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Research Article

Adaptive *Q-S* Synchronization of Fractional-Order Chaotic Systems with Nonidentical Structures

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This paper investigates the adaptive *Q-S* synchronization of the fractional-order chaotic systems with nonidentical structures. Based on the stability of fractional-order systems and adaptive control technique, a general formula for designing the controller and parameters update law is proposed to achieve adaptive *Q-S* synchronization between two different chaotic systems with different structures. The effective scheme parameters identification and *Q-S* synchronization of chaotic systems can be realized simultaneously. Furthermore, two typical illustrative numerical simulations are given to demonstrate the effectiveness of the proposed scheme, for each case, we design the controller and parameter update laws in detail. The numerical simulations are performed to verify the effectiveness of the theoretical results.

1. Introduction

Since the pioneering work of Pecora and Carroll, chaos control and synchronization have received particularly attention among scientists from various research fields including secure communication, and information science [1–4]. From then, many kinds of synchronization have been proposed in dynamical systems. For instance, Li et al. [5] pointed out that coexistence of complete synchronization in coupled identical Chen systems is by linear control. In [6], they presented an idea of stochastic phase synchronization about the dynamical evolution of the underlying system. Lu [7] proposed a new general scheme which was discussed for the generalized synchronization of discrete-time chaotic and hyperchaotic systems. Later on, Li et al. [8] studied antisynchronization of two different chaotic systems. In 1999, Mainieri and Rehacek first proposed the concept of projective synchronization which characterized that the drive and response systems could be synchronized up to a scaling constant matrix [9]. Recently, Li [10] proposed the *Q-S* synchronization that is a more general definition of projective synchronization when the response system contains scaling matrix. In the application of secure communication, more scaling matrices may also be a useful utility to improve the security of the secure communication scheme [11–16]. Amongst all kinds of chaos synchronization, projective synchronization is the most noticeable one because of its proportional feature between the synchronized dynamical states and, hence, it has received extensive research. To our best of knowledge, *Q-S* synchronization of the fractional-order chaotic systems has not considered adequately.

On the other hand, most of existing synchronization methods are mainly concerned with the synchronization of two special identical or similar or with mismatched systems. Zhang et al. [17] discussed the complete synchronization of a coupled fractional-order system, they have found a kind of interesting nonlinear phenomenon-hybrid synchronization in linearly coupled fractional-order chaotic systems. Recently, Mainieri and Rehacek [18] studied the hybrid projective synchronization of fractional-order chaotic system. They realized the slave system can be synchronized with the projection of the master system generated through state transformation. Despite these many results for the fractional-order systems, but in fact, in many practical problems, the synchronization is carried out even though the oscillators have different structures; therefore, it is extremely necessary and important to study the synchronization of the fractional-order systems with nonidentical structures. At the same time, most of the methods mentioned above have ignored the uncertainties, but in real situations, this assumption cannot be satisfied in many real situations because it is hard to known all the system parameters in advance [19–23]. In this case, the applications of the existing methods are somewhat limited. Hence, the synchronization and identification of chaotic systems with uncertainties are more essential work for research.

Motivated by the above discussion, this paper investigates the adaptive Q-S synchronization of fractional-order chaotic systems via increased order. Moreover, based on the stability theory of fractional order systems, an adaptive synchronization controller and adaptive laws of parameters are developed; numerical simulations are carried out to demonstrate the effectiveness and flexibility for the controllers.

The organization of this paper is as follows. Preliminaries and model description are given in Section 2. In Section 3 based on the stability theory, a general Q-S synchronization approach of fractional-order chaotic systems with unknown parameters is presented. Section 4 shows the effectiveness of the approach for the extensive simulation studies. Finally, Section 5 concludes the paper.

2. Preliminaries and Model Description

The fractional calculus is a generalization of an integration and differentiation to a noninteger-order integro-differential operator which can be denoted by a fundamental operator as follows [23]:

$$ {}_aD_t^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \displaystyle\int_{\alpha}^{t} (d\tau)^{-\alpha}, & R(\alpha) < 0. \end{cases} \quad (1) $$

There exist many definitions for fractional derivatives. The Riemann-Liouvile definition and the Caputo definition are the two most commonly used ones. In this paper, the Caputo definition is adopted for derivatives which is introduced as follows:

$$ D_*^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-n+1} f^{(n)}(\tau)\, d\tau, \quad (2) $$

for $n-1 \le \alpha < n$, where $\Gamma(\cdot)$ is the Gamma function,

$$ \Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \quad (3) $$

The drive system and response system are described by

$$ D_*^q \mathbf{x} = f(\mathbf{x}), \quad (4) $$

$$ D_*^q \mathbf{y} = g(\mathbf{y}) + U(t, \mathbf{x}, \mathbf{y}), \quad (5) $$

where $\mathbf{x}, \mathbf{y}$ are the state vectors; $\mathbf{f}, \mathbf{g}$ are differentiable vector functions; and $\mathbf{U}(t, \mathbf{x}, \mathbf{y})$ is a controll function. We denote the vector error state be $\mathbf{e} = \mathbf{Qx} - \mathbf{Sy}$, where $\mathbf{Q}, \mathbf{S}$ are scaling matrices.

