Research Article

A System of Generalized Variational-Hemivariational Inequalities with Set-Valued Mappings

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By using surjectivity theorem of pseudomonotone and coercive operators rather than the KKM theorem and fixed point theorem used in recent literatures, we obtain some conditions under which a system of generalized variational-hemivariational inequalities concerning set-valued mappings, which includes as special cases many problems of hemivariational inequalities studied in recent literatures, is solvable. As an application, we prove an existence theorem of solutions for a system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives.

1. Introduction

Let $V_1, V_2, \ldots, V_n$ be real, separable, reflexive Banach spaces with dual spaces $V_1^*, V_2^*, \ldots, V_n^*$, and let $X_1, X_2, \ldots, X_n$ be real reflexive Banach spaces with dual spaces $X_1^*, X_2^*, \ldots, X_n^*$ such that there exist linear continuous and compact operators $T_i : V_i \rightarrow X_i$ for $i = 1, 2, \ldots, n$. We denote by $\langle \cdot, \cdot \rangle_{E^* \times E}$ the duality pairing between Banach space E and its dual E^* and by $\| \cdot \|_E, \| \cdot \|_{E^*}$ the norms on the space E and its dual space E^*, respectively, where $E = \{V_i, X_i, i = 1, 2, \ldots, n\}$. Assume that $A_i : \prod_{k=1}^n V_k \rightarrow 2^{V_i^*}$, $i = 1, 2, \ldots, n$ are set-valued mappings on the product space $\prod_{k=1}^n V_k$ of Banach spaces $V_1, \ldots, V_n$ and that $J : \prod_{k=1}^n X_k \rightarrow R$ is a functional on the product space $\prod_{k=1}^n X_k$ of Banach spaces $X_1, \ldots, X_n$, which is locally Lipschitz with respect to each component; that is, for all $i = 1, 2, \ldots, n$, the functionals $J(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) : X_i \rightarrow R$ are locally Lipschitz for all fixed x_j, $j \neq i$, and $G_i : V_i \rightarrow R \cup \{+\infty\}$, $i = 1, 2, \ldots, n$ are proper, convex, and lower semicontinuous

functionals. In this paper, we study a system of generalized variational-hemivariational inequalities concerning set-valued mappings, which is specified as follows.

For all $i = 1, 2, \ldots, n$, find $u_i \in V_i$ and $\mu_i \in A_i(\mathbf{u})$ such that

$$\begin{aligned}
(\text{P}) \quad & \langle \mu_i, v_i - u_i \rangle_{V_i^* \times V_i} + J_i^\circ (\widehat{\mathbf{u}}; \widehat{v}_i - \widehat{u}_i) + G_i(v_i) \\
& - G_i(u_i) \geq 0, \quad \forall v_i \in V_i,
\end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \prod_{k=1}^n V_k$, $\widehat{\mathbf{u}} = (\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n) = (T_1 u_1, T_2 u_2, \ldots, T_n u_n)$, $\widehat{v}_i = T_i v_i$ and $J_i^\circ(\widehat{\mathbf{u}}; \widehat{v}_i - \widehat{u}_i)$ is the partial generalized directional derivative (in the sense of Clarke) of the functional J, which is locally Lipschitz for each component, with respect to the ith component at the point $\widehat{u}_i \in X_i$ in the direction $\widehat{v}_i - \widehat{u}_i$ for all given $\widehat{u}_j \in X_j$, $j \neq i$, which can be defined by

$$\begin{aligned}
& J_i^\circ (\widehat{\mathbf{u}}; \widehat{v}_i - \widehat{u}_i) \\
& = \limsup_{x \rightarrow \widehat{u}_i, \lambda \downarrow 0} \frac{J(\widehat{u}_1, \ldots, \widehat{u}_{i-1}, x + \lambda(\widehat{v}_i - \widehat{u}_i), \widehat{u}_{i+1}, \ldots, \widehat{u}_n)}{\lambda}
\end{aligned}$$

$$- \frac{J\left(\widehat{u}_1, \ldots, \widehat{u}_{i-1}, x, \widehat{u}_{i+1}, \ldots, \widehat{u}_n\right)}{\lambda}. \tag{2}$$

