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*Research Article*

# Uniqueness and Existence of Positive Solutions for Singular Differential Systems with Coupled Integral Boundary Value Problems

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This paper provides sufficient conditions for the existence and uniqueness of positive solutions to a singular differential system with integral boundary value. The emphasis here is that the boundary conditions are coupled and this is where the main novelty of this work lies. By mixed monotone method, the existence and uniqueness results of the problem are established. An example is given to demonstrate the main results.

## 1. Introduction

In recent years, differential system has been studied extensively in the literature (see, for instance, [1–17] and their references). Most of the results told us that the equations had at least single and multiple positive solutions. In papers [6], the authors obtained some of the newest results for differential system with four-point coupled boundary conditions. But there is no result on the uniqueness of solution in them.

In this paper, we discuss the existence and uniqueness of the positive solutions for a new class of boundary value problems of singular differential system. Precisely, we consider the following problem:

$$\begin{aligned} &-x''(t) = f(t, x(t), y(t)), \quad t \in (0,1), \\ &-y''(t) = g(t, x(t), y(t)), \quad t \in (0,1), \\ &x(0) = \int_0^1 y(t)\, d\alpha(t), \qquad y(0) = \int_0^1 x(t)\, d\beta(t), \\ &x(1) = y(1) = 0, \end{aligned} \tag{1}$$

where $\alpha$ and $\beta$ are right continuous on $[0,1)$, left continuous at $t = 1$, and nondecreasing on $[0,1]$, with $\alpha(0) = \beta(0) = 0$; $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ denote the Riemann-Stieltjes integrals of $u$ with respect to $\alpha$ and $\beta$, respectively; $f \in C((0,1) \times [0,+\infty) \times (0,+\infty), [0,+\infty))$, $g \in C((0,1) \times (0,+\infty) \times [0,+\infty), [0,+\infty))$; that is, $f(t,x,y)$ may be singular at $t = 0$, $t = 1$, and $y = 0$ and $g(t,x,y)$ may be singular at $t = 0$, $t = 1$, and $x = 0$. By a positive solution of the system (1), we mean that $(x,y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1))$, $(x,y)$ satisfies (1), and $x > 0$ and $y > 0$ on $[0,1)$.

## 2. Preliminaries

For each $u \in E := C[0,1]$, we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a Banach space. Similarly, for each $(x,y) \in E \times E$, we write $\|(x,y)\|_1 = \max\{\|x\|, \|y\|\}$. Clearly, $(E \times E, \|\cdot\|_1)$ is a Banach space.

Throughout this paper, we shall use the following notation:

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2}$$

It is well known that $k(t,s)$ is the Green function of the following second order boundary value problem:

$$-x''(t) = 0, \quad 0 < t < 1, \qquad x(0) = x(1) = 0, \tag{3}$$

and $k(t,s)$ is nonnegative continuous function. It is easy to verify that for $t, s \in [0,1]\times[0,1]$,

$$k(t,t)k(s,s) = t(1-t)s(1-s) \le k(t,s) \le t(1-t)\ (\text{or } s(1-s)). \tag{4}$$

We first list the following assumptions for convenience.

$(H_1)$ $f \in C((0,1)\times[0,+\infty)\times(0,+\infty),[0,+\infty))$, $f(t,x,y)$ is nondecreasing in $x$ and nonincreasing in $y$, and there exist $\lambda_1, \mu_1 \in [0,1)$ such that

$$c^{\lambda_1} f(t,x,y) \le f(t,cx,y), \quad \forall x,y>0,\ c\in(0,1), \tag{5}$$

$$f(t,x,cy) \le c^{-\mu_1} f(t,x,y), \quad \forall x,y>0,\ c\in(0,1); \tag{6}$$

$g \in C((0,1)\times(0,+\infty)\times[0,+\infty),[0,+\infty))$, $g(t,x,y)$ is nonincreasing in $x$ and nondecreasing in $y$, and there exist $\lambda_2, \mu_2 \in [0,1)$ such that

$$c^{\lambda_2} g(t,x,y) \le g(t,x,cy), \quad \forall x,y>0,\ c\in(0,1), \tag{7}$$

$$g(t,cx,y) \le c^{-\mu_2} g(t,x,y), \quad \forall x,y>0,\ c\in(0,1). \tag{8}$$

$(H_2)$ $0 < \int_0^1 f(t,1,1-t)dt < +\infty$, $0 < \int_0^1 g(t,1-t,1)dt < +\infty$.

$(H_3)$ $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa > 0$, where

$$\kappa_1 = \int_0^1 (1-t)\,d\alpha(t), \qquad \kappa_2 = \int_0^1 (1-t)\,d\beta(t), \qquad \kappa = 1 - \kappa_1\kappa_2. \tag{9}$$

*Remark 1.* By $(H_1)$ and $(H_2)$, we can get

$$0 < \int_0^1 f(t,1-t,1)\,dt < +\infty, \qquad 0 < \int_0^1 g(t,1,1-t)\,dt < +\infty. \tag{10}$$

*Remark 2.* (i) (5) and (6) imply that

$$f(t,cx,y) \le c^{\lambda_1} f(t,x,y), \quad \forall x,y>0,\ c>1, \tag{11}$$

$$f(t,x,y) \le c^{\mu_1} f(t,x,cy), \quad \forall x,y>0,\ c>1. \tag{12}$$

Conversely, (11) implies (5) and (12) implies (6).

