}, \\ &\qquad i = 1, 2, \quad i \neq j. \end{aligned} \tag{48}$$

Suppose that

$$\begin{aligned} Y &= (y_1(n), y_2(n), u_1(n), u_2(n)), \\ Z &= (z_1(n), z_2(n), v_1(n), v_2(n)) \end{aligned} \tag{49}$$

are any two solutions of system (46) defined on $\mathbb{S}$, then

$$\|Y\| \le B, \qquad \|Z\| \le B, \tag{50}$$

where $B = \sum_{i=1}^{2}\{A_i + B_i\}$, and

$$\begin{aligned} \mathbb{S} = \{&(y_1(n), y_2(n), u_1(n), u_2(n)) \mid \ln x_{i*} \le y_i(n) \le \ln x_i^*, \\ &u_{i*} \le u_i(n) \le u_i^*, i = 1, 2, n \in \mathbb{Z}^+\}. \end{aligned} \tag{51}$$

Let us construct the following Lyapunov function defined on $\mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$:

$$V(n, Y, Z) = \sum_{i=1}^{2}\left\{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\right\}. \tag{52}$$

Obviously, $\|Y - Z\| = \sum_{i=1}^{2}\{|y_i(n) - z_i(n)| + |u_i(n) - v_i(n)|\}$ is equivalent to $\|Y - Z\|_* = \{\sum_{i=1}^{2}[(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2]\}^{1/2}$; that is, there exist two constants $D_1 > 0$, $D_2 > 0$, such that

$$D_1\|Y - Z\| \le \|Y - Z\|_* \le D_2\|Y - Z\|. \tag{53}$$

Consequently,

$$(D_1\|Y - Z\|)^2 \le V(n, Y, Z) \le (D_2\|Y - Z\|)^2. \tag{54}$$

Denote $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = D_1^2x^2$, $b(x) = D_2^2x^2$, thus the condition (i) of Lemma 4 is satisfied.

In addition, for any $(n, Y, Z), (n, \overline{Y}, \overline{Z}) \in \mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$, we find that

$$\begin{aligned} &\left|V(n, Y, Z) - V(n, \overline{Y}, \overline{Z})\right| \\ &\quad = \left|\sum_{i=1}^{2}\left\{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\right\}\right. \\ &\qquad \left. - \sum_{i=1}^{2}\left\{(\overline{y}_i(n) - \overline{z}_i(n))^2 + (\overline{u}_i(n) - \overline{v}_i(n))^2\right\}\right| \\ &\quad \le \sum_{i=1}^{2}\left|(y_i(n) - z_i(n))^2 - (\overline{y}_i(n) - \overline{z}_i(n))^2\right| \\ &\qquad + \sum_{i=1}^{2}\left|(u_i(n) - v_i(n))^2 - (\overline{u}_i(n) - \overline{v}_i(n))^2\right| \\ &\quad = \sum_{i=1}^{2}\left\{\left|(y_i(n) - z_i(n)) + (\overline{y}_i(n) - \overline{z}_i(n))\right|\right. \\ &\qquad \left. \cdot\left|(y_i(n) - z_i(n)) - (\overline{y}_i(n) - \overline{z}_i(n))\right|\right\} \\ &\qquad + \sum_{i=1}^{2}\left\{\left|(u_i(n) - v_i(n)) + (\overline{u}_i(n) - \overline{v}_i(n))\right|\right. \\ &\qquad \left. \cdot\left|(u_i(n) - v_i(n)) - (\overline{u}_i(n) - \overline{v}_i(n))\right|\right\} \\ &\quad \le \sum_{i=1}^{2}\left\{\left(|y_i(n)| + |z_i(n)| + |\overline{y}_i(n)| + |\overline{z}_i(n)|\right)\right. \\ &\qquad \left. \cdot\left(|y_i(n) - \overline{y}_i(n)| + |z_i(n) - \overline{z}_i(n)|\right)\right\} \\ &\qquad + \sum_{i=1}^{2}\left\{\left(|u_i(n)| + |v_i(n)| + |\overline{u}_i(n)| + |\overline{v}_i(n)|\right)\right. \\ &\qquad \left. \cdot\left(|u_i(n) - \overline{u}_i(n)| + |v_i(n) - \overline{v}_i(n)|\right)\right\} \\ &\quad \le \lambda\left\{\sum_{i=1}^{2}\left\{|y_i(n) - \overline{y}_i(n)| + |u_i(n) - \overline{u}_i(n)|\right\}\right. \\ &\qquad \left. + \sum_{i=1}^{2}\left\{|z_i(n) - \overline{z}_i(n)| + |v_i(n) - \overline{v}_i(n)|\right\}\right\} \\ &\quad = \lambda\left\{\left\|Y - \overline{Y}\right\| + \left\|Z - \overline{Z}\right\|\right\}, \end{aligned} \tag{55}$$

where $\overline{Y} = (\overline{y}_1(n), \overline{y}_2(n), \overline{u}_1(n), \overline{u}_2(n)), \overline{Z} = (\overline{z}_1(n), \overline{z}_2(n), \overline{v}_1(n), \overline{v}_2(n))$, and $\lambda = 4\max\{A_i, B_i\}$ $(i = 1, 2)$. Hence, the condition (ii) of Lemma 4 is satisfied.