In the last few years, there are many researchers who dedicated themselves to the study of various types of hemivariational inequalities and systems of hemivariational inequalities, which are a generalization of the variational inequalities, and related problems such as equilibrium problems. In these papers, based on Clarke's generalized directional derivative and Clarke's generalized gradient for locally Lipchitz functions, the researchers study the existence and uniqueness of solution by mainly using KKM theorems, surjectivity theorems for pseudomonotone and coercive operators, fixed point theorems, critical point theory, and so on. We refer readers for the study of hemivariational inequalities to monographs of Carl et al. [1], Migórski et al. [2], Naniewicz and Panagiotopoulos [3], and Panagiotopoulos [4]. For the system of hemivariational inequalities, Denkowski and Migórski [5] studied a dynamic thermoviscoelastic frictional contact problem which was modeled by a system of evolution hemivariational inequalities. They proved the existence and uniqueness of the weak solution for the problem by using a surjectivity result for operators of pseudomonotone type. In 2011, Repovš and Varga [6] studied the Nash equilibrium point by using the Ky Fan version of the KKM theorem and the Tarafdar fixed point theorem for a class of hemivariational inequality system. It is obvious that some problems studied in literatures are special cases of our system of generalized variational-hemivariational inequalities under some special conditions, such as $n = 1$, A_i are single-valued, or G_i are indicators of some convex subsets K_i for $i = 1, 2, \ldots, n$. Although it seems that our problem (P) cannot include the problem studied in [6] as a special case, we remark here that, in essence, the problem (P) is a generalization of the problem in [6] since, when A_i $(i = 1, 2, \ldots, n)$ are single-valued and G_i are the indicators of the convex subsets K_i, the problem (P) reduces to the problem studied by Repovš and Varga [6] with $T_i = S_i$ and $A_i^\circ(Tu; T_i v_i - T_i u_i)$ being incorporated into $J_i^\circ(Su; S_i v_i - S_i u_i)$ under the regularity condition. For more information on the research of hemivariational inequalities and systems of hemivariational inequalities, we can refer to [7–16] and references therein.

It is well known that, by surjectivity theorem of pseudomonotone and coercive operators, there exists solution u_i to each variational-hemivariational inequality in the system (1) for all $u_j \in V_j$, $j \neq i$ under some suitable conditions on the operators A_i, J, and G_i. A natural question is whether these conditions are sufficient for the existence of solutions to the system (1) which is combined by solvable variational-hemivariational inequalities. If not, what other stronger conditions do we need to guarantee the solvability of the system (1)? In this paper, we are devoted to these questions by using surjectivity theorem of pseudomonotone and coercive operators rather than the KKM theorem, and the fixed point theorem used by Repovš and Varga in [6] to obtain the existence of the solutions to the problem (P) of a system of generalized variational-hemivariational inequalities concerning set-valued mappings.

As will be seen in the proof of our main theorem in Section 3, the case where $n = k$ for any finite positive integer $k > 2$ is a natural generalization of the case where $n = 2$. Therefore, in what follows, We will focus on the problem of a system of two generalized variational-hemivariational inequalities, which can be reformulated as follows. Consider $u_1 \in V_1$, $u_2 \in V_2$, $\mu_1 \in A_1(u)$ and $\mu_2 \in A_2(u)$ such that

$$(\mathrm{P}') \begin{cases} \langle \mu_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(\widehat{u}; \widehat{v}_1 - \widehat{u}_1) \\ \quad + G_1(v_1) - G_1(u_1) \geq 0, & \forall v_1 \in V_1, \\ \langle \mu_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(\widehat{u}; \widehat{v}_2 - \widehat{u}_2) \\ \quad + G_2(v_2) - G_2(u_2) \geq 0, & \forall v_2 \in V_2, \end{cases} \tag{3}$$

where $u = (u_1, u_2)$, $\widehat{u} = (\widehat{u}_1, \widehat{u}_2) = (T_1 u_1, T_2 u_2)$ and $\widehat{v}_i = T_i v_i$ for $i = 1, 2$.

The paper is structured as follows. In Section 2, we recall some preliminary material. Section 3 gives conditions under which the problem (P) of a system of generalized variational-hemivariational inequalities concerning set-valued mapping is solvable by considering the simple case, the problem (P') of a system of two generalized variational-hemivariational inequalities. At last, in Section 4, we are concerned with an application of our results to a system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives.

2. Preliminaries

In this section, we recall some important notations and useful results on nonlinear analysis, nonsmooth analysis, and operators of monotone type, which can be found in [2, 3, 17, 18].

Without confusion of symbols, we suppose, just in this section, that X is a Banach space with its dual X^* and duality paring $\langle \cdot, \cdot \rangle$ between X^* and X, $G : X \to R \cup \{+\infty\}$ is a proper and convex functional, and $J : X \to R$ is a locally Lipschitz functional with Clarke's generalized directional derivative $J^\circ(u, v)$. We denote by $\widehat{\partial} G(u) : X \to 2^{X^*} \setminus \{\emptyset\}$ and $\partial J(u) : X \to 2^{X^*} \setminus \{\emptyset\}$ the subgradient of the convex functional G in the sense of convex analysis and Clarke's generalized gradient of the locally Lipschitz functional J, respectively. Then,

$$\widehat{\partial} G(u) = \{u^* \in X^* : G(v) - G(u) \\ \geq \langle u^*, v - u \rangle, \ \forall v \in X\}, \tag{4}$$

$$\partial J(u) = \{\omega \in X^* : J^\circ(u, v) \geq \langle \omega, v \rangle, \ \forall v \in X\}.$$

We have the following basic properties on Clarke's generalized directional derivative and Clarke's generalized gradient (see, e.g., [2, 17]).