Definition 1. For the given drive system (4) and response system (5), there exist two real matrices $\mathbf{Q}, \mathbf{S}$, such that $\lim_{t\to\infty}\|\mathbf{e}\| = \|\mathbf{Qx} - \mathbf{Sy}\| = 0$. It is to say that the Q-S synchronization is achieved between system (4) and system (5).

Some remarks on Definition 1.

Remark 2. It is easy to see that the definition of Q-S synchronization encompasses complete synchronization, anti-synchronization, and projective synchronization when matrices Q and S are selected special values, respectively. In short, Q-S synchronization is a more general form that includes many kinds of synchronization as its special items.

Remark 3. For simplicity for further discussion, we suppose that matrix S is row full rank, which guarantee the inverse matrix exists when the matrix is a square matrix.

Lemma 4 (see [24]). *For the nonlinear fractional-order system $D_*^q \mathbf{x} = f(\mathbf{x})$ or $D_*^q \mathbf{x} = \mathbf{Ax}$ with the order as $0 < q \le 1$, if there exists a real symmetric positive definite matrix $\mathbf{P}$ such that the equation $J = \mathbf{x}^T \mathbf{P} D_*^q \mathbf{x} \le 0$ always holds for any states $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, then the above fractional-order system is asymptotically locally stable.*

3. Theoretical Results

Consider two general fractional-order uncertain chaotic systems which are referred to as the drive and response systems, respectively, in the form of

$$ D_*^q \mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\,\mathbf{\Phi}, \quad (6) $$

$$ D_*^q \mathbf{y} = \mathbf{g}(\mathbf{y}) + \mathbf{G}(\mathbf{y})\,\mathbf{\Theta} + \mathbf{U}, \quad (7) $$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^{n_1}$ and $\mathbf{y}(t) = (y_1, y_2, \ldots, y_n)^T \in \mathbf{R}^{n_2}$ are the state vectors, $\mathbf{f} : \mathbf{R}^{n_1} \to \mathbf{R}^{n_1}$ and $\mathbf{g} : \mathbf{R}^{n_2} \to \mathbf{R}^{n_2}$ are continuous vector functions, including nonlinear terms, $\mathbf{F} : \mathbf{R}^{n_1} \to \mathbf{R}^{n_1 \times m_1}$ and $\mathbf{G} : \mathbf{R}^{n_2} \to \mathbf{R}^{n_2 \times m_2}$ are matrix functions, $\mathbf{\Phi} \in \mathbf{R}^{m_1}$ and $\mathbf{\Theta} \in \mathbf{R}^{m_2}$ are the parameter vectors, and $\mathbf{U} \in \mathbf{R}^n$ is a controller to be designed later.

When $n_1 = n_2$, $m_1 = m_2$, the structure of the drive system is identical to that of the response system, this problem has been solved. Therefore, when the dimension of drive system is not equal to that of the response system, there is no doubt that it is an interesting problem. So we will investigate the following two cases.

Case 1 ($n_1 > n_2$). In this case, the dimension of the drive system is greater than that of the response system. For realizing the Q-S synchronization between the drive system (6) and response system (7) with different structures, so we must add extra auxiliary state(s) to the response system.

We define the auxiliary state(s) $\mathbf{y}' \in \mathbf{R}^{n_1 - n_2}$ then we obtain a new state vector $\overline{\mathbf{y}} = (\mathbf{y}, \mathbf{y}')^T$, thus, the response system (7) is rewritten as follows:

$$ D_*^q \overline{\mathbf{y}} = \overline{\mathbf{g}}(\overline{\mathbf{y}}) + \overline{\mathbf{G}}(\overline{\mathbf{y}})\,\mathbf{\Theta} + \overline{\mathbf{U}}(t, x, y), \quad (8) $$

where $\overline{\mathbf{y}} = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}' \end{pmatrix}$, $\overline{\mathbf{g}}(\overline{\mathbf{y}}) = \begin{pmatrix} \mathbf{g}(\mathbf{y}) \\ 0 \end{pmatrix}$, $\overline{\mathbf{G}}(\overline{\mathbf{y}}) = \begin{pmatrix} \mathbf{G}(\mathbf{y}) \\ 0 \end{pmatrix}$, $\overline{\mathbf{U}}(t,x,y) = \begin{pmatrix} \mathbf{U}(t,x,y) \\ \mathbf{U}'(t,x,y) \end{pmatrix}$, $\mathbf{y}'$, and $\mathbf{U}'(t,x,y) \in \mathbf{R}^{n_1-n_2}$.

Our goal is to design a suitable controller to realize the Q-S synchronization between drive system (6) and response system (8) with different arbitrary scaling matrices Q, S. The following theorem can be obtained.