At last, we calculate the $\Delta V(n, Y, Z)$ of $V(n, Y, Z)$ along the solutions of system (48) and obtain that

$$\begin{aligned} &\Delta V_{(48)}(n, Y, Z) \\ &\quad = V(n+1, Y, Z) - V(n, Y, Z) \end{aligned}$$

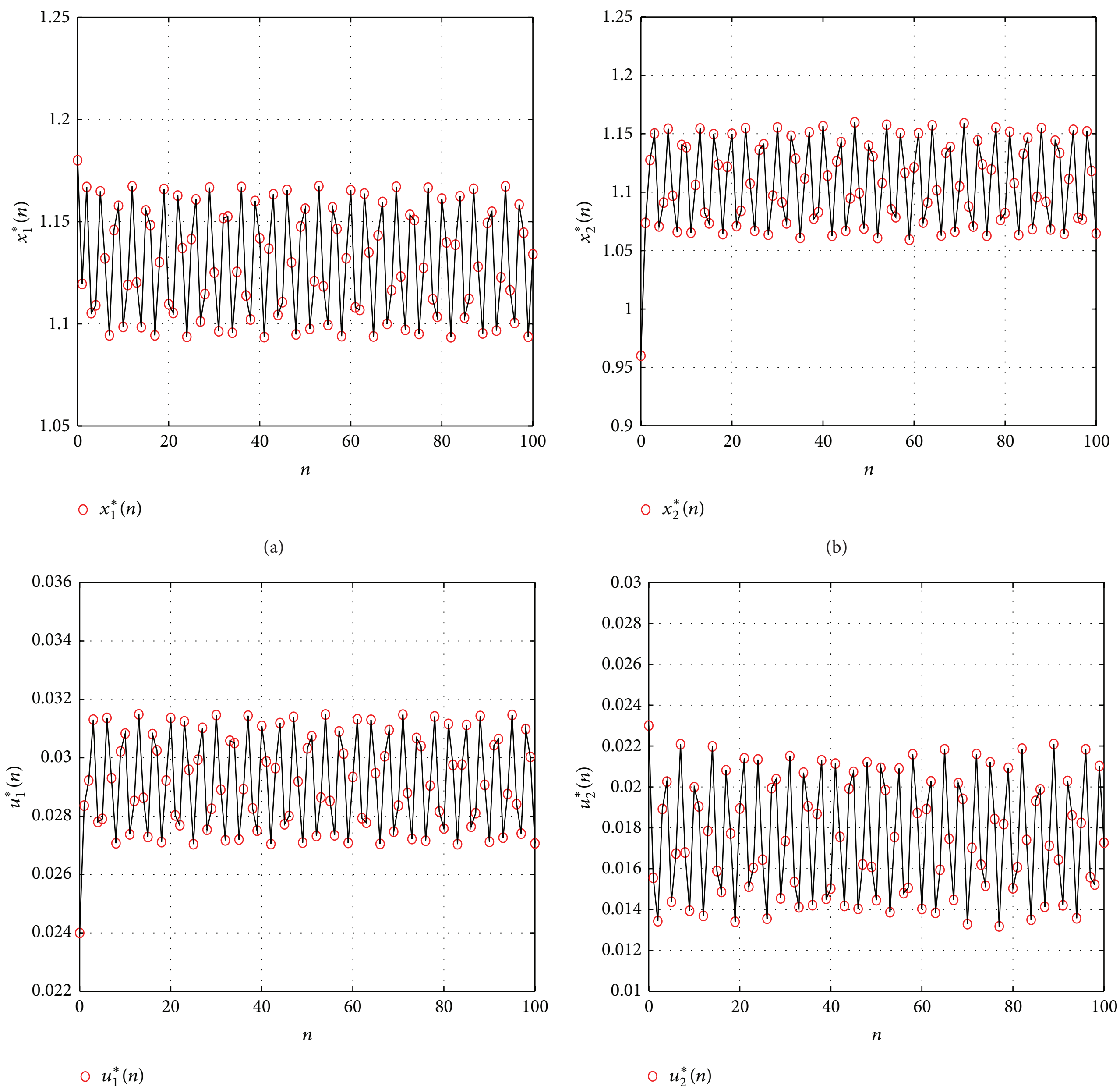
$$
\begin{aligned}
&= \sum_{i=1}^{2} \left\{ \left(y_i(n+1) - z_i(n+1)\right)^2 + \left(u_i(n+1) - v_i(n+1)\right)^2 \right\} \\
&\quad - \sum_{i=1}^{2} \left\{ \left(y_i(n) - z_i(n)\right)^2 + \left(u_i(n) - v_i(n)\right)^2 \right\} \\
&= \sum_{i=1}^{2} \left\{ \left(y_i(n+1) - z_i(n+1)\right)^2 - \left(y_i(n) - z_i(n)\right)^2 \right. \\
&\qquad \left. + \left(u_i(n+1) - v_i(n+1)\right)^2 - \left(u_i(n) - v_i(n)\right)^2 \right\} \\
&= \sum_{i=1}^{2} \left\{ \left[\left(y_i(n) - z_i(n)\right) - a_i(n)\left(e^{y_i(n)} - e^{z_i(n)}\right) \right. \right. \\
&\qquad - e_i(n)\left(u_i(n) - v_i(n)\right) \\
&\qquad \left. - c_j(n)\left(\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}}\right)\right]^2 \\
&\qquad - \left(y_i(n) - z_i(n)\right)^2 + \left[\left(1 - b_i(n)\right)\left(u_i(n) - v_i(n)\right) \right. \\
&\qquad\qquad \left. + d_i(n)\left(e^{y_i(n)} - e^{z_i(n)}\right)\right]^2 \\
&\qquad \left. - \left(u_i(n) - v_i(n)\right)^2 \right\} \\
&= \sum_{i=1}^{2} \left\{ -2a_i(n)\left(y_i(n) - z_i(n)\right)\left(e^{y_i(n)} - e^{z_i(n)}\right) \right. \\
&\qquad - 2e_i(n)\left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right) \\
&\qquad - 2c_j(n)\left(y_i(n) - z_i(n)\right)\left(\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}}\right) \\
&\qquad + a_i^2(n)\left(e^{y_i(n)} - e^{z_i(n)}\right)^2 + 2a_i(n)e_i(n) \\
&\qquad \times \left(u_i(n) - v_i(n)\right)\left(e^{y_i(n)} - e^{z_i(n)}\right) + 2a_i(n)c_j(n) \\
&\qquad \times \left(\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}}\right)\left(e^{y_i(n)} - e^{z_i(n)}\right) \\
&\qquad + e_i^2(n)\left(u_i(n) - v_i(n)\right)^2 + 2e_i(n)c_j(n) \\
&\qquad \times \left(\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}}\right)\left(u_i(n) - v_i(n)\right) \\
&\qquad + c_j^2(n)\left(\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}}\right)^2 \\
&\qquad + b_i(n)\left(b_i(n) - 2\right)\left(u_i(n) - v_i(n)\right)^2 + 2d_i(n) \\
&\qquad \times \left(1 - b_i(n)\right)\left(e^{y_i(n)} - e^{z_i(n)}\right)\left(u_i(n) - v_i(n)\right) \\
&\qquad \left. + d_i^2(n)\left(e^{y_i(n)} - e^{z_i(n)}\right)^2 \right\}.
\end{aligned} \tag{56}
$$