(ii) (7) and (8) implies that

$$g(t,cx,y) \le c^{\lambda_2} g(t,x,y), \quad \forall x,y>0,\ c>1, \tag{13}$$

$$g(t,x,y) \le c^{\mu_2} g(t,x,cy), \quad \forall x,y>0,\ c>1. \tag{14}$$

Conversely, (13) implies (7) and (14) implies (8).

**Lemma 3.** *Assume that $(H_3)$ holds. Let $u, v \in L[0,1]\cap C(0,1)$; then the system of BVPs*

$$-x''(t) = u(t), \quad -y''(t) = v(t), \quad t\in(0,1),$$
$$x(0) = \int_0^1 y(t)\,d\alpha(t), \qquad y(0) = \int_0^1 x(t)\,d\beta(t), \tag{15}$$
$$x(1) = y(1) = 0$$

*has integral representation*

$$x(t) = \int_0^1 G_1(t,s)u(s)\,ds + \int_0^1 H_1(t,s)v(s)\,ds,$$
$$y(t) = \int_0^1 G_2(t,s)v(s)\,ds + \int_0^1 H_2(t,s)u(s)\,ds, \tag{16}$$

*where*

$$G_1(t,s) = \frac{k_1(1-t)}{\kappa}\int_0^1 k(s,\tau)\,d\beta(\tau) + k(t,s),$$
$$H_1(t,s) = \frac{1-t}{\kappa}\int_0^1 k(s,\tau)\,d\alpha(\tau),$$
$$G_2(t,s) = \frac{k_2(1-t)}{\kappa}\int_0^1 k(s,\tau)\,d\alpha(\tau) + k(t,s),$$
$$H_2(t,s) = \frac{1-t}{\kappa}\int_0^1 k(s,\tau)\,d\beta(\tau). \tag{17}$$

*Proof.* It is easy to see that (15) is equivalent to the system of integral equations

$$x(t) = x(0)(1-t) + \int_0^1 k(t,s)u(s)\,ds, \quad t\in[0,1], \tag{18}$$

$$y(t) = y(0)(1-t) + \int_0^1 k(t,s)v(s)\,ds, \quad t\in[0,1]. \tag{19}$$

Integrating (18) and (19) with respect to $d\beta(t)$ and $d\alpha(t)$, respectively, on $[0,1]$ gives

$$\int_0^1 x(t)\,d\beta(t) = x(0)\int_0^1 (1-t)\,d\beta(t) + \iint_0^1 k(t,s)u(s)\,ds\,d\beta(t),$$
$$\int_0^1 y(t)\,d\alpha(t) = y(0)\int_0^1 (1-t)\,d\alpha(t) + \iint_0^1 k(t,s)v(s)\,ds\,d\alpha(t). \tag{20}$$

Therefore,

$$\begin{pmatrix} -\kappa_2 & 1 \\ 1 & -\kappa_1 \end{pmatrix}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \iint_0^1 k(t,s)u(s)\,ds\,d\beta(t) \\ \iint_0^1 k(t,s)v(s)\,ds\,d\alpha(t) \end{pmatrix}, \tag{21}$$

and so

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \frac{1}{\kappa}\begin{pmatrix} -\kappa_1 & 1 \\ 1 & -\kappa_2 \end{pmatrix}\begin{pmatrix} \iint_0^1 k(t,s)\,u(s)\,ds\,d\beta(t) \\ \iint_0^1 k(t,s)\,v(s)\,ds\,d\alpha(t) \end{pmatrix}. \tag{22}$$

Substituting (22) into (18) and (19), we have

$$\begin{aligned} x(t) &= \frac{\kappa_1(1-t)}{\kappa}\iint_0^1 k(t,s)\,u(s)\,ds\,d\beta(t) \\ &\quad + \frac{1-t}{\kappa}\iint_0^1 k(t,s)\,v(s)\,ds\,d\alpha(t) \\ &\quad + \int_0^1 k(t,s)\,u(s)\,ds, \\ y(t) &= \frac{1-t}{\kappa}\iint_0^1 k(t,s)\,u(s)\,ds\,d\beta(t) \\ &\quad + \frac{\kappa_2(1-t)}{\kappa}\iint_0^1 k(t,s)\,v(s)\,ds\,d\alpha(t) \\ &\quad + \int_0^1 k(t,s)\,v(s)\,ds, \end{aligned} \tag{23}$$