Theorem 5. *The Q-S synchronization of fractional-order systems between systems* (6) *and* (8) *can be achieved, if the active control under the hypothesis is given by* (9) *and the updating laws of the estimated parameter are given by* (10)

$$\mathbf{U} = -\overline{\mathbf{g}}(\overline{\mathbf{y}}) - \overline{\mathbf{G}}(\overline{\mathbf{y}})\widehat{\boldsymbol{\Theta}} + \mathbf{S}^{-1}\mathbf{Q}\left(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\widehat{\boldsymbol{\Phi}} + \mathbf{K}\mathbf{e}\right), \quad (9)$$

$$\begin{aligned} D_*^q\widehat{\boldsymbol{\Phi}} &= -[\mathbf{Q}\mathbf{F}(\mathbf{x})]^T\mathbf{e}, \\ D_*^q\widehat{\boldsymbol{\Theta}} &= \left[\mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\right]^T\mathbf{e}, \end{aligned} \quad (10)$$

where $\mathbf{K} = \operatorname{diag}(k_1, k_2, \ldots, k_n)$ *is a gain matrix for each state controller,* $\widehat{\boldsymbol{\Phi}}$ *and* $\widehat{\boldsymbol{\Theta}}$ *are the estimated vectors of unknown parameters, and* $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi} - \widehat{\boldsymbol{\Phi}}$, $\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta} - \widehat{\boldsymbol{\Theta}}$.

Proof. We have the vector error state $\mathbf{e} = \mathbf{Q}\mathbf{x} - \mathbf{S}\mathbf{y}$, then we can obtain the error dynamical system as follows:

$$D_*^q\mathbf{e}(t) = \mathbf{Q}(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\Phi}) - \mathbf{S}\left(\overline{\mathbf{g}}(\overline{\mathbf{y}}) + \overline{\mathbf{G}}(\overline{\mathbf{y}})\boldsymbol{\Theta} + \mathbf{U}\right). \quad (11)$$

According to Lemma 4, combining (9) and (10) with (11), one has

$$\mathbf{J} = \begin{bmatrix} \mathbf{e}^T & \widetilde{\boldsymbol{\Phi}}^T & \widetilde{\boldsymbol{\Theta}}^T \end{bmatrix} \begin{bmatrix} D_*^q\mathbf{e} \\ D_*^q\widetilde{\boldsymbol{\Phi}} \\ D_*^q\widetilde{\boldsymbol{\Theta}} \end{bmatrix} = \mathbf{e}^T D_*^q\mathbf{e} + \widetilde{\boldsymbol{\Phi}}^T D_*^q\widetilde{\boldsymbol{\Phi}} + \widetilde{\boldsymbol{\Theta}}^T D_*^q\widetilde{\boldsymbol{\Theta}}. \quad (12)$$

From (6)–(11), we can get that

$$\begin{aligned} &\mathbf{e}^T D_*^q\mathbf{e} + \widetilde{\boldsymbol{\Phi}}^T D_*^q\widetilde{\boldsymbol{\Phi}} + \widetilde{\boldsymbol{\Theta}}^T D_*^q\widetilde{\boldsymbol{\Theta}} \\ &= \mathbf{e}^T\left[\mathbf{Q}(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\Phi}) - \mathbf{S}\left(\overline{\mathbf{g}}(\overline{\mathbf{y}}) + \overline{\mathbf{G}}(\overline{\mathbf{y}})\boldsymbol{\Theta} + \mathbf{U}\right)\right] \\ &\quad + \widetilde{\boldsymbol{\Phi}}^T\left([-\mathbf{Q}\mathbf{F}(\mathbf{x})]^T\mathbf{e}\right) + \widetilde{\boldsymbol{\Theta}}^T\left(\left[\mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\right]^T\mathbf{e}\right) \\ &= \mathbf{e}^T\left[\mathbf{Q}(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\Phi}) - \mathbf{S}\overline{\mathbf{g}}(\overline{\mathbf{y}}) - \mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\boldsymbol{\Theta} + \mathbf{S}\overline{\mathbf{g}}(\overline{\mathbf{y}})\right. \\ &\qquad \left. + \mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\left(\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\right) - \mathbf{Q}\mathbf{f}(\mathbf{x}) - \mathbf{Q}\mathbf{F}(\mathbf{x})\left(\boldsymbol{\Phi} - \widetilde{\boldsymbol{\Phi}}\right) - \mathbf{K}\mathbf{e}\right] \\ &\quad + \widetilde{\boldsymbol{\Phi}}^T\left(-(\mathbf{Q}\mathbf{F}(\mathbf{x}))^T\mathbf{e}\right) + \widetilde{\boldsymbol{\Theta}}^T\left(\left(\mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\right)^T\mathbf{e}\right) \\ &= -\mathbf{e}^T\mathbf{K}\mathbf{e}. \end{aligned} \quad (13)$$

Suppose we select an appropriate matrix $\mathbf{K}$, such that $-\mathbf{e}^T\mathbf{K}\mathbf{e} \le 0$. From Lemma 4, the origin of error dynamical system (11) is asymptotically stable, it is to say that the response system (6) can synchronize the drive system (8) globally and asymptotically. □

Remark 6. In order to identify the unknown parameters, we suppose that the nonlinear vector functions $-[\mathbf{Q}\mathbf{F}(\mathbf{x})]^T$ and $[\mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})]^T$ should be linearly independent.