By the mean-value theorem, one has

$$
\begin{gathered}
e^{y_i(n)} - e^{z_i(n)} = \theta_i(n)\left(y_i(n) - z_i(n)\right), \\
\frac{e^{y_j(n)}}{1+e^{y_j(n)}} - \frac{e^{z_j(n)}}{1+e^{z_j(n)}} = \frac{\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2}\left(y_j(n) - z_j(n)\right), \\
i, j = 1, 2, \quad i \neq j,
\end{gathered} \tag{57}
$$

where $\theta_i(n)$ and $\sigma_j(n)$ lie between $e^{y_i(n)}$ and $e^{z_i(n)}$ and $e^{y_j(n)}$ and $e^{z_j(n)}$, respectively. Then $x_{i*} \leq \theta_i(n) \leq x_i^*$, $x_{j*} \leq \sigma_j(n) \leq x_j^*$, $i, j = 1, 2$, $i \neq j$, $n \in \mathbb{Z}^+$. Substituting (57) into (56), one has

$$
\begin{aligned}
&\Delta V_{(48)}(n, Y, Z) \\
&\quad = \sum_{i=1}^{2} \left\{ -2a_i(n)\theta_i(n)\left(y_i(n) - z_i(n)\right)^2 \right. \\
&\qquad - 2e_i(n)\left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right) \\
&\qquad - \frac{2c_j(n)\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2}\left(y_i(n) - z_i(n)\right)\left(y_j(n) - z_j(n)\right) \\
&\qquad + a_i^2(n)\theta_i^2(n)\left(y_i(n) - z_i(n)\right)^2 \\
&\qquad + 2a_i(n)e_i(n)\theta_i(n)\left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right) \\
&\qquad + \frac{2a_i(n)c_j(n)\theta_i(n)\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2}\left(y_i(n) - z_i(n)\right) \\
&\qquad \times \left(y_j(n) - z_j(n)\right) + e_i^2(n)\left(u_i(n) - v_i(n)\right)^2 \\
&\qquad + \frac{2e_i(n)c_j(n)\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2}\left(y_j(n) - z_j(n)\right) \\
&\qquad \times \left(u_i(n) - v_i(n)\right) + \frac{c_j^2(n)\sigma_j^2(n)}{\left(1+\sigma_j(n)\right)^4}\left(y_j(n) - z_j(n)\right)^2 \\
&\qquad + b_i(n)\left(b_i(n) - 2\right)\left(u_i(n) - v_i(n)\right)^2 \\
&\qquad + 2d_i(n)\left(1 - b_i(n)\right)\theta_i(n)\left(y_i(n) - z_i(n)\right) \\
&\qquad \left. \times \left(u_i(n) - v_i(n)\right) + d_i^2(n)\theta_i^2(n)\left(y_i(n) - z_i(n)\right)^2 \right\} \\
&\quad = \sum_{i=1}^{2} \left\{ \left(-2a_i(n)\theta_i(n) + a_i^2(n)\theta_i^2(n) + d_i^2(n)\theta_i^2(n)\right) \right. \\
&\qquad \times \left(y_i(n) - z_i(n)\right)^2 + 2\left(\frac{a_i(n)c_j(n)\theta_i(n)\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2} \right. \\
&\qquad\qquad \left. - \frac{c_j(n)\sigma_j(n)}{\left(1+\sigma_j(n)\right)^2}\right)
\end{aligned}
$$



