

Research Article

A Study of Nonlinear Fractional q -Difference Equations with Nonlocal Integral Boundary Conditions

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This paper is concerned with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional q -difference equations with nonlocal integral boundary conditions. The existence results are obtained by applying some well-known fixed point theorems and illustrated with examples.

1. Introduction

Several kinds of boundary value problems of fractional-order have recently been investigated by many researchers. Fractional derivatives appear naturally in the mathematical modelling of dynamical systems involving fractals and chaos. In fact, the concept of fractional calculus has played a key role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering. This might have been due to the fact that fractional-differential operators help to understand the hereditary phenomena in many materials and processes in a better way than the corresponding integer-order differential operators. Examples include physics, chemistry, biology, biophysics, blood flow phenomena, control theory, signal and image processing, and economics [1–4]. For some recent results on the topic, see a series of papers [5–12] and the references therein.

Fractional q -difference equations, regarded as fractional analogue of q -difference equations, have been studied by several researchers. For some earlier work on the subject, we refer to [13, 14], whereas the recent development on the existence theory of fractional q -difference equations can be found in [15–25]. In a recent paper [26], the authors investigated a nonlocal boundary value problem of nonlinear fractional q -difference equations:

$${}^c D_q^\alpha x(t) = f(t, x(t)), \quad 0 \leq t \leq 1, \quad 1 < \alpha \leq 2,$$

$$\begin{aligned} \alpha_1 x(0) - \beta_1 D_q x(0) &= \gamma_1 x(\eta_1), \\ \alpha_2 x(1) + \beta_2 D_q x(1) &= \gamma_2 x(\eta_2), \end{aligned} \quad (1)$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, ${}^c D_q^\alpha$ is the fractional q -derivative of the Caputo type, D_q is q -derivative, and $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $\eta_i \in (0, 1)$, $i = 1, 2$.

The purpose of the present paper is to study the following nonlocal boundary value problem of nonlinear fractional q -difference equations:

$$\begin{aligned} {}^c D_q^\beta x(t) + \lambda x(t) &= f(t, x(t)), \\ t \in [0, 1], \quad 1 < \beta \leq 2, \quad 0 < q < 1, \\ x(0) = 0, \quad x(1) &= I_q x(\eta) = \int_0^\eta x(s) d_q s, \quad 0 < \eta < 1, \end{aligned} \quad (2)$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, ${}^c D_q^\beta$ is the fractional q -derivative of the Caputo type, and λ is a real number.

The paper is organized as follows. Section 2 contains some necessary background material on the topic, while the main results are presented in Section 3. We make use of Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative to establish the existence results for the problem at hand.

2. Preliminaries on Fractional q -Calculus

Here we recall some definitions and fundamental results on fractional q -calculus.

Definition 1 (see [13]). Let $\beta \geq 0$, let $0 < q < 1$, and let f be a function defined on $[0, 1]$. The fractional q -integral of the Riemann-Liouville type is $(I_q^\beta f)(t) = f(t)$ and

$$(I_q^\beta f)(t) = \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q s, \quad \beta > 0, \quad t \in [0, 1], \quad (3)$$

where

$$\Gamma_q(\beta) = \frac{(1 - q)^{(\beta-1)}}{(1 - q)^{\beta-1}}, \quad 0 < q < 1. \quad (4)$$

Recall that $\Gamma_q(\beta + 1) = [\beta]_q \Gamma_q(\beta)$, with

$$[\beta]_q = \frac{q^\beta - 1}{q - 1}, \quad (1 - q)^{(0)} = 1, \quad (5)$$

$$(1 - q)^{(n)} = \prod_{k=0}^{n-1} (1 - q^{k+1}), \quad n \in \mathbb{N}.$$

More generally, if $\beta \in \mathbb{R}$, then

$$(1 - q)^{(\beta)} = \prod_{i=0}^{\infty} \frac{(1 - q^{i+1})}{(1 - q^{\beta+i+1})}. \quad (6)$$