Proposition 1. *Let X be Banach space, and let $u, v \in X$, and J be locally Lipschitz functional defined on X. Then one has the following.*

(1) *The function $v \mapsto J^\circ(u, v)$ is finite, positively homogeneous, subadditive, and then convex on X;*

(2) $J^\circ(u, v)$ is upper semicontinuous as a function of (u, v), but as a function of v alone, it is Lipschitz continuous on X.

(3) $\partial J(u)$ is a nonempty, convex, bounded, and weak*-compact subset of X^*.

(4) For every $v \in X$, one has

$$J^\circ(u, v) = \max\{\langle \xi, v \rangle : \xi \in \partial J(u)\}. \tag{5}$$

(5) The graph of the Clarke generalized gradient $\partial J(u)$ is closed in $X \times (w^* - X^*)$ topology.

(6) The multifunction $X \ni u \to \partial J(u) \subseteq X^*$ is upper semicontinuous from X into $w^* - X^*$.

Definition 2. A locally Lipschitz functional $J : X \to R$ is said to be regular (in the sense of Clarke) at $u \in X$ if

(i) for all $v \in X$ the directional derivative $J'(u, v)$ exists;

(ii) for all $v \in X$, $J'(u, v) = J^\circ(u, v)$,

where $J'(u, v)$ is directional derivative of J at $u \in X$ in the direction $v \in X$, which is defined by

$$J'(u, v) = \lim_{\lambda \downarrow 0} \frac{J(u + \lambda v) - J(u)}{\lambda}, \tag{6}$$

whenever this limit exists. The functional J is regular (in the sense of Clarke) on X if it is regular at every point $u \in X$.

Proposition 3. Let X_1 and X_2 be two Banach spaces. If $J : X_1 \times X_2 \to R$ is locally Lipschitz and either J or $-J$ is regular at $u = (u_1, u_2) \in X_1 \times X_2$, then

$$\partial J(u_1, u_2) \subseteq \partial_1 J(u_1, u_2) \times \partial_2 J(u_1, u_2), \tag{7}$$

or equivalently one has

$$J^\circ(u_1, u_2; v_1, v_2) \le J_1^\circ(u_1, u_2; v_1) + J_2^\circ(u_1, u_2; v_2),$$
$$\forall (v_1, v_2) \in X_1 \times X_2, \tag{8}$$

where $\partial_1(u_1, u_2)$ (resp., $\partial_2(u_1, u_2)$) represents the partial generalized subgradient of $J(\cdot, u_2)$ (resp., $J(u_1, \cdot)$) and $J_1^\circ(u_1, u_2; v_1)$ (resp., $J_2^\circ(u_1, u_2; v_2)$) denotes the partial generalized directional derivative of $J(\cdot, u_2)$ (resp., $J(u_1, \cdot)$) at the point u_1 (resp., u_2) in the direction v_1 (resp., v_2), but the converse of inclusion (7) and inequality (8) is not true in general.

Definition 4. Let X be real reflexive Banach space with dual X^*. A mapping T from X into 2^{X^*} is said to be pseudomonotone if

(1) the set Tu is nonempty, bounded, closed, and convex for all $u \in X$;

(2) T is upper semicontinuous from each finite dimensional subspace of X to X^* endowed with the weak topology;

(3) $\{u_i\}$ is a sequence in X converging weakly to u, and $u_i^* \in Tu_i$ is such that

$$\limsup \langle u_i^*, u_i - u \rangle \le 0, \tag{9}$$

then for each element $v \in X$ there exists $u^*(v) \in Tu$ such that

$$\liminf \langle u_i^*, u_i - v \rangle \ge \langle u^*(v), u - v \rangle. \tag{10}$$

Definition 5. Let X be real reflexive Banach space with dual X^*. A mapping T from X into 2^{X^*} is said to be generalized pseudomonotone if for any sequences $\{u_i\} \subset X$, $\{u_i^*\} \subset X^*$ with $u_i^* \in Tu_i$, $u_i \to u$ weakly in X, $u_i^* \to u^*$ weakly in X^* and

$$\limsup \langle u_i^*, u_i - u \rangle \le 0; \tag{11}$$

then one has $u^* \in Tu$ and