(a) (b) (c) (d)

Figure 1: Positive almost periodic solution of system (62). (a)–(d) Time-series of $x_1^*(n)$, $x_2^*(n)$, $u_1^*(n)$, and $u_2^*(n)$ with initial values $x_1^*(0) = 1.18$, $x_2^*(0) = 0.96$, $u_1^*(0) = 0.024$, and $u_2^*(0) = 0.023$ for $n \in [0, 100]$, respectively.

$$\times \left(y_i(n) - z_i(n)\right)\left(y_j(n) - z_j(n)\right)$$
$$+ 2\left(a_i(n) e_i(n) \theta_i(n) + d_i(n)\left(1 - b_i(n)\right)\right.$$
$$\left. \times \theta_i(n) - e_i(n)\right)\left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right)$$
$$+ \frac{2e_i(n) c_j(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2}\left(y_j(n) - z_j(n)\right)$$
$$\times \left(u_i(n) - v_i(n)\right) + \frac{c_j^2(n) \sigma_j^2(n)}{\left(1 + \sigma_j(n)\right)^4}\left(y_j(n) - z_j(n)\right)^2$$
$$\left. + \left(e_i^2(n) + b_i(n)\left(b_i(n) - 2\right)\right)\left(u_i(n) - v_i(n)\right)^2 \right\}$$
$$\leq \sum_{i=1}^{2} \left\{ \left(-2a_i(n) \theta_i(n) + a_i^2(n) \theta_i^2(n) + d_i^2(n) \theta_i^2(n)\right) \right.$$
$$\times \left(y_i(n) - z_i(n)\right)^2$$
$$+ 2\left|\left(\frac{a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2} - \frac{c_j(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2}\right)\right.$$

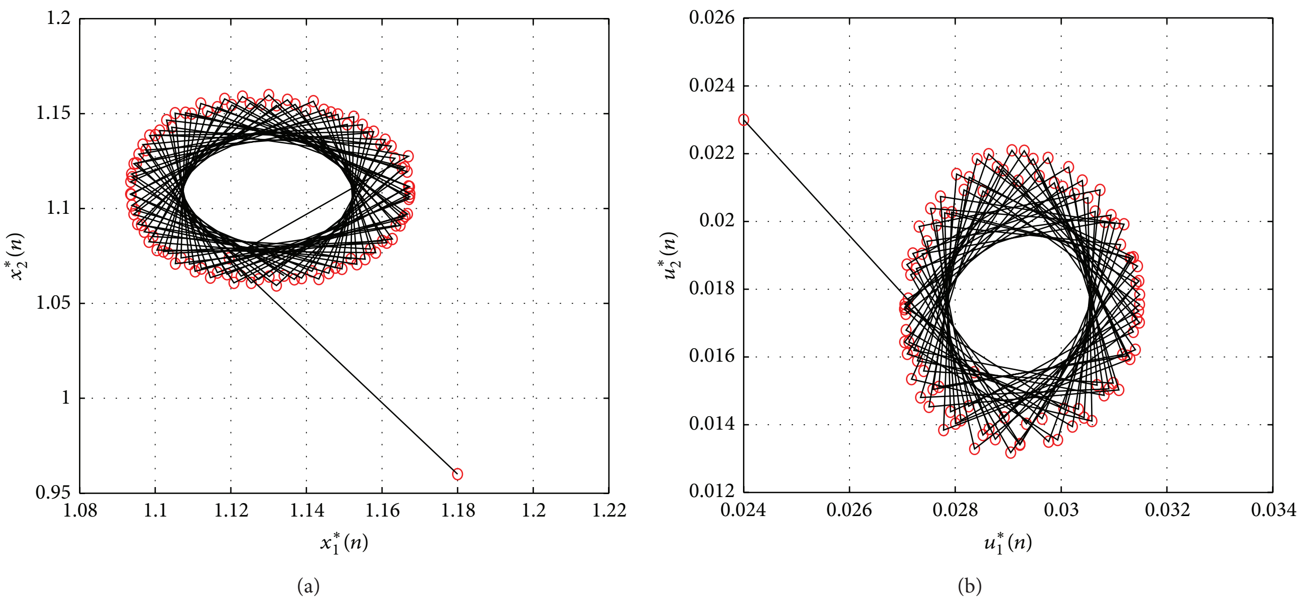


FIGURE 2: 2-dimensional phase portrait. (a) Phase portrait of $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(0) = 1.18$, $x_2^*(0) = 0.96$ for $n \in [0, 100]$, respectively. (b) Phase portrait of $u_1^*(n)$ and $u_2^*(n)$ with initial values $u_1^*(0) = 0.024$, $u_2^*(0) = 0.023$ for $n \in [0, 100]$, respectively.

$$
\begin{aligned}
&\times \left(y_i(n) - z_i(n)\right)\left(y_j(n) - z_j(n)\right)\Bigg| \\
&+ 2\Big|\left(a_i(n) e_i(n) \theta_i(n) + d_i(n)\left(1 - b_i(n)\right)\ \theta_i(n)\right. \\
&\qquad \left. - e_i(n)\right)\left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right)\Big| \\
&+ 2\left|\frac{e_i(n) c_j(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2}\left(u_i(n) - v_i(n)\right)\right. \\
&\qquad \left. \times \left(y_j(n) - z_j(n)\right)\right| + \frac{c_j^2(n) \sigma_j^2(n)}{\left(1 + \sigma_j(n)\right)^4} \\
&\times \left(y_j(n) - z_j(n)\right)^2 + \left(e_i^2(n) + b_i^2(n) - 2b_i(n)\right) \\
&\left.\times \left(u_i(n) - v_i(n)\right)^2\right\}.
\end{aligned} \tag{58}
$$