Lemma 2 (see [27]). For $\beta \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid:

$$I_q^\beta ((t - a)^{(\lambda)}) = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\beta + \lambda + 1)} (t - a)^{(\beta+\lambda)}, \quad 0 < a < t < b. \quad (7)$$

In particular, for $\lambda = 0$, $a = 0$, using q -integration by parts, we have

$$\begin{aligned} (I_q^\beta 1)(t) &= \frac{1}{\Gamma_q(\beta)} \int_0^t (t - qs)^{(\beta-1)} d_q s \\ &= \frac{1}{\Gamma_q(\beta)} \int_0^t \frac{D_q((t - s)^{(\beta)})}{-[\beta]_q} d_q s \\ &= \frac{-1}{\Gamma_q(\beta + 1)} \int_0^t D_q((t - s)^{(\beta)}) d_q s \\ &= \frac{1}{\Gamma_q(\beta + 1)} t^{(\beta)}. \end{aligned} \quad (8)$$

For $0 < q < 1$, we define the q -derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \neq 0, \quad (9)$$

$$D_q f(0) = \lim_{n \rightarrow \infty} \frac{f(sq^n) - f(0)}{sq^n}, \quad s \neq 0.$$

For more details, see [28].

Definition 3 (see [29]). The fractional q -derivative of the Riemann-Liouville type of order $\beta \geq 0$ is defined by $(D_q^0 f)(t) = f(t)$ and

$$(D_q^\beta f)(t) = (D_q^{[\beta]} I_q^{[\beta]-\beta} f)(t), \quad \beta > 0, \quad (10)$$

where $[\beta]$ is the smallest integer greater than or equal to β .

Definition 4 (see [29]). The fractional q -derivative of the Caputo type of order $\beta \geq 0$ is defined by

$$({}^c D_q^\beta f)(t) = (I_q^{[\beta]-\beta} D_q^{[\beta]} f)(t), \quad \beta > 0, \quad (11)$$

where $[\beta]$ is the smallest integer greater than or equal to β .

Now we state some known results involving q -derivatives and q -integrals.

Lemma 5 (see [29]). Let $\beta, \gamma \geq 0$, and let f be a function defined on $[0, 1]$. Then

- (i) $(I_q^\beta I_q^\gamma f)(t) = (I_q^{\beta+\gamma} f)(t)$,
- (ii) $(D_q^\beta I_q^\beta f)(t) = f(t)$,
- (iii) $(I_q^\beta {}^c D_q^\beta f)(t) = f(t) - \sum_{k=0}^{[\beta]-1} (t^k / \Gamma_q(k + 1)) ({}^c D_q^k f)(0)$, $\beta > 0$.

Lemma 6 (see [19]). Let $\beta \geq 0$ and $n \in \mathbb{N}$. Then, the following equality holds:

$$(I_q^\beta D_q^n f)(t) = D_q^n I_q^\beta f(t) - \sum_{k=0}^{[\beta]-1} \frac{t^{\beta-n+k}}{\Gamma_q(\beta - n + k)} (D_q^k f)(0). \quad (12)$$

In the forthcoming analysis, the following lemma plays a pivotal role.

Lemma 7. Let $h \in C([0, 1], \mathbb{R})$ be a given function. Then the unique solution of the boundary value problem,

$$\begin{aligned} {}^c D_q^\beta x(t) + \lambda x(t) &= h(t), \\ t \in [0, 1], \quad 1 < \beta \leq 2, \quad 0 < q < 1, \end{aligned} \quad (13)$$

$$x(0) = 0, \quad x(1) = \int_0^\eta x(s) d_q s, \quad 0 < \eta < 1,$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} (h(s) - \lambda x(s)) d_q s \\ &+ \delta_1 t \int_0^\eta \frac{(\eta - qs)^{(\beta)}}{\Gamma_q(\beta + 1)} (h(s) - \lambda x(s)) d_q s \\ &- \delta_1 t \int_0^1 \frac{(1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} (h(s) - \lambda x(s)) d_q s, \end{aligned} \quad (14)$$

where

$$\delta_1 = \frac{1 + q}{1 + q - \eta^2}. \quad (15)$$

Proof. Using Lemma 5, we can write the solution of fractional q -difference equation in (13) as

$$x(t) = \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (h(s) - \lambda x(s)) d_qs + c_0 t + c_1, \quad t \in [0, 1]. \tag{16}$$