which is equivalent to the system (16). □

*Remark 4.* From (4) and $(H_3)$, for $t \in [0,1]$, we have

$$\begin{aligned} &G_i(t,s) \le \rho s(1-s)\ (\text{or } \rho(1-t)), \\ &H_i(t,s) \le \rho s(1-s)\ (\text{or } \rho(1-t)), \\ &\qquad\qquad i = 1,2, \\ &G_i(t,s) \ge \nu(1-t)\,s(1-s), \\ &H_i(t,s) \ge \nu(1-t)\,s(1-s), \\ &\qquad\qquad i = 1,2, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \rho &= \max\left\{\frac{\kappa_1}{\kappa}\int_0^1 d\beta(\tau) + 1, \frac{\kappa_2}{\kappa}\int_0^1 d\alpha(\tau) + 1, \right. \\ &\qquad \left. \frac{1}{\kappa}\int_0^1 d\beta(\tau), \frac{1}{\kappa}\int_0^1 d\alpha(\tau)\right\}, \\ \nu &= \min\left\{\frac{\kappa_1}{\kappa}\int_0^1 \tau(1-\tau)\,d\beta(\tau), \right. \\ &\qquad \frac{\kappa_2}{\kappa}\int_0^1 \tau(1-\tau)\,d\alpha(\tau), \\ &\qquad \frac{1}{\kappa}\int_0^1 \tau(1-\tau)\,d\beta(\tau), \\ &\qquad \left. \frac{1}{\kappa}\int_0^1 \tau(1-\tau)\,d\alpha(\tau)\right\}. \end{aligned} \tag{25}$$

Denote

$$\begin{aligned} P = \{&(x,y) \in E\times E : x(t) \ge \gamma(1-t)\,\|(x,y)\|_1, \\ &y(t) \ge \gamma(1-t)\,\|(x,y)\|_1,\ t \in [0,1]\}, \end{aligned} \tag{26}$$

where $\gamma = \nu/\rho \in (0,1)$. It can be easily seen that $P$ is a cone in $E \times E$. For any real constant $r > 0$, define $P_r = \{(x,y) \in P : \|(x,y)\|_1 < r\}$.

Define an operator $T : P \setminus \{\theta\} \to P$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)), \tag{27}$$

where operators $T_1, T_2 : P\setminus\{\theta\} \to Q = \{u \in E \mid u(t) \ge 0, t \in [0,1]\}$ are defined by

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 G_1(t,s)\,f(s,x(s),y(s))\,ds \\ &\quad + \int_0^1 H_1(t,s)\,g(s,x(s),y(s))\,ds, \quad t \in [0,1], \\ T_2(x,y)(t) &= \int_0^1 G_2(t,s)\,g(s,x(s),y(s))\,ds \\ &\quad + \int_0^1 H_2(t,s)\,f(s,x(s),y(s))\,ds, \quad t \in [0,1]. \end{aligned} \tag{28}$$

Now we claim that $T(x,y)$ is well defined for $(x,y) \in P\setminus\{\theta\}$. In fact, since $(x,y) \in P\setminus\{\theta\}$, we can see that

$$\begin{aligned} \gamma(1-t)\,\|(x,y)\|_1 &\le x(t), \\ y(t) &\le \|(x,y)\|_1, \quad t \in [0,1]. \end{aligned} \tag{29}$$

Let $c$ be a positive number such that $\|(x,y)\|_1/c < 1$ and $c > 1$. From $(H_1)$ and Remark 2, we have

$$\begin{aligned} f(t,x(t),y(t)) &\le f(t,c,\gamma\|(x,y)\|_1(1-t)) \\ &\le c^{\lambda_1} f\left(t,1,\frac{\gamma\|(x,y)\|_1}{c}(1-t)\right) \\ &\le c^{\lambda_1+\mu_1}\left(\gamma\|(x,y)\|_1\right)^{-\mu_1} f(t,1,1-t), \\ g(t,x(t),y(t)) &\le c^{\lambda_2+\mu_2}\left(\gamma\|(x,y)\|_1\right)^{-\mu_2} f(t,1,1-t). \end{aligned} \tag{30}$$

Hence, for any $t \in [0,1]$, by Remark 4 and equation (30), we get

$$\begin{aligned} &T_i(x,y)(t) \\ &\quad \le \rho\int_0^1 f(s,x(s),y(s))\,ds + \rho\int_0^1 g(s,x(s),y(s))\,ds \\ &\quad \le \rho c^{\lambda_1+\mu_1}\left(\gamma\|(x,y)\|_1\right)^{-\mu_1}\int_0^1 f(s,1,1-s)\,ds \\ &\qquad + \rho c^{\lambda_2+\mu_2}\left(\gamma\|(x,y)\|_1\right)^{-\mu_2}\int_0^1 g(s,1-s,1)\,ds \\ &\quad < +\infty, \quad i = 1,2. \end{aligned} \tag{31}$$

Thus, $T$ is well defined on $P\setminus\{\theta\}$.

**Lemma 5.** *Assume that $(H_1)$, $(H_2)$, and $(H_3)$ hold. Then, for any $0 < a < b < +\infty$, $T : (\overline{P_b} \setminus P_a) \to P$ is a completely continuous operator.*

*Proof.* Firstly, we show that $T(\overline{P_b} \setminus P_a) \subset P$. By Remark 4, for $\tau, t, s \in [0,1]$, we obtain

$$\begin{aligned} &G_i(t,s) \geq \gamma(1-t)\, G_i(\tau,s), \\ &H_i(t,s) \geq \gamma(1-t)\, H_i(\tau,s), \\ &\qquad\qquad i = 1,2, \\ &H_1(t,s) \geq \gamma(1-t)\, G_2(\tau,s), \\ &G_1(t,s) \geq \gamma(1-t)\, H_2(\tau,s), \\ &H_2(t,s) \geq \gamma(1-t)\, G_1(\tau,s), \\ &G_2(t,s) \geq \gamma(1-t)\, H_1(\tau,s). \end{aligned} \tag{32}$$