Based on Theorem 5, two corollaries can be easily derived as below.

Corollary 7. *Suppose the parameters in the drive system* $\boldsymbol{\Phi}$ *are known, the controller can be designed as follows:*

$$\mathbf{U} = -\overline{\mathbf{g}}(\overline{\mathbf{y}}) - \overline{\mathbf{G}}(\overline{\mathbf{y}})\widehat{\boldsymbol{\Theta}} + \mathbf{S}^{-1}\mathbf{Q}(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\Phi} + \mathbf{K}\mathbf{e}). \quad (14)$$

Moreover, the parameter update laws are degraded as

$$D_*^q\widehat{\boldsymbol{\Theta}} = \left[\mathbf{S}\overline{\mathbf{G}}(\overline{\mathbf{y}})\right]^T\mathbf{e}. \quad (15)$$

Therefore, the drive system and response system can achieve Q-S synchronization.

Corollary 8. *Suppose the parameters* $\boldsymbol{\Theta}$ *in the response system are known, then the controller is modified as*

$$\mathbf{U} = -\overline{\mathbf{g}}(\overline{\mathbf{y}}) - \overline{\mathbf{G}}(\overline{\mathbf{y}})\boldsymbol{\Theta} + \mathbf{S}^{-1}\mathbf{Q}\left(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\widehat{\boldsymbol{\Phi}} + \mathbf{K}\mathbf{e}\right). \quad (16)$$

Moreover, the parameter update laws can be taken as

$$D_*^q\widehat{\boldsymbol{\Phi}} = -[\mathbf{Q}\mathbf{F}(\mathbf{x})]^T\mathbf{e}. \quad (17)$$

Therefore, the drive system and response system can achieve Q-S synchronization.

Case 2 $(n_1 < n_2)$. That is, since the order of the drive system is lower than that of the response system, the redundant state(s) in the response system should synchronize other artificially built state(s) of the drive system.

Denote the auxiliary state as $\mathbf{x}' \in \mathbf{R}^{n_2-n_1}$, then we can get a new n_2 dimension state vector $\overline{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}$, and the new drive system can rewrite as

$$D_*^q\overline{\mathbf{x}} = \overline{\mathbf{f}}_1(\overline{\mathbf{x}}) + \overline{\mathbf{F}}_1(\overline{\mathbf{x}})\boldsymbol{\Theta}, \quad (18)$$

where $\overline{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}$, $\overline{\mathbf{f}}_1(\mathbf{x}) = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}) \\ \mathbf{f}_1'(\mathbf{x}) \end{pmatrix}$, and $\overline{\mathbf{F}}_1(\mathbf{x}) = \begin{pmatrix} \mathbf{F}_1(\mathbf{x}) \\ \mathbf{F}_1'(\mathbf{x}) \end{pmatrix}$.

Theorem 9. *For the given scaling matrices Q, S, the Q-S synchronization of fractional-order systems between system* (6) *and* (8) *can be achieved, if the active control under the hypothesis is given by* (19) *and the updating laws of the estimated parameter are given by* (20)

$$\mathbf{U} = -g(\mathbf{y}) - \mathbf{G}(\mathbf{y})\widehat{\boldsymbol{\Theta}} + \mathbf{S}^{-1}\left(\mathbf{Q}\left(\overline{\mathbf{f}}(\overline{\mathbf{x}}) + \overline{\mathbf{F}}(\overline{\mathbf{x}})\widehat{\boldsymbol{\Phi}}\right) + \mathbf{K}\mathbf{e}\right), \quad (19)$$

$$\begin{aligned} D_*^q\widehat{\boldsymbol{\Phi}} &= -\left[\mathbf{Q}\overline{\mathbf{F}}(\mathbf{x})\right]^T\mathbf{e}, \\ D_*^q\widehat{\boldsymbol{\Theta}} &= [\mathbf{S}\mathbf{G}(\mathbf{y})]^T\mathbf{e}. \end{aligned} \quad (20)$$

The proof is the same as in Case 1, so we omit it.

Similarly, from Theorem 9, we can also obtain the following corollaries.

Corollary 10. *Suppose the parameters in the drive system* $\mathbf{\Phi}$ *are known, the controller can be designed as follows:*

$$\mathbf{U} = -g(\mathbf{y}) - \mathbf{G}(\mathbf{y})\widehat{\mathbf{\Theta}} + \mathbf{S}^{-1}\left(\mathbf{Q}\left(\overline{\mathbf{f}}(\overline{\mathbf{x}}) + \overline{\mathbf{F}}(\overline{\mathbf{x}})\mathbf{\Phi}\right) + \mathbf{K}\mathbf{e}\right). \tag{21}$$

Moreover, the parameter update laws are degraded as

$$D_*^q\widehat{\mathbf{\Theta}} = \left[\mathbf{S}\mathbf{G}(\mathbf{y})\right]^T\mathbf{e}. \tag{22}$$

Therefore, the drive system and response system can achieve Q-S synchronization.