Using the boundary conditions of (13) in (16), we get $c_1 = 0$ and

$$c_0 = \delta_1 \left[\int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (h(s) - \lambda x(s)) d_qs - \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (h(s) - \lambda x(s)) d_qs \right], \tag{17}$$

where δ_1 is given by (15). Substituting the values of c_0, c_1 in (16), we obtain (14). \square

Let $\mathcal{C} := C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1]$ to \mathbb{R} endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$.

By virtue of Lemma 7, we define an operator $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}$ as

$$\begin{aligned} (\mathcal{G}x)(t) &= \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \\ &+ \delta_1 t \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (f(s, x(s)) - \lambda x(s)) d_qs \\ &- \delta_1 t \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \end{aligned} \tag{18}$$

and note that the given problem (2) has solutions only if the operator equation $\mathcal{G}x = x$ has fixed points.

3. Main Results

In the sequel, we assume that

(A₁) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and that there exists a q -integrable function $\zeta : [0, 1] \rightarrow \mathbb{R}$ such that $|f(t, x) - f(t, y)| \leq \zeta(t)|x - y|, t \in [0, 1], x, y \in \mathbb{R}$.

For computational convenience, we introduce the notations:

$$\Lambda = \mu_1 + |\lambda| \mu_2, \tag{19}$$

where

$$\begin{aligned} \mu_1 &:= (1 + |\delta_1|) (I_q^\beta \zeta)(1) + |\delta_1| (I_q^{\beta+1} \zeta)(\eta), \\ \mu_2 &:= \left(\frac{1}{\Gamma_q(\beta+1)} (1 + |\delta_1|) + \frac{|\delta_1| \eta^{\beta+1}}{\Gamma_q(\beta+2)} \right). \end{aligned} \tag{20}$$

Our first existence result is based on Leray-Schauder nonlinear alternative.

Lemma 8 (nonlinear alternative for single valued maps [30]). *Let E be a Banach space, C a closed, convex subset of E , and V an open subset of C with $0 \in V$. Suppose that $\mathcal{G} : \bar{V} \rightarrow C$ is a continuous, compact (i.e., $\mathcal{G}(\bar{V})$ is a relatively compact subset of C) map. Then either*

- (i) \mathcal{G} has a fixed point in \bar{V} , or
- (ii) there is $x \in \partial V$ (the boundary of V in C) and $\varepsilon \in (0, 1)$ with $x = \varepsilon \mathcal{G}(x)$.

Theorem 9. *Suppose that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. In addition, it is assumed that*

- (A₂) *there exist functions $h_1, h_2 \in L^1([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $|f(t, x)| \leq h_1(t)\vartheta(|x|) + h_2(t)$, for $(t, x) \in [0, 1] \times \mathbb{R}$.*

Then the problem (2) has at least one solution on $[0, 1]$ if there exists a positive number ω such that

$$\omega > \frac{\vartheta(\omega) \nu_1 + \nu_2}{1 - |\lambda| \mu_2}, \tag{21}$$

where

$$|\lambda| \neq \frac{1}{\mu_2},$$

$$\nu_i := (1 + |\delta_1|) (I_q^\beta h_i)(1) + |\delta_1| (I_q^{\beta+1} h_i)(\eta), \quad i = 1, 2. \tag{22}$$