Hence, for $(x,y) \in \overline{P_b} \setminus P_a$, $t \in [0,1]$, we have

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 G_1(t,s) f(s, x(s), y(s))\, ds \\ &\quad + \int_0^1 H_1(t,s) g(s, x(s), y(s))\, ds \\ &\geq \gamma(1-t) \int_0^1 G_1(\tau,s) f(s, x(s), y(s))\, ds \\ &\quad + \gamma(1-t) \int_0^1 H_1(\tau,s) g(s, x(s), y(s))\, ds \\ &= \gamma(1-t)\, T_1(x,y)(\tau), \quad \forall \tau \in [0,1], \\ T_1(x,y)(t) &= \int_0^1 G_1(t,s) f(s, x(s), y(s))\, ds \\ &\quad + \int_0^1 H_1(t,s) g(s, x(s), y(s))\, ds \\ &\geq \gamma(1-t) \int_0^1 H_2(\tau,s) f(s, x(s), y(s))\, ds \\ &\quad + \gamma(1-t) \int_0^1 G_2(\tau,s) g(s, x(s), y(s))\, ds \\ &= \gamma(1-t)\, T_2(x,y)(\tau), \quad \forall \tau \in [0,1]. \end{aligned} \tag{33}$$

Then $T_1(x,y)(t) \geq \gamma(1-t)\|T_1(x,y)\|$ and $T_1(x,y)(t) \geq \gamma(1-t)\|T_2(x,y)\|$; that is, $T_1(x,y)(t) \geq \gamma(1-t)\|(T_1(x,y), T_2(x,y))\|_1$. In the same way, we can prove that $T_2(x,y)(t) \geq \gamma(1-t)\|(T_1(x,y), T_2(x,y))\|_1$. Therefore, $T(\overline{P_b} \setminus P_a) \subset P$.

Next, we prove that $T$ is a compact operator. That is, for any bounded subset $V \subset \overline{P_b} \setminus P_a$, we show that $T(V)$ is relatively compact in $E \times E$. Since $V \subset \overline{P_b} \setminus P_a$ is a bounded subset, there exists a constant $c > 1$ such that $\|(x,y)\|_1 = \max\{\|x\|, \|y\|\} \leq c$ for all $(x,y) \in V$. Notice that, for any $(x,y) \in V$, we have

$$\|T(x,y)\|_1 = \max\{\|T_1(x,y)\|, \|T_2(x,y)\|\} \tag{34}$$

and from $(H_1)$, $(H_2)$, Remarks 2 and 4, (16), and (18), we obtain

$$\begin{aligned} &T_i(x,y)(t) \\ &\quad \leq \rho \int_0^1 f(s, x(s), y(s))\, ds + \rho \int_0^1 g(s, x(s), y(s))\, ds \\ &\quad \leq \rho c^{\lambda_1+\mu_1} (\gamma a)^{-\mu_1} \int_0^1 f(s, 1, 1-s)\, ds \\ &\qquad + \rho c^{\lambda_2+\mu_2} (\gamma a)^{-\mu_2} \int_0^1 g(s, 1-s, 1)\, ds \\ &\quad < +\infty, \quad i = 1,2. \end{aligned} \tag{35}$$

Therefore, $T(V)$ is uniformly bounded.

In the following, we shall show that $T(V)$ is equicontinuous on $[0,1]$.

For $(x,y) \in V$, $t \in [0,1]$, using Lemma 3, we have

$$\begin{aligned} &T_1(x,y)(t) \\ &\quad = \int_0^1 G_1(t,s) f(s, x(s), y(s))\, ds \\ &\qquad + \int_0^1 H_1(t,s) g(s, x(s), y(s))\, ds \\ &\quad = \frac{\kappa_1(1-t)}{\kappa} \int_0^1 \left( \int_0^1 k(s,\tau)\, d\beta(\tau) \right) f(s, x(s), y(s))\, ds \\ &\qquad + \int_0^t s(1-t) f(s, x(s), y(s))\, ds \\ &\qquad + \int_t^1 t(1-s) f(s, x(s), y(s))\, ds \\ &\qquad + \frac{1-t}{\kappa} \int_0^1 \left( \int_0^1 k(s,\tau)\, d\alpha(\tau) \right) g(s, x(s), y(s))\, ds. \end{aligned} \tag{36}$$