Now, we set that

$$
\Delta V_{(48)}(n, Y, Z) = \sum_{i=1}^{2}\left\{V_{1i} + V_{2ij} + V_{3i} + V_{4ij} + V_{5j} + V_{6i}\right\}, \quad i, j = 1, 2, \quad i \neq j, \tag{59}
$$

where

$$
\begin{aligned}
V_{1i} &= \left(-2a_i(n)\theta_i(n) + a_i^2(n)\theta_i^2(n) + d_i^2(n)\theta_i^2(n)\right) \\
&\quad \times \left(y_i(n) - z_i(n)\right)^2 \\
&\leq \left(-2a_i^L x_{i*} + \left(a_i^U\right)^2\left(x_i^*\right)^2 + \left(d_i^U\right)^2\left(x_i^*\right)^2\right) \\
&\quad \times \left(y_i(n) - z_i(n)\right)^2, \\
V_{2ij} &= 2\left|\left(\frac{a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2} - \frac{c_j(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2}\right)\right. \\
&\quad \left. \times \left(y_i(n) - z_i(n)\right)\left(y_j(n) - z_j(n)\right)\right| \\
&\leq \left(\frac{a_i^U c_j^U x_i^* x_j^*}{\left(1 + x_{j*}\right)^2} + \frac{c_j^U x_j^*}{\left(1 + x_{j*}\right)^2}\right) \\
&\quad \times \left\{\left(y_i(n) - z_i(n)\right)^2 + \left(y_j(n) - z_j(n)\right)^2\right\}, \\
V_{3i} &= 2\left|\left(a_i(n) e_i(n) \theta_i(n) + d_i(n)\left(1 - b_i(n)\right)\theta_i(n) - e_i(n)\right)\right. \\
&\quad \left. \times \left(y_i(n) - z_i(n)\right)\left(u_i(n) - v_i(n)\right)\right| \\
&\leq \left(a_i^U e_i^U x_i^* + d_i^U\left(1 - b_i^L\right)x_i^* + e_i^U\right) \\
&\quad \times \left\{\left(y_i(n) - z_i(n)\right)^2 + \left(u_i(n) - v_i(n)\right)^2\right\}, \\
V_{4ij} &= 2\left|\frac{e_i(n) c_j(n) \sigma_j(n)}{\left(1 + \sigma_j(n)\right)^2}\left(u_i(n) - v_i(n)\right)\left(y_j(n) - z_j(n)\right)\right| \\
&\leq \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2}\left\{\left(u_i(n) - v_i(n)\right)^2 + \left(y_j(n) - z_j(n)\right)^2\right\}, \\
V_{5j} &= \frac{c_j^2(n) \sigma_j^2(n)}{\left(1 + \sigma_j(n)\right)^4}\left(y_j(n) - z_j(n)\right)^2
\end{aligned}
$$