Proof. As a first step, it will be shown that the operator $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}$ (defined by (18)) maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$. Notice that \mathcal{G} is continuous. For a positive number σ , let $\bar{B}_\sigma = \{x \in C([0, 1], \mathbb{R}) : \|x\| \leq \sigma\}$ be a bounded set in $C([0, 1], \mathbb{R})$. Then, for any $x \in \bar{B}_\sigma$, we find that

$$\begin{aligned} |(\mathcal{G}x)(t)| &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (|f(s, x(s))| + |\lambda| |x(s)|) d_qs \\ &+ |\delta_1| |t| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (|f(s, x(s))| + |\lambda| |x(s)|) d_qs \\ &+ |\delta_1| |t| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (|f(s, x(s))| + |\lambda| |x(s)|) d_qs \\ &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} [h_1(s) \vartheta(\|x\|) + h_2(s)] d_qs \\ &+ |\delta_1| |t| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} [h_1(s) \vartheta(\|x\|) + h_2(s)] d_qs \\ &+ |\delta_1| |t| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} [h_1(s) \vartheta(\|x\|) + h_2(s)] d_qs \end{aligned}$$

$$\begin{aligned}
&\leq \vartheta(\sigma) \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_1(s) d_qs \right. \\
&\quad + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} h_1(s) d_qs \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_1(s) d_qs \right\} \\
&+ \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_2(s) d_qs \right. \\
&\quad + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} h_2(s) d_qs \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_2(s) d_qs \right\} \\
&+ \sigma |\lambda| \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs \right. \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right\} \\
&\leq \vartheta(\sigma) \nu_1 + \nu_2 + \sigma |\lambda| \mu_2. \tag{23}
\end{aligned}$$

This establishes our assertion.

Next, we show that the operator \mathcal{E} maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$. Taking $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and $x \in \overline{B}_\sigma$ together with the inequality: $(t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)} \leq (t_2 - t_1)$ for $0 < t_1 < t_2$ (see [19] p. 4), we obtain

$$\begin{aligned}
&|(\mathcal{E}x)(t_2) - (\mathcal{E}x)(t_1)| \\
&\leq \left| \int_0^{t_2} \frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \right. \\
&\quad \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \right| \\
&\quad + |\delta_1| (t_2 - t_1) \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} \\
&\quad \quad \times (|f(s, x(s))| + |\lambda| |x(s)|) d_qs. \\
&\quad + |\delta_1| (t_2 - t_1) \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
&\quad \quad \times (|f(s, x(s))| + |\lambda| |x(s)|) d_qs.
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_0^{t_1} \frac{(t_2-t_1)}{\Gamma_q(\beta)} [h_1(s) \vartheta(\sigma) + h_2(s) + |\lambda| \sigma] d_qs \right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)} [h_1(s) \vartheta(\sigma) + h_2(s) + |\lambda| \sigma] d_qs \right| \\
&\quad + |\delta_1| (t_2 - t_1) \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} \\
&\quad \quad \times [h_1(s) \vartheta(\sigma) + h_2(s) + |\lambda| \sigma] d_qs. \\
&\quad + |\delta_1| (t_2 - t_1) \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
&\quad \quad \times [h_1(s) \vartheta(\sigma) + h_2(s) + |\lambda| \sigma] d_qs. \tag{24}
\end{aligned}$$

It is obvious that the right hand side of the above inequality tends to zero independently of $x \in \overline{B}_\sigma$ as $t_2 \rightarrow t_1$. Therefore the operator \mathcal{E} is completely continuous by the Arzelá-Ascoli theorem.

Thus the operator \mathcal{E} satisfies the hypothesis of Lemma 8 and hence by its conclusion, either condition (i) or condition (ii) holds. We claim that the conclusion (ii) is not possible.