Differentiating with respect to $t$ and combining $(H_1)$ and $(H_2)$, we obtain

$$\begin{aligned} &\left| T_1(x,y)'(t) \right| \\ &\quad = \left| -\frac{\kappa_1}{\kappa} \int_0^1 \left( \int_0^1 k(s,\tau)\, d\beta(\tau) \right) f(s, x(s), y(s))\, ds \right. \\ &\qquad - \int_0^t s f(s, x(s), y(s))\, ds \\ &\qquad + \int_t^1 (1-s) f(s, x(s), y(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\kappa}\int_0^1\left(\int_0^1 k(s,\tau)\,d\alpha(\tau)\right)g(s,x(s),y(s))\,ds\Bigg| \\
&\leq \frac{\kappa_1}{\kappa}\int_0^1\left(\int_0^1 k(s,\tau)\,d\beta(\tau)\right)f(s,x(s),y(s))\,ds \\
&\quad+\int_0^t sf(s,x(s),y(s))\,ds \\
&\quad+\int_t^1 (1-s)f(s,x(s),y(s))\,ds \\
&\quad+\frac{1}{\kappa}\int_0^1\left(\int_0^1 k(s,\tau)\,d\alpha(\tau)\right)g(s,x(s),y(s))\,ds \\
&\leq c^{\lambda_1+\mu_1}(\gamma a)^{-\mu_1}\left(\frac{\rho\kappa_1}{\kappa}\int_0^1 f(s,1,1-s)\,ds\right. \\
&\qquad+\int_0^t sf(s,1,1-s)\,ds \\
&\qquad\left.+\int_t^1 (1-s)f(s,1,1-s)\,ds\right) \\
&\quad+\frac{\rho}{\kappa}c^{\lambda_2+\mu_2}(\gamma a)^{-\mu_2}\int_0^1 g(s,1-s,1)\,ds =: K(t).
\end{aligned}
\tag{37}
$$

Exchanging the integral order, we have

$$
\begin{aligned}
&\int_0^1 K(t)\,dt \\
&= c^{\lambda_1+\mu_1}(\gamma a)^{-\mu_1}\left(\frac{\rho\kappa_1}{\kappa}\int_0^1 f(s,1,1-s)\,ds\right. \\
&\qquad\left.+2\int_0^1 s(1-s)f(s,1,1-s)\,ds\right) \\
&\quad+\frac{\rho}{\kappa}c^{\lambda_2+\mu_2}(\gamma a)^{-\mu_2}\int_0^1 g(s,1-s,s)\,ds < +\infty.
\end{aligned}
\tag{38}
$$

From the absolute continuity of the integral, we know that $T_1(V)$ is equicontinuous on $[0,1]$. Thus, according to the Ascoli-Arzela theorem, $T_1(V)$ is a relatively compact set. In the same way, we can prove that $T_2(V)$ is relatively compact. Therefore, $T(V)$ is relatively compact.

Finally, it remains to show that $T$ is continuous. We need to prove only that $T_1, T_2 : \overline{P_b} \setminus P_a \to Q$ are continuous. Suppose that $(x_m, y_m), (x_0, y_0) \in \overline{P_b} \setminus P_a$, and $\|(x_m, y_m) - (x_0, y_0)\|_1 \to 0$ $(m \to \infty)$. Let $L = \sup\{\|(x_m, y_m)\|_1,\ m = 0,1,2,\ldots\}$. Then we may still choose positive constants $c$ such that $L/c < 1$ and $c > 1$. From $(H_1)$ and Remark 2, we get

$$
\begin{aligned}
&f(t,x_m(t),y_m(t)) \leq c^{\lambda_1+\mu_1}(\gamma a)^{-\mu_1}f(t,1,1-t), \\
&\qquad m = 0,1,2\ldots, \\
&g(t,x_m(t),y_m(t)) \leq c^{\lambda_2+\mu_2}(\gamma a)^{-\mu_2}g(t,1-t,1), \\
&\qquad m = 0,1,2\ldots, \\
&\left|T_1(x_m,y_m)(t) - T_1(x_0,y_0)(t)\right| \\
&\leq \rho\int_0^1 \left|f(s,x_m(s),y_m(s))\right. \\
&\qquad\left.-f(s,x_0(s),y_0(s))\right|ds \\
&\quad+\rho\int_0^1 \left|g(s,x_m(s),y_m(s))\right. \\
&\qquad\left.-g(s,x_0(s),y_0(s))\right|ds.
\end{aligned}
\tag{39}
$$

For any $\epsilon > 0$, by $(H_2)$, there exists a positive number $\delta \in (0,1/2)$ such that

$$
\begin{aligned}
&\int_{[0,\delta]\cup[1-\delta,1]} \rho c^{\lambda_1+\mu_1}(\gamma a)^{-\mu_1}f(s,1,1-s)\,ds < \frac{\varepsilon}{4}, \\
&\int_{[0,\delta]\cup[1-\delta,1]} \rho c^{\lambda_2+\mu_2}(\gamma a)^{-\mu_2}g(s,1-s,1)\,ds < \frac{\varepsilon}{4}.
\end{aligned}
\tag{40}
$$

On the other hand, for $(x,y) \in \overline{P_b} \setminus P_a$ and $t \in [\delta, 1-\delta]$, we have

$$
a\gamma\delta \leq x(t), \qquad y(t) \leq b. \tag{41}
$$

Since $f(t,x,y)$ and $g(t,x,y)$ are uniformly continuous in $[\delta,1-\delta]\times[a\gamma\delta,b]\times[a\gamma\delta,b]$, we have

$$
\begin{aligned}
&\lim_{m\to+\infty}\left|f(t,x_m(t),y_m(t)) - f(t,x_0(t),y_0(t))\right| \\
&\quad= \lim_{m\to+\infty}\left|g(t,x_m(t),y_m(t)) - g(t,x_0(t),y_0(t))\right| \\
&\quad= 0
\end{aligned}
\tag{42}
$$

holds uniformly on $t \in [\delta, 1-\delta]$. Then the Lebesgue dominated convergence theorem yields that

$$
\begin{aligned}
&\int_\delta^{1-\delta}\left|f(s,x_m(s),y_m(s)) - f(s,x_0(s),y_0(s))\right|ds \to 0 \\
&\int_\delta^{1-\delta}\left|g(s,x_m(s),y_m(s)) - g(s,x_0(s),y_0(s))\right|ds \to 0, \\
&\qquad m \to +\infty.
\end{aligned}
\tag{43}
$$