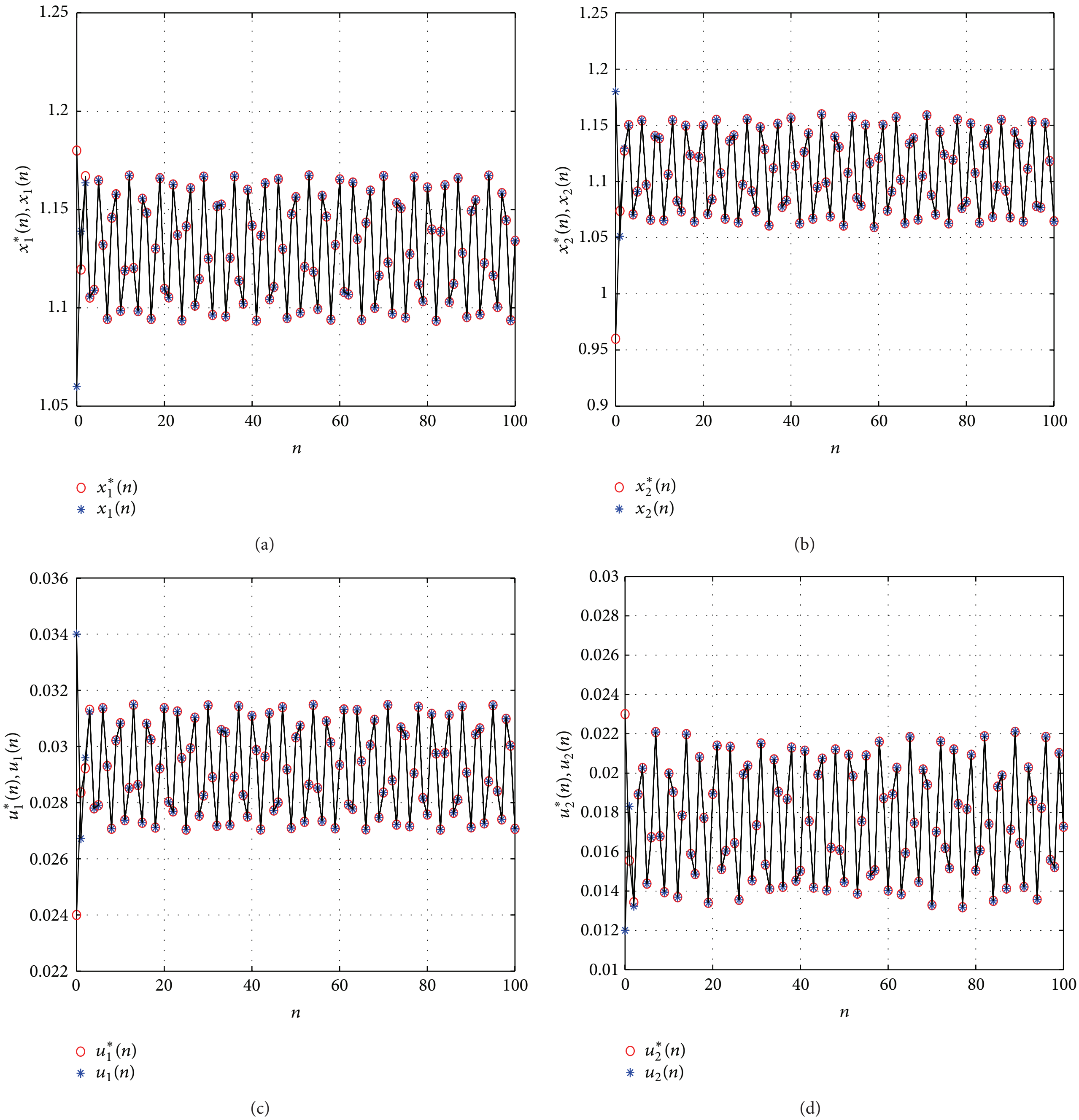


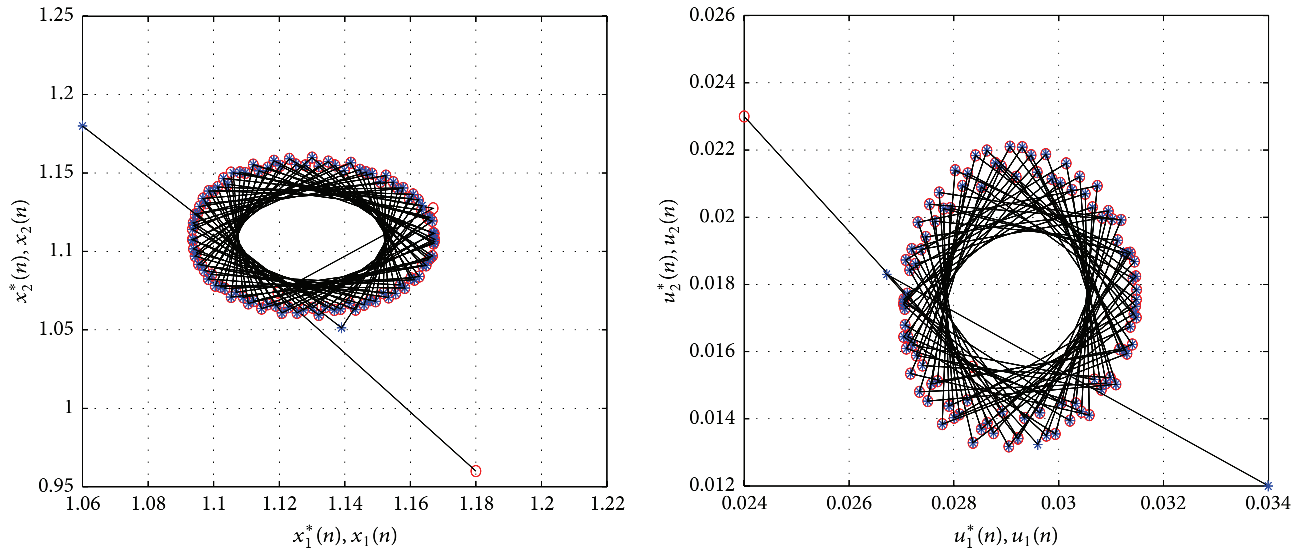
FIGURE 3: Uniformly asymptotic stability. (a)–(d) Time-series of $x_1^*(n)$, $x_2^*(n)$, $u_1^*(n)$ and $u_2^*(n)$ with initial values $x_1^*(0) = 1.18$, $x_2^*(0) = 0.96$, $u_1^*(0) = 0.024$, and $u_2^*(0) = 0.023$ and $x_1(n), x_2(n), u_1(n)$, and $u_2(n)$ with initial values $x_1(0) = 1.06$, $x_2(0) = 1.18$, $u_1(0) = 0.034$, and $u_2(0) = 0.012$ for $n \in [0, 100]$, respectively.

$$\le \frac{\left(c_j^U\right)^2\left(x_j^*\right)^2}{\left(1+x_{j*}\right)^4}\left(y_j(n)-z_j(n)\right)^2,$$

$$\begin{aligned} V_{6i} &= \left(e_i^2(n)+b_i^2(n)-2b_i(n)\right)\left(u_i(n)-v_i(n)\right)^2 \\ &\le \left(\left(e_i^U\right)^2+\left(b_i^U\right)^2-2b_i^L\right)\left(u_i(n)-v_i(n)\right)^2, \end{aligned} \tag{60}$$

which, together with (58), yields that

$$\begin{aligned} &\Delta V_{(48)}(n, Y, Z) \\ &\le \sum_{i=1}^{2}\Bigg\{\Bigg(-2a_i^L x_{i*}+\left(a_i^U\right)^2\left(x_i^*\right)^2+\left(d_i^U\right)^2\left(x_i^*\right)^2 \\ &\qquad +\frac{a_i^U c_j^U x_i^* x_j^*}{\left(1+x_{j*}\right)^2}+\frac{c_j^U x_j^*}{\left(1+x_{j*}\right)^2}+a_i^U e_i^U x_i^* \end{aligned}$$