Corollary 11. *Suppose the parameters* $\mathbf{\Theta}$ *in the response system are known, then the controller is modified as*

$$\mathbf{U} = -g(\mathbf{y}) - \mathbf{G}(\mathbf{y})\mathbf{\Theta} + \mathbf{S}^{-1}\left(\mathbf{Q}\left(\overline{\mathbf{f}}(\overline{\mathbf{x}}) + \overline{\mathbf{F}}(\overline{\mathbf{x}})\widehat{\mathbf{\Phi}}\right) + \mathbf{K}\mathbf{e}\right). \tag{23}$$

Moreover, the parameter update laws can be taken as

$$D_*^q\widehat{\mathbf{\Phi}} = -\left[\mathbf{Q}\overline{\mathbf{F}}(\mathbf{x})\right]^T\mathbf{e}. \tag{24}$$

Therefore, the drive system and response system can achieve the adaptive Q-S synchronization.

4. Simulation Results

In this section, we will present two numerical examples to verify the effectiveness of the adaptive *Q-S* synchronization scheme with different structures proposed in Section 3. All the following numerical simulations are performed via the predictor-corrector algorithm [23].

4.1. Adaptive Q-S Synchronization between the Fractional-Order Hyperchaotic Lorenz System and Fractional-Order Financial System ($n_1 > n_2$). The fractional-order Lorenz system can be described as follows:

$$\begin{aligned} D_*^q x_1 &= a(x_2 - x_1) + x_4, \\ D_*^q x_2 &= cx_1 - x_2 - x_1x_3, \\ D_*^q x_3 &= x_1x_2 - bx_3, \\ D_*^q x_4 &= -x_2x_3 + rx_4, \end{aligned} \tag{25}$$

when $q = 0.98$ and $(a, b, c, r) = (10, 8/3, 28, -1)$, the fractional-order Lorenz displays hyperchaotic attractors.

The fractional-order financial system is described by

$$\begin{aligned} D_*^q y_1 &= y_3 + (y_2 - d)y_1, \\ D_*^q y_2 &= 1 - fy_2 - y_1^2, \\ D_*^q y_3 &= -y_1 - hy_3, \end{aligned} \tag{26}$$

when $q = 0.98$, $(d, f, h) = (3, 0.1, 1)$, and the fractional-order financial system shows chaotic attractors.

We take the fractional-order Lorenz system as drive system, then the response system is given as

$$\begin{aligned} D_*^q y_1 &= y_3 + (y_2 - d)y_1 + u_1, \\ D_*^q y_2 &= 1 - fy_2 - y_1^2 + u_2, \\ D_*^q y_3 &= -y_1 + hy_3 + u_3. \end{aligned} \tag{27}$$

In order to realize the *Q-S* synchronization between the above two different structure systems, we construct an auxiliary state $y_4 = 0$ to the response system, so the response system can be rewritten as the following form:

$$\begin{aligned} D_*^q y_1 &= y_3 + (y_2 - d)y_1 + u_1, \\ D_*^q y_2 &= 1 - fy_2 - y_1^2 + u_2, \\ D_*^q y_3 &= -y_1 - hy_3 + u_3, \\ D_*^q y_4 &= u_4. \end{aligned} \tag{28}$$

For simplicity, we choose the matrices $Q = S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $S^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

From the prefer theorem, we can obtain that

$$\begin{gathered} \mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_4 \\ x_2 - x_1x_3 \\ x_1x_2 \\ -x_2x_3 \end{bmatrix}, \\ \mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \\ 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}, \qquad \mathbf{\Phi} = \begin{bmatrix} a \\ b \\ c \\ r \end{bmatrix}, \\ g(\overline{\mathbf{y}}) = \begin{bmatrix} y_3 + y_2y_1 \\ 1 - y_1^2 \\ -y_1 \\ 0 \end{bmatrix}, \\ G(\overline{\mathbf{y}}) = \begin{bmatrix} -y_1 & 0 & 0 & 0 \\ 0 & -y_2 & 0 & 0 \\ 0 & 0 & -y_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{\Theta} = \begin{bmatrix} d \\ f \\ h \\ 0 \end{bmatrix}. \end{gathered} \tag{29}$$

According to Definition 1, we choose the following scaling matrices:

$$\mathbf{Q}\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \qquad \mathbf{S}\mathbf{y} = \begin{pmatrix} y_1 \\ 2y_2 + y_3 \\ 2y_3 \\ y_4 \end{pmatrix}. \tag{30}$$