Thus, for above $\varepsilon > 0$, there exists a natural number $N$, for $m > N$; we have

$$
\begin{aligned}
&\int_\delta^{1-\delta}\rho\left|f(s,x_m(s),y_m(s)) - f(s,x_0(s),y_0(s))\right|ds < \frac{\varepsilon}{4}, \\
&\int_\delta^{1-\delta}\rho\left|g(s,x_m(s),y_m(s)) - g(s,x_0(s),y_0(s))\right|ds < \frac{\varepsilon}{4}.
\end{aligned}
\tag{44}
$$

It follows from (39)–(44) that when $m > N$

$$\begin{aligned}
&\|T_1(x_m, y_m) - T_1(x_0, y_0)\| \\
&\quad \le \rho \int_0^1 \left| f(s, x_m(s), y_m(s)) - f(s, x_0(s), y_0(s)) \right| ds \\
&\qquad + \rho \int_0^1 \left| g(s, x_m(s), y_m(s)) - g(s, x_0(s), y_0(s)) \right| ds \\
&\quad \le \int_{[0,\delta]\cup[1-\delta,1]} \rho c^{\lambda_1+\mu_1} (\gamma a)^{-\mu_1} f(s, 1, 1-s)\, ds \\
&\qquad + \int_{[0,\delta]\cup[1-\delta,1]} \rho c^{\lambda_2+\mu_2} (\gamma a)^{-\mu_2} g(s, 1-s, 1)\, ds \\
&\qquad + \int_\delta^{1-\delta} \rho \left| f(s, x_m(s), y_m(s)) \right. \\
&\qquad\qquad \left. - g(s, x_0(s), y_0(s)) \right| ds \\
&\qquad + \int_\delta^{1-\delta} \rho \left| g(s, x_m(s), y_m(s)) \right. \\
&\qquad\qquad \left. - g(s, x_0(s), y_0(s)) \right| ds < \varepsilon.
\end{aligned} \tag{45}$$

This implies that $T_1 : \overline{P_b} \setminus P_a \to Q$ is continuous. Similarly, we can prove that $T_2 : \overline{P_b} \setminus P_a \to Q$ is continuous. So, $T : \overline{P_b} \setminus P_a \to P$ is continuous. Summing up, $T : \overline{P_b} \setminus P_a \to P$ is completely continuous. □

Our main tool of this paper is the following cone compression and expansion fixed point theorem.

**Lemma 6** (see [18]). *Let $E$ be a Banach space and $P$ a cone in $E$. Suppose that $\Omega_1$ and $\Omega_2$ are two bounded open subsets of $E$ with $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator satisfying*

$$\begin{aligned}
\|Tx\| &\ge \|x\|, \quad \text{for } x \in P \cap \partial\Omega_1, \\
\|Tx\| &\le \|x\|, \quad \text{for } x \in P \cap \partial\Omega_2,
\end{aligned} \tag{46}$$

*then $T$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

## 3. Main Results

In this section, we present our main results.

**Theorem 7.** *Suppose that conditions $(H_1)$, $(H_2)$, and $(H_3)$ hold. Then, if $\lambda_1 + \mu_1 < 1$ and $\lambda_2 + \mu_2 < 1$, the differential system* (1) *has a unique positive solution $(x^*, y^*)$.*

*Proof.* We divide the rather long proof into three steps.

(i) The differential system (1) has at least one positive solution $(x^*, y^*)$.

Choose r, $R$ such that

$$\begin{aligned}
0 < r &\le \min\left\{ \left( \frac{\nu}{4} \gamma^{\lambda_1} \int_0^1 s(1-s) \right.\right. \\
&\qquad \left.\left. \times f(s, 1-s, 1)\, ds \right)^{1/(1-\lambda_1)}, \frac{1}{2} \right\}, \\
R &\ge \max\left\{ \left( \rho \int_0^1 f(s, 1, 1-s)\, ds \right.\right. \\
&\qquad \left.\left. + \rho \int_0^1 g(s, 1-s, 1)\, ds \right)^{1/(1-\max\{\lambda_1,\lambda_2\})}, \frac{1}{\gamma}, 2 \right\}.
\end{aligned} \tag{47}$$

Clearly $0 < r < 1 < R$. By Lemma 5, $T : \overline{P_R} \setminus P_r \to P$ is completely continuous.

Extend $T$ (denote $T$ yet) to $T : \overline{P_R} \to P$ which is completely continuous.