(a) (b)

Figure 4: 2-dimensional phase portrait. (a) Phase portrait of $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(0) = 1.18$, $x_2^*(0) = 0.96$ and $x_1(n)$ and $x_2(n)$ with initial values $x_1(0) = 1.06$, $x_2(0) = 1.18$ for $n \in [0, 100]$, respectively. (b) Phase portrait of $u_1^*(n)$ and $u_2^*(n)$ with initial values $u_1^*(0) = 0.024$, $u_2^*(0) = 0.023$ and $u_1(n)$ and $u_2(n)$ with initial values $u_1(0) = 0.034$, $u_2(0) = 0.012$ for $n \in [0, 100]$, respectively.

$$
\begin{aligned}
&\left.+ d_i^U\left(1 - b_i^L\right)x_i^* + e_i^U\right)\left(y_i(n) - z_i(n)\right)^2 \\
&+ \left(\frac{a_i^U c_j^U x_i^* x_j^*}{\left(1 + x_{j*}\right)^2} + \frac{c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} + \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2}\right. \\
&\qquad\left. + \frac{\left(c_j^U\right)^2\left(x_j^*\right)^2}{\left(1 + x_{j*}\right)^4}\right)\left(y_j(n) - z_j(n)\right)^2 \\
&+ \left(a_i^U e_i^U x_i^* + d_i^U\left(1 - b_i^L\right)x_i^* + e_i^U + \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2}\right. \\
&\qquad\left.\left. + \left(e_i^U\right)^2 + \left(b_i^U\right)^2 - 2b_i^L\right)\left(u_i(n) - v_i(n)\right)^2\right\} \\
&= \sum_{i=1}^{2}\left\{\left(\left(- 2a_i^L x_{i*} + \left(a_i^U\right)^2\left(x_i^*\right)^2 + \left(d_i^U\right)^2\left(x_i^*\right)^2\right.\right.\right. \\
&\qquad + \frac{a_i^U c_j^U x_i^* x_j^*}{\left(1 + x_{j*}\right)^2} + \frac{c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} + a_i^U e_i^U x_i^* \\
&\qquad\left. + d_i^U\left(1 - b_i^L\right)x_i^* + e_i^U\right)\left(y_i(n) - z_i(n)\right)^2 \\
&\qquad + \left(\frac{a_j^U c_i^U x_i^* x_j^*}{\left(1 + x_{i*}\right)^2} + \frac{c_i^U x_i^*}{\left(1 + x_{i*}\right)^2} + \frac{e_j^U c_i^U x_i^*}{\left(1 + x_{i*}\right)^2}\right. \\
&\qquad\left. + \frac{\left(c_i^U\right)^2\left(x_i^*\right)^2}{\left(1 + x_{i*}\right)^4}\right)\left(y_i(n) - z_i(n)\right)^2 \\
&\qquad + \left(a_i^U e_i^U x_i^* + d_i^U\left(1 - b_i^L\right)x_i^* + e_i^U + \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2}\right. \\
&\qquad\left.\left. + \left(e_i^U\right)^2 + \left(b_i^U\right)^2 - 2b_i^L\right)\left(u_i(n) - v_i(n)\right)^2\right\} \\
&= -\sum_{i=1}^{2}\left\{\left(2a_i^L x_{i*} - \left(a_i^U\right)^2\left(x_i^*\right)^2 - \left(d_i^U\right)^2\left(x_i^*\right)^2\right.\right. \\
&\qquad - \frac{a_i^U c_j^U x_i^* x_j^*}{\left(1 + x_{j*}\right)^2} - \frac{c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} - a_i^U e_i^U x_i^* \\
&\qquad - d_i^U\left(1 - b_i^L\right)x_i^* - e_i^U - \frac{a_j^U c_i^U x_i^* x_j^*}{\left(1 + x_{i*}\right)^2} \\
&\qquad\left. - \frac{c_i^U x_i^*}{\left(1 + x_{i*}\right)^2} - \frac{e_j^U c_i^U x_i^*}{\left(1 + x_{i*}\right)^2} - \frac{\left(c_i^U\right)^2\left(x_i^*\right)^2}{\left(1 + x_{i*}\right)^4}\right) \\
&\qquad \times \left(y_i(n) - z_i(n)\right)^2 \\
&\qquad + \left(2b_i^L - a_i^U e_i^U x_i^* - d_i^U\left(1 - b_i^L\right)x_i^* - e_i^U\right. \\
&\qquad\left. - \frac{e_i^U c_j^U x_j^*}{\left(1 + x_{j*}\right)^2} - \left(e_i^U\right)^2 - \left(b_i^U\right)^2\right) \\
&\qquad\left. \times \left(u_i(n) - v_i(n)\right)^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\left\{s_{ij}\left(y_i(n)-z_i(n)\right)^2 + s_{ij}^{*}\left(u_i(n)-v_i(n)\right)^2\right\} \\
&\leq -\gamma\sum_{i=1}^{2}\left\{\left(y_i(n)-z_i(n)\right)^2 + \left(u_i(n)-v_i(n)\right)^2\right\} \\
&= -\gamma V(n),
\end{aligned}
\tag{61}
$$