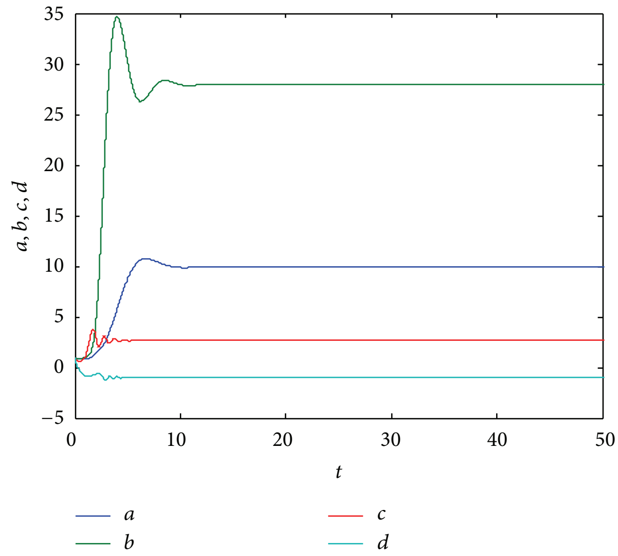

Figure 1: Time evolution of parameter estimations for system (25).

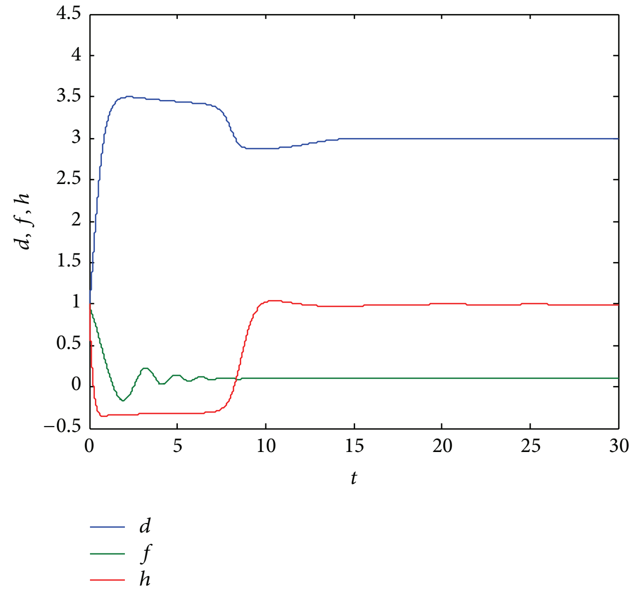

Figure 2: Time evolution of parameter estimations for system (27).

From (9), we have

$$\begin{aligned} u_1 &= -y_3 - y_1 y_2 + \hat{d} y_1 + x_4 + \hat{a}(x_2 - x_4) + k_1 e_1, \\ u_2 &= y_1^2 - 1 + \hat{f} y_2 - x_2 - x_1 x_3 + \hat{c} x_1 + k_2 e_2, \\ u_3 &= y_1 + x_1 x_2 - \hat{h} y_3 - \hat{b} x_3 + k_3 e_3, \\ u_4 &= -x_2 x_3 + \hat{r} x_4 + k_4 e_4, \end{aligned} \tag{31}$$

and the parameters estimation update laws as follows:

$$\begin{aligned} D_*^q \hat{a} &= (x_4 - x_2) e_1, \\ D_*^q \hat{b} &= x_3 e_3, \\ D_*^q \hat{c} &= -x_1 e_2, \\ D_*^q \hat{r} &= -x_4 e_4, \\ D_*^q \hat{d} &= y_1 e_1, \\ D_*^q \hat{f} &= y_2 e_2, \\ D_*^q \hat{h} &= y_3 e_3. \end{aligned} \tag{32}$$

It selects the parameters of the drive and response systems as $(a, b, c, r) = (10, 8/3, 28, -1)$, $d = 3$, $f = 0.1$, and $h = 1$ to ensure the chaotic behavior, the initial conditions are $(x_1(0), x_2(0), x_3(0)) = (1, 2, 3)$ and $(y_1(0), y_2(0), y_3(0)) = (0.1, 0.1, 0.1)$, and the initial values for the estimations of unknown parameters are $\hat{a} = 1$, $\hat{b} = 1$, $\hat{c} = 1$, $\hat{r} = 1$, $\hat{d} = 0.1$, $\hat{f} = 0.1$, $\hat{h} = 0.1$, respectively. The feedback gain $\mathbf{K} = (1, 1, 1, 1)$. Figures 1 and 2 show that all the unknown parameters in the system are identified to their true values as time increase. It is shown in Figure 3 that the synchronization errors between the two different structures fractional-order systems (25) and (27) converge to zero with time passing.

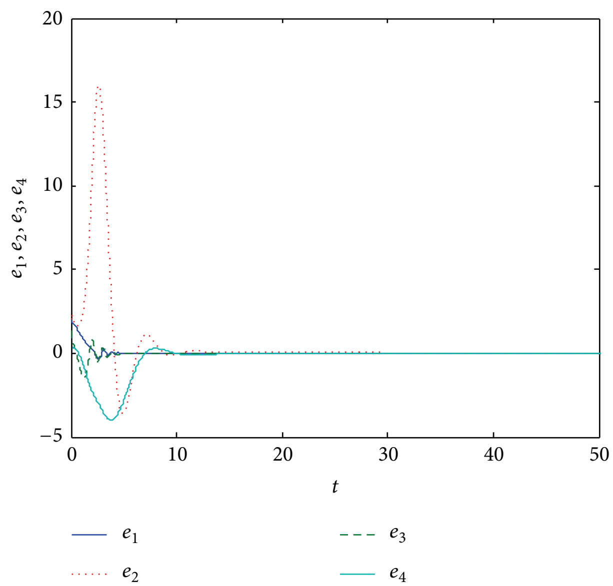

Figure 3: Time evolution of synchronization error between system (25) and system (27).