Let $V = \{x \in C([0, 1], \mathbb{R}) : \|x\| < \omega\}$ with ω given by (21). Then we will show that $\|\mathcal{E}x\| < \omega$. Indeed, by means of (A_4) , we get

$$\begin{aligned}
&\|\mathcal{E}x\| \\
&\leq \vartheta(\|x\|) \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_1(s) d_qs \right. \\
&\quad + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} h_1(s) d_qs \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_1(s) d_qs \right\} \\
&\quad + \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_2(s) d_qs \right. \\
&\quad + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} h_2(s) d_qs \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} h_2(s) d_qs \right\} \\
&\quad + \omega |\lambda| \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs \right. \\
&\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right\} \\
&\leq \vartheta(\omega) \nu_1 + \nu_2 + \omega |\lambda| \mu_2 < \omega. \tag{25}
\end{aligned}$$

Assume that there exist $x \in \partial V$ and $\varepsilon \in (0, 1)$ such that $x = \varepsilon \mathcal{G}x$. Then for such a choice of x and ε , we get a contradiction:

$$\begin{aligned} \omega = \|x\| &= \varepsilon \|\mathcal{G}x\| < \vartheta(\|x\|) \nu_1 + \nu_2 + \omega |\lambda| \mu_2 \\ &= \vartheta(\omega) \nu_1 + \nu_2 + \omega |\lambda| \mu_2 < \omega. \end{aligned} \tag{26}$$

Thus it follows by Lemma 8 that \mathcal{G} has a fixed point $x \in \bar{V}$ which is a solution of the problem (2). This completes the proof. \square

Remark 10. In case we take h_1, h_2 in (A_2) to be continuous, then $\nu_i = \mu_2 \|h_i\|$, $i = 1, 2$, where μ_2 is given by (20).

Our next result deals with existence and uniqueness of solutions for the problem (2) and is based on Banach's fixed point theorem.

Theorem 11. *Suppose that the assumption (A_1) holds and that $\Lambda < 1$, where Λ is given by (19). Then the boundary value problem (2) has a unique solution.*

Proof. Fix $N = \sup_{t \in [0,1]} |f(t, 0)| < \infty$, $r \geq (N\mu_2/1 - \Lambda)$ (μ_2 is given by (20)), and define $\bar{B}_r = \{x \in \mathcal{E} : \|x\| \leq r\}$. We will show that $\mathcal{G}\bar{B}_r \subset \bar{B}_r$, where \mathcal{G} is defined by (18). For $x \in \bar{B}_r$, $t \in [0, 1]$, it follows by the assumption (A_1) that

$$\begin{aligned} |f(t, x(t))| &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq \zeta(t) |x(t)| + |f(t, 0)| \leq \zeta(t) r + N. \end{aligned} \tag{27}$$

Then, for $x \in \bar{B}_r$, $t \in [0, 1]$, we have

$$\begin{aligned} |(\mathcal{G}x)(t)| &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} [\zeta(s)r + N] d_qs \\ &\quad + r|\lambda| \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs + |t| |\delta_1| \\ &\quad \times \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} [\zeta(s)r + N] d_qs \\ &\quad + r|\lambda| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs + |t| |\delta_1| \\ &\quad \times \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} [\zeta(s)r + N] d_qs \\ &\quad + r|\lambda| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \\ &\leq r \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(s) d_qs \right. \\ &\quad \left. + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} \zeta(s) d_qs \right. \\ &\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(s) d_qs \right\} \end{aligned}$$

$$\begin{aligned} &+ N \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs \right. \\ &\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right\} \\ &+ r|\lambda| \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs \right. \\ &\quad \left. + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right\} \\ &\leq r \left\{ (I_q^\beta \zeta)(1) + |\delta_1| (I_q^{\beta+1} \zeta)(\eta) + |\delta_1| (I_q^\beta \zeta)(1) \right\} \\ &\quad + N \left\{ \frac{1}{\Gamma_q(\beta+1)} + \frac{|\delta_1| \eta^{\beta+1}}{\Gamma_q(\beta+2)} + \frac{|\delta_1|}{\Gamma_q(\beta+1)} \right\} \\ &\quad + r|\lambda| \left\{ \frac{1}{\Gamma_q(\beta+1)} + \frac{|\delta_1| \eta^{\beta+1}}{\Gamma_q(\beta+2)} + \frac{|\delta_1|}{\Gamma_q(\beta+1)} \right\}, \end{aligned} \tag{28}$$

which, in view of (19), implies that

$$\|\mathcal{G}x\| \leq N\mu_2 + r\Lambda \leq r. \tag{29}$$

This shows that $\mathcal{G}\bar{B}_r \subset \bar{B}_r$.