where $\gamma = \min\{s_{ij}, s_{ij}^{*}\}$, $i, j = 1, 2$, $i \neq j$. It follows from the conditions of Theorem 8 that we have $0 < \gamma < 1$, then the condition (iii) of Lemma 4 is satisfied. Therefore, we have from Lemma 4 that there exists a unique uniformly asymptotically stable almost periodic solution $(y_1^{*}(n), y_2^{*}(n), u_1^{*}(n), u_2^{*}(n))$ of system (46) which is bounded by $\mathbb{S}$ for all $n \in \mathbb{Z}^{+}$, which implies that there exists a unique uniformly asymptotically stable positive almost periodic solution $(x_1^{*}(n), x_2^{*}(n), u_1^{*}(n), u_2^{*}(n))$ of system (2) which is bounded by Ω for all $n \in \mathbb{Z}^{+}$. This completed the proof. □

Remark 9. If we neglect the role of feedback controls, that is, $b_i(n) = 0$, $d_i(n) = 0$ and $e_i(n) = 0$, $i = 1, 2$, then system (2) can be reduced to system (1). Propositions 5 and 6 and Theorem 8 can come down to our corresponding main results (see [1]). The fact shows that the feedback controls have influence on the existence and uniformly asymptotic stability of unique positive almost solution of system (2).

4. Numerical Simulations

In this section we give a numerical example in support to our analytical findings.

Example 10. Consider the following discrete system with feedback controls:

$$
\begin{aligned}
x_1(n+1) = x_1(n)\exp\Bigg[&1.20-0.02\sin\left(\sqrt{2}n\pi\right) \\
&-\left(1.05+0.01\sin\left(\sqrt{2}n\pi\right)\right)x_1(n) \\
&-\frac{\left(0.025+0.002\cos\left(\sqrt{2}n\pi\right)\right)x_2(n)}{1+x_2(n)} \\
&-\left(0.015+0.001\sin\left(\sqrt{3}n\pi\right)\right)u_1(n)\Bigg], \\
x_2(n+1) = x_2(n)\exp\Bigg[&1.15-0.02\cos\left(\sqrt{2}n\pi\right) \\
&-\left(1.02+0.02\cos\left(\sqrt{2}n\pi\right)\right)x_2(n) \\
&-\frac{\left(0.035+0.005\sin\left(\sqrt{3}n\pi\right)\right)x_1(n)}{1+x_1(n)} \\
&-\left(0.025+0.003\cos\left(\sqrt{3}n\pi\right)\right)u_2(n)\Bigg], \\
\Delta u_1(n) = -&\left(0.93-0.03\cos\left(\sqrt{2}n\pi\right)\right)u_1(n) \\
&+\left(0.024-0.002\cos\left(\sqrt{2}n\pi\right)\right)x_1(n), \\
\Delta u_2(n) = -&\left(0.95-0.03\sin\left(\sqrt{3}n\pi\right)\right)u_2(n) \\
&+\left(0.015+0.003\sin\left(\sqrt{2}n\pi\right)\right)x_2(n).
\end{aligned}
\tag{62}
$$

A simple computation shows that

$$
\begin{aligned}
x_1^{*} &\approx 1.1982, \qquad x_2^{*} \approx 1.1853, \\
u_1^{*} &\approx 0.0346, \qquad u_2^{*} \approx 0.0232, \\
x_{1*} &\approx 0.9665, \qquad x_{2*} \approx 0.9076, \\
u_{1*} &\approx 0.0221, \qquad u_{2*} \approx 0.0111, \\
r_1^{L} - e_1^{U}u_1^{*} - c_2^{U} &\approx 1.1524 > 0, \\
r_2^{L} - e_2^{U}u_2^{*} - c_1^{U} &\approx 1.0894 > 0.
\end{aligned}
\tag{63}
$$

Obviously, the assumptions in (19) are satisfied, and moreover, one has

$$
\begin{aligned}
s_{12} &\approx 0.3086, \qquad s_{21} \approx 0.1831, \\
s_{12}^{*} &\approx 0.8386, \qquad s_{21}^{*} \approx 0.8142,
\end{aligned}
\tag{64}
$$

that is, $0 < \gamma = \min\{s_{12}, s_{21}, s_{12}^{*}, s_{21}^{*}\} \approx 0.1831 < 1$, so the assumptions of Theorem 8 are satisfied. Thus, there exists a unique uniformly asymptotically stable positive almost periodic solution of system (62). From Figure 1, we can easily see that system (62) exists a positive almost periodic solution $(x_1^{*}(t), x_2^{*}(t), u_1^{*}(n), u_2^{*}(n))$, and the 2-dimensional phase portraits of almost periodic system (62) are displayed in Figure 2, respectively. Figure 3 shows that any positive solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ tends to the pervious almost periodic solution $(x_1^{*}(n), x_2^{*}(n), u_1^{*}(n), u_2^{*}(n))$; furthermore, the 2-dimensional phase portraits reflect the fact in Figure 4.

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