Then, for $(x, y) \in \partial P_r$, we have

$$r\gamma(1-t) \le x(t), \quad y(t) \le r, \quad t \in [0,1]. \tag{48}$$

By Remarks 1 and 2, $(H_1)$, and $(H_2)$, we get

$$\begin{aligned}
T_i(x, y)(t) &\ge \frac{\nu}{4} \int_0^1 s(1-s) f(s, \gamma r(1-s), r)\, ds \\
&\ge \frac{\nu}{4} \int_0^1 s(1-s) f(s, \gamma r(1-s), 1)\, ds \\
&\ge \frac{\nu}{4} \gamma^{\lambda_1} r^{\lambda_1} \int_0^1 s(1-s) f(s, 1-s, 1)\, ds \\
&\ge r = \|(x, y)\|_1, \quad i = 1, 2,\ t \in \left[0, \frac{3}{4}\right].
\end{aligned} \tag{49}$$

This guarantees that

$$\|T(x, y)\|_1 \ge \|(x, y)\|_1, \quad \forall (x, y) \in \partial P_r. \tag{50}$$

On the other hand, for $(x, y) \in \partial P_R$, we have

$$R\gamma(1-t) \le x(t), \quad y(t) \le R, \quad t \in [0,1]. \tag{51}$$

Therefore,

$$\begin{aligned}
&T_i(x, y)(t) \\
&\quad \le \rho \int_0^1 f(s, R, \gamma R(1-s))\, ds \\
&\qquad + \rho \int_0^1 g(s, \gamma R(1-s), R)\, ds
\end{aligned}$$

$$\begin{aligned}
&\leq \rho \int_0^1 f(s, R, 1-s)\, ds + \rho \int_0^1 g(s, 1-s, R)\, ds \\
&\leq \rho R^{\lambda_1} \int_0^1 f(s, 1, 1-s)\, ds + \rho R^{\lambda_2} \int_0^1 g(s, 1-s, 1)\, ds \\
&\leq \rho R^{\max\{\lambda_1, \lambda_2\}} \left( \int_0^1 f(s, 1, 1-s)\, ds \right. \\
&\qquad \left. + \int_0^1 g(s, 1-s, 1)\, ds \right) \\
&\leq R = \|(x, y)\|_1, \quad i = 1, 2,\ t \in [0, 1].
\end{aligned} \tag{52}$$

This guarantees that

$$\|T(x, y)\|_1 \leq \|(x, y)\|_1, \quad \forall (x, y) \in \partial P_R. \tag{53}$$

By the complete continuity of $T$, (50) and (53), and Lemma 6, we obtain that $T$ has a fixed point $(x^*, y^*)$ in $\overline{P_R} \setminus P_r$. Consequently, (1) has a positive solution $(x^*, y^*)$ in $\overline{P_R} \setminus P_r$.

(ii) Suppose that $(x, y)$ is a positive solution of the differential system (1).

Then there exist real numbers $0 < m < 1$ such that

$$\begin{aligned}
m(1-t) &\leq x(t) \leq \frac{1}{m}(1-t), \\
m(1-t) &\leq y(t) \leq \frac{1}{m}(1-t), \quad \forall t \in [0, 1].
\end{aligned} \tag{54}$$

From Lemma 5, we know that $(x, y) \in P \setminus \{\theta\}$. So, we have

$$\gamma \|(x, y)\|_1 (1-t) \leq x(t), \qquad y(t) \leq \|(x, y)\|_1. \tag{55}$$

Let $c$ be a constant such that $\|(x, y)\|_1 / c < 1$ and $c > 1/\gamma$. By Lemma 3, we get

$$\begin{aligned}
x(t) &\leq \rho (1-t) \int_0^1 f\left(s, c, \frac{\gamma \|(x, y)\|_1}{c}(1-s)\right) ds \\
&\quad + \rho (1-t) \int_0^1 g\left(s, \frac{\gamma \|(x, y)\|_1}{c}(1-s), c\right) ds \\
&\leq c^{\lambda_1+\mu_1} (\gamma \|(x, y)\|_1)^{-\mu_1} \rho (1-t) \int_0^1 f(s, 1, 1-s)\, ds \\
&\quad + c^{\lambda_2+\mu_2} (\gamma \|(x, y)\|_1)^{-\mu_2} \rho (1-t) \int_0^1 g(s, 1-s, 1)\, ds \\
&=: C(1-t), \quad t \in [0, 1].
\end{aligned} \tag{56}$$

In the same way, we can prove that $y(t) \leq C(1-t)$, $t \in [0, 1]$. Then we may pick out $m$ such that $m = \min\{\gamma \|(x, y)\|_1, 1/C, 1/2\}$, which implies that (54) holds.

(iii) The differential system (1) has a unique positive solution $(x^*, y^*)$.