Now, for $x, y \in \mathcal{E}$, we have

$$\begin{aligned} \|(\mathcal{G}x) - (\mathcal{G}y)\| &\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \right. \\ &\quad \times (|f(s, x(s)) - f(s, y(s))| \\ &\quad \left. + |\lambda| |x(s) - y(s)|) d_qs \right. \\ &\quad \left. + |\delta_1| |t| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (|f(s, x(s)) - f(s, y(s))| \right. \\ &\quad \left. + |\lambda| |x(s) - y(s)|) d_qs \right. \\ &\quad \left. + |\delta_1| |t| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \right. \\ &\quad \times (|f(s, x(s)) - f(s, y(s))| \\ &\quad \left. + |\lambda| |x(s) - y(s)|) d_qs \right\} \\ &\leq \|x - y\| \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(s) d_qs \right. \\ &\quad \left. + |\delta_1| |t| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} \zeta(s) d_qs \right. \end{aligned}$$

$$\begin{aligned}
& + |\delta_1| |t| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(s) d_qs \\
& + |\lambda| \left(\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right. \\
& \quad + |\delta_1| |t| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} d_qs \\
& \quad \left. + |\delta_1| |t| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right) \Bigg\}, \tag{30}
\end{aligned}$$

which, by (19), takes the form

$$\|(\mathcal{G}x)(t) - (\mathcal{G}y)(t)\| \leq \Lambda \|x - y\|. \tag{31}$$

As $\Lambda < 1$ (the given assumption), therefore \mathcal{G} is a contraction. Hence, by Banach's contraction mapping principle, the problem (2) has a unique solution. \square

If we take $\zeta(t) = L$ (L is a positive constant), the condition $\Lambda < 1$ becomes $L < ((1/\mu_2) - |\lambda|)$ and Theorem 11 can be phrased as follows.

Corollary 12. *Let there exist a constant $L \in (0, (1/\mu_2) - |\lambda|)$ with μ_2 given by (20), and let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the assumption: $|f(t, x) - f(t, y)| \leq L|x - y|$, $t \in [0, 1]$, $x, y \in \mathbb{R}$. Then the boundary value problem (2) has a unique solution.*

Our last result relies on Krasnoselskii's fixed point theorem [31].

Lemma 13 (Krasnoselskii). *Let Y be a closed, bounded, convex, and nonempty subset of a Banach space X . Let Q_1, Q_2 be two operators such that (i) $Q_1x + Q_2y \in Y$ whenever $x, y \in Y$; (ii) Q_1 is compact and continuous; (iii) Q_2 is a contraction mapping. Then there exists $z \in Y$ such that $z = Q_1z + Q_2z$.*

Theorem 14. *Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (A_1) . Furthermore, it is assumed that*

(A_3) *there exist a function $\xi \in C([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\chi \in C([0, 1], \mathbb{R}^+)$ such that*

$$|f(t, x)| \leq \xi(t) \chi(|x|), \quad (t, x) \in [0, 1] \times \mathbb{R}. \tag{32}$$

If

$$\begin{aligned}
& |\delta_1| \left[\int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} \zeta(s) d_qs + \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \zeta(s) d_qs \right] \\
& + |\lambda| |\delta_1| \left[\frac{\eta^{\beta+1}}{\Gamma_q(\beta+2)} + \frac{1}{\Gamma_q(\beta+1)} \right] < 1, \tag{33}
\end{aligned}$$

with $|\lambda|\mu_2 < 1$, then the boundary value problem (2) has at least one solution on $[0, 1]$.