4.2. Adaptive Q-S Synchronization between Arneodo System and Hyperchaotic Chen System ($n_1 < n_2$). In this subsection, we will study the Q-S synchronization between Arneodo system and hyperchaotic Chen system, the drive system is Arneodo system and the response system is hyperchaotic Chen system.

The fractional order Arneodo system are described by

$$\begin{aligned} D_*^q x_1 &= x_2, \\ D_*^q x_2 &= x_3, \\ D_*^q x_3 &= -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^3, \end{aligned} \tag{33}$$

where the true values for each parameter are $(\beta_1, \beta_2, \beta_3, \beta_4) = (-5.5, 3.5, 1, -1)$ and the order $q = 0.97$, system (33) displays chaotic behaviors.

The controlled hyperchaotic fractional-order Chen system is described as

$$\begin{aligned} D_*^q y_1 &= a_1 (y_2 - y_1) + y_4 + u_1, \\ D_*^q y_2 &= b_1 y_1 - y_1 y_3 + c_1 y_2 + u_2, \\ D_*^q y_3 &= y_1 y_2 - d_1 y_3 + u_3, \\ D_*^q y_4 &= y_2 y_3 + 0.5 y_4 + u_4. \end{aligned} \tag{34}$$

Based on the proposed scheme in Section 3, we should add an auxiliary state variable to the drive system.

For the convenience of controller form, we can get

$$\begin{gathered} \mathbf{f}(\bar{\mathbf{x}}) = \begin{bmatrix} x_2 \\ x_3 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{F}(\bar{\mathbf{x}}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & -x_3 & x_1^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{\Phi} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \\ g(\mathbf{y}) = \begin{bmatrix} y_4 \\ -y_1 y_3 - y_2 \\ y_1 y_2 \\ y_2 y_3 + 0.5 y_4 \end{bmatrix}, \\ G(\mathbf{y}) = \begin{bmatrix} y_2 - y_1 & 0 & 0 & 0 \\ 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & -y_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{\Theta} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}. \end{gathered} \tag{35}$$

According to Definition 1, we choose the scaling matrices

$$\mathbf{Qx} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \qquad \mathbf{Sy} = \begin{pmatrix} y_1 \\ 2y_2 + y_3 \\ 2y_3 \\ y_4 \end{pmatrix}. \tag{36}$$

So we can obtain

$$\begin{gathered} \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\ \mathbf{S}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{gathered} \tag{37}$$

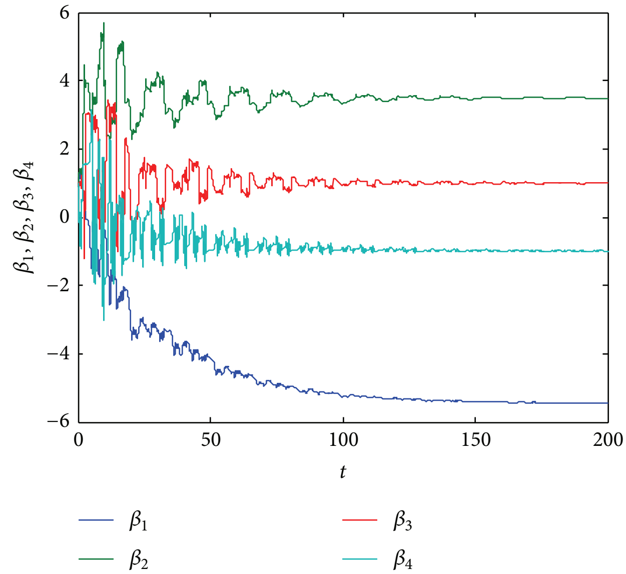

FIGURE 4: Time evolution of parameter estimations for system (33).

Based on the proposed scheme, the controllers are designed in the following form:

$$\begin{aligned} u_1 &= -y_4 - (y_2 - y_1)\hat{a}_1 - x_2 + k_1 e_1, \\ u_2 &= y_1 y_3 - \hat{b}_1 y_1 - \hat{c}_1 y_2 - x_2 - x_3 + 1 + k_2 e_2, \\ u_3 &= -y_1 y_2 + y_3 \hat{d}_1 - x_1 \hat{\beta}_1 - x_2 \hat{\beta}_2 - x_3 \hat{\beta}_3 + x_1^3 \hat{\beta}_4 + k_3 e_3, \\ u_4 &= -y_2 y_3 - \hat{r} y_4 + 1 + k_4 e_4, \end{aligned} \tag{38}$$

and the parameters estimation update laws as follows:

$$\begin{aligned} D_*^q \hat{a}_1 &= -(y_2 - y_1) e_1, \\ D_*^q \hat{b}_1 &= -y_1 e_2, \\ D_*^q \hat{c}_1 &= -y_2 e_2, \\ D_*^q \hat{d}_1 &= -2 y_3 e_3, \\ D_*^q \hat{\beta}_1 &= x_1 e_3, \\ D_*^q \hat{\beta}_2 &= x_2 e_3, \\ D_*^q \hat{\beta}_3 &= x_3 e_3, \\ D_*^q \hat{\beta}_4 &= -x_1^3 e_3. \end{aligned} \tag{39}$$

Analogously, we also would like to give the numerical simulations to verify the effectiveness of the above-designed controller and the update laws. In these numerical simulations, we take the initial states as the initial conditions are $(x_1(0), x_2(0), x_3(0)) = (1, -2, 5)$ and $(y_1(0), y_2(0), y_3(0)) = (1, 1, 1)$, and the initial values for the estimations of unknown parameters are $\hat{\beta}_1 = 1$, $\hat{\beta}_2 = 1$, $\hat{\beta}_3 = 1$, $\hat{\beta}_4 = 1$, $\hat{a}_1 = 1$, $\hat{b}_1 = 0.1$, $\hat{c}_1 = 1$, $\hat{d}_1 = 0.1$. The feedback gain $\mathbf{K} = (5, 5, 5, 5)$. From Figures 4 and 5, it can be clearly seen that all

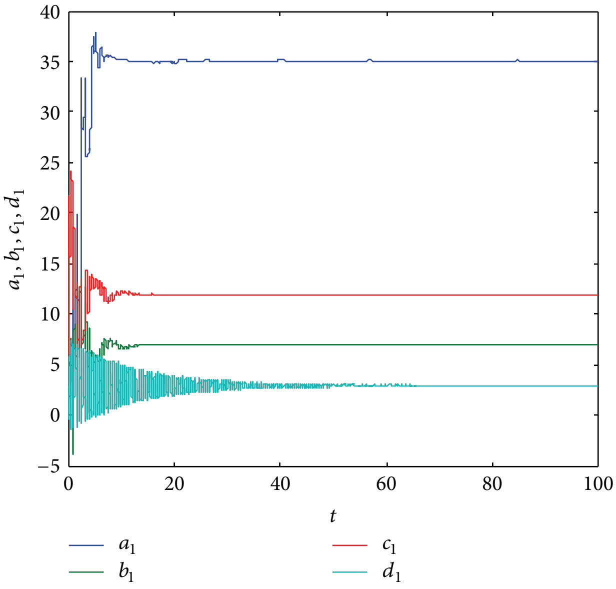

FIGURE 5: Time evolution of parameter estimations for system (34).

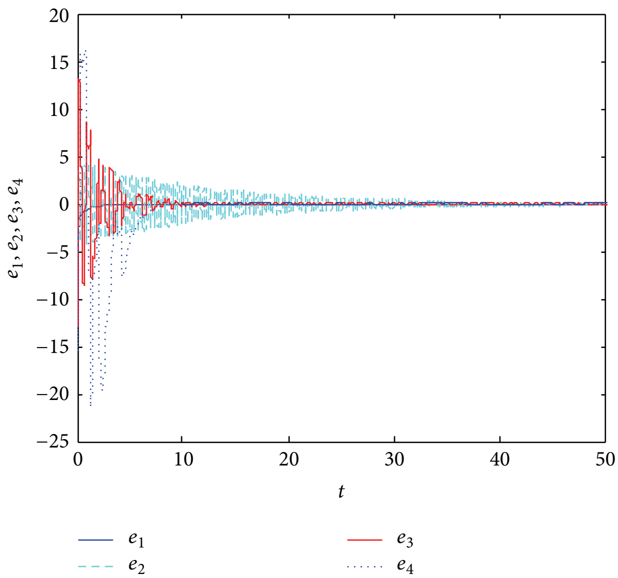

FIGURE 6: Time evolution of synchronization error between system (33) and system (34).

the unknown parameters in the drive and response systems are identified to their true values as time increase. It is shown in Figure 6 that the synchronization errors between the two different structures fractional-order systems (33) and (34) converge to zero with time passing, which means that the *Q*-*S* synchronization between the fractional-order system (33) and system (34) with different dimension is achieved.

5. Conclusion

In this paper, adaptive *Q*-*S* synchronization of fractional-order chaotic systems with different structures is investigated. Based on the stability theory and adaptive control method, a general approach for suitable controller and adaptive laws is provided to realize the *Q*-*S* synchronization. Typical examples are taken to display the applications of the proposed scheme. For each case, the controller and parameter update laws are designed in detail. Meanwhile, results could extend to other fractional-chaotic systems with different structure and uncertain parameters. Numerical simulations show the effectiveness and feasibility of the controllers and identification rules.

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