Assuming the contrary, we find that the differential system (1) has a positive solution $(x_*, y_*)$ different from $(x^*, y^*)$. By (54), there exist $\delta_1, \delta_2 > 0$, such that

$$\begin{aligned}
&\delta_1 (1-t) \leq x^*(t), \qquad y^*(t) \leq \frac{1}{\delta_1}(1-t), \quad \forall t \in [0, 1], \\
&\delta_2 (1-t) \leq x_*(t), \qquad y_*(t) \leq \frac{1}{\delta_2}(1-t), \quad \forall t \in [0, 1].
\end{aligned} \tag{57}$$

Hence, we have

$$\begin{aligned}
\delta_1 \delta_2 x_*(t) &\leq x^*(t) \leq \frac{1}{\delta_1 \delta_2} x_*(t), \\
\delta_1 \delta_2 y_*(t) &\leq y^*(t) \leq \frac{1}{\delta_1 \delta_2} y_*(t), \\
&\forall t \in [0, 1].
\end{aligned} \tag{58}$$

Clearly, $\delta_1 \delta_2 \neq 1$. Put

$$\begin{aligned}
\delta^* = \sup \Big\{ \delta \mid \delta x_*(t) &\leq x^*(t) \leq \frac{1}{\delta} x_*(t), \\
\delta y_*(t) &\leq y^*(t) \leq \frac{1}{\delta} y_*(t), \forall t \in [0, 1] \Big\}.
\end{aligned} \tag{59}$$

It is easy to see that $1 > \delta^* \geq \delta_1 \delta_2 > 0$, and

$$\begin{aligned}
\delta^* x_*(t) &\leq x^*(t) \leq \frac{1}{\delta^*} x_*(t), \\
\delta^* y_*(t) &\leq y^*(t) \leq \frac{1}{\delta^*} y_*(t), \quad \forall t \in [0, 1].
\end{aligned} \tag{60}$$

So, by $(H_1)$, we have

$$\begin{aligned}
f(t, x^*(t), y^*(t)) &\geq f\left(t, \delta^* x_*(t), \frac{1}{\delta^*} y_*(t)\right) \\
&\geq \delta^{*\lambda_1+\mu_1} f(t, x_*(t), y_*(t)) \\
&\geq \delta^{*\sigma} f(t, x_*(t), y_*(t)), \\
g(t, x^*(t), y^*(t)) &\geq \delta^{*\lambda_2+\mu_2} g(t, x_*(t), y_*(t)) \\
&\geq \delta^{*\sigma} g(t, x_*(t), y_*(t)),
\end{aligned} \tag{61}$$

where $\sigma = \max\{\lambda_1 + \mu_1, \lambda_2 + \mu_2\}$ such that $\sigma < 1$. Therefore, we have

$$\begin{aligned}
x^*(t) &= T_1(x^*, y^*)(t) \\
&= \int_0^1 G_1(t, s) f(s, x^*(s), y^*(s))\, ds \\
&\quad + \int_0^1 H_1(t, s) g(s, x^*(s), y^*(s))\, ds \\
&\geq \delta^{*\sigma} \int_0^1 G_1(t, s) f(s, x_*(s), y_*(s))\, ds \\
&\quad + \int_0^1 H_1(t, s) g(s, x_*(s), y_*(s))\, ds \\
&= \delta^{*\sigma} T_1(x_*, y_*)(t) = \delta^{*\sigma} x_*(t).
\end{aligned} \tag{62}$$

Similarly, we can get

$$y^*(t) \geq \delta^{*\sigma} y_*(t),$$
$$x_*(t) \geq \delta^{*\sigma} x^*(t), \qquad y_*(t) \geq \delta^{*\sigma} y^*(t). \tag{63}$$

Noticing that $0 < \delta^*, \sigma < 1$, we get to a contradiction with the maximality of $\delta^*$. Thus, the differential system (1) has a unique positive solution $(x^*, y^*)$. This completes the proof of Theorem 7. □

## 4. An Example

In this section, we give an example to illustrate the usefulness of our main results. Let us consider the singular differential system with couple boundary value problem

$$-x'' = \frac{\sqrt{x}}{\sqrt[3]{yt(1-t)}}, \qquad -y'' = \frac{\sqrt[3]{y}}{\sqrt{x}},$$
$$x(1) = y(1) = 0, \qquad x(0) = y\left(\frac{1}{3}\right) + y\left(\frac{1}{2}\right), \tag{64}$$
$$y(0) = \int_0^1 x(t)\,dt^2.$$

Let

$$f(t,x,y) = \frac{\sqrt{x}}{\sqrt[3]{yt(1-t)}}, \qquad g(t,x,y) = \frac{\sqrt[3]{y}}{\sqrt{x}},$$
$$\alpha(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{3}\right), \\ 1, & t \in \left[\frac{1}{3}, \frac{1}{2}\right), \\ 2, & t \in \left[\frac{1}{2}, 1\right], \end{cases} \tag{65}$$
$$\beta(t) = t^2, \qquad \lambda_1 = \mu_2 = \frac{1}{2}, \qquad \lambda_2 = \mu_1 = \frac{1}{3};$$

then

$$\kappa_1 = \frac{7}{6}, \qquad \kappa_2 = \frac{1}{3}, \qquad \kappa = 1 - \kappa_1\kappa_2 = \frac{11}{18},$$
$$\int_0^1 f(s, 1, 1-s)\,ds = B\left(\frac{2}{3}, \frac{1}{6}\right), \tag{66}$$
$$\int_0^1 g(s, 1-s, 1)\,ds = B\left(1, \frac{1}{2}\right).$$

So all conditions of Theorem 7 are satisfied for (64), and our conclusion follows from Theorem 7.

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