Proof. Let us consider the set $\bar{B}_{\bar{r}} = \{x \in \mathcal{E} : \|x\| \leq \bar{r}\}$, where \bar{r} is given by

$$\bar{r} \geq \frac{\chi(\bar{r}) \|\xi\|}{1 - |\lambda|\mu_2}, \quad \|\xi\| = \sup_{t \in [0, t]} |\xi(t)|, \tag{34}$$

and introduce the operators \mathcal{G}_1 and \mathcal{G}_2 on $\bar{B}_{\bar{r}}$ as

$$\begin{aligned}
(\mathcal{G}_1x)(t) &= \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs, \\
t &\in [0, 1],
\end{aligned}$$

$$\begin{aligned}
(\mathcal{G}_2x)(t) &= \delta_1 t \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (f(s, x(s)) - \lambda x(s)) d_qs \\
&\quad - \delta_1 t \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs, \\
t &\in [0, 1]. \tag{35}
\end{aligned}$$

In order to show the hypothesis of Krasnoselskii's fixed point theorem, we proceed as follows.

(i) For $x, y \in \bar{B}_{\bar{r}}$, we find that

$$\begin{aligned}
& |(\mathcal{G}_1x + \mathcal{G}_2y)(t)| \\
& \leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (\xi(s) \chi(|x(s)|) + |\lambda| |x(s)|) d_qs \\
& \quad + |\delta_1| \int_0^\eta \frac{(\eta-qs)^{(\beta)}}{\Gamma_q(\beta+1)} (\xi(s) \chi(|y(s)|) + |\lambda| |y(s)|) d_qs \\
& \quad + |\delta_1| \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} (\xi(s) \chi(|y(s)|) + |\lambda| |y(s)|) d_qs \\
& \leq \chi(\bar{r}) \|\xi\| \left[\frac{1}{\Gamma_q(\beta+1)} (1 + |\delta_1|) + |\delta_1| \frac{\eta^{\beta+1}}{\Gamma_q(\beta+2)} \right] \\
& \quad + \bar{r} |\lambda| \left[\frac{1}{\Gamma_q(\beta+1)} (1 + |\delta_1|) + |\delta_1| \frac{\eta^{\beta+1}}{\Gamma_q(\beta+2)} \right] \\
& \leq \chi(\bar{r}) \|\xi\| \mu_2 + \bar{r} |\lambda| \mu_2 \leq \bar{r}. \tag{36}
\end{aligned}$$

This implies that $\mathcal{G}_1x + \mathcal{G}_2y \in \bar{B}_{\bar{r}}$.

(ii) From the continuity of f , it follows that the operator \mathcal{G}_1 is continuous. Also, \mathcal{G}_1 is uniformly bounded on $\bar{B}_{\bar{r}}$ as

$$\|\mathcal{G}_1x\| \leq \frac{\chi(\bar{r}) \|\xi\|}{\Gamma_q(\beta+1)} + \frac{|\lambda| \bar{r}}{\Gamma_q(\beta+1)}. \tag{37}$$

Next, for any $x \in \bar{B}_{\bar{r}}$, and $\tau_1, \tau_2 \in [0, 1]$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
 & |(\mathcal{I}_1 x)(\tau_2) - (\mathcal{I}_1 x)(\tau_1)| \\
 &= \left| \int_0^{\tau_2} \frac{(\tau_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \right. \\
 &\quad \left. - \int_0^{\tau_1} \frac{(\tau_1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \right| \\
 &\leq \left| \int_0^{\tau_1} \frac{(\tau_2 - qs)^{(\beta-1)} - (\tau_1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \right. \\
 &\quad \left. \times (f(s, x(s)) - \lambda x(s)) d_qs \right| \tag{38} \\
 &\quad + \left| \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} (f(s, x(s)) - \lambda x(s)) d_qs \right| \\
 &\leq (\chi(\bar{r}) \|\xi\| + |\lambda| \bar{r}) \\
 &\quad \times \left[\int_0^{\tau_1} \frac{(\tau_2 - qs)^{(\beta-1)} - (\tau_1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right. \\
 &\quad \left. + \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_qs \right]
 \end{aligned}$$

which is independent of x and tends to zero as $\tau_2 \rightarrow \tau_1$. Thus, \mathcal{I}_1 is equicontinuous. So \mathcal{I}_1 is relatively compact on $\bar{B}_{\bar{r}}$. Hence, by the Arzelá-Ascoli theorem, \mathcal{I}_1 is compact on $\bar{B}_{\bar{r}}$.

(iii) From (A_1) and (33) it follows that \mathcal{I}_2 is a contraction mapping.

Thus all the conditions of Lemma 13 are satisfied. Hence, by the conclusion of Lemma 13, the problem (2) has at least one solution on $[0, 1]$. \square

As a special case, for $\chi(x) \equiv 1$, there always exists a positive \bar{r} such that (34) holds true. In consequence, we have the following corollary.

Corollary 15. *Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (A_1) . Furthermore, $|f(t, x)| \leq \xi(t)$, $\forall (t, x) \in [0, 1] \times \mathbb{R}$, and $\xi \in C([0, 1], \mathbb{R}^+)$. Then the boundary value problem (2) has at least one solution on $[0, 1]$ provided (33) holds.*

4. Examples

Example 1. Consider the problem

$$\begin{aligned}
 & {}^c D_q^{3/2} x(t) + \frac{1}{4} x(t) = \frac{1}{5} x + \cos x, \quad 0 \leq t \leq 1, \\
 & x(0) = 0, \quad x(1) = I_q x\left(\frac{1}{3}\right), \tag{39}
 \end{aligned}$$

where $q = 1/2, \lambda = 1/4, \eta = 1/3$. It is easy to find that $\delta_1 = 27/25$, and

$$|f(t, x)| = \left| \frac{1}{5} x + \cos x \right| \leq \frac{1}{5} |x| + 1. \tag{40}$$

Obviously $h_1 = 1/5, h_2 = 1, \vartheta(\omega) = \omega$. In consequence, $\nu_1 = 0.3002935758, \nu_2 = 1.501467879$, and condition (21) yields $\omega > 4.629306104$. Thus, the hypothesis of Theorem 9 is satisfied. Hence it follows by the conclusion of Theorem 9 that there exists at least one solution for the problem (39).

Example 2. Consider the nonlocal boundary value problem given by

$$\begin{aligned}
 & {}^c D_q^{3/2} x(t) + \frac{1}{4} x(t) = L(\cos t + \tan^{-1} x), \quad 0 \leq t \leq 1, \\
 & x(0) = 0, \quad x(1) = I_q x\left(\frac{1}{3}\right). \tag{41}
 \end{aligned}$$

Here $q = 1/2, \lambda = 1/4, \eta = 1/3$ and L is a constant to be fixed later on. With the given data, it is found that $\delta_1 = 27/25, |f(t, x) - f(t, y)| \leq L|x - y|$ and

$$\Lambda = \frac{1}{\Gamma_{1/2}(5/2)} \left(1 + \frac{27\sqrt{3}(4\sqrt{2}-1) + 6\sqrt{2}}{25\sqrt{3}(4\sqrt{2}-1)} \right) \left(L + \frac{1}{4} \right). \tag{42}$$

Letting

$$L < \left[\frac{1}{\Gamma_{1/2}(5/2)} \left(1 + \frac{27\sqrt{3}(4\sqrt{2}-1) + 6\sqrt{2}}{25\sqrt{3}(4\sqrt{2}-1)} \right) \right]^{-1} - \frac{1}{4}, \tag{43}$$

all the conditions of Corollary 12 are satisfied. Therefore, the conclusion of Corollary 12 applies to the problem (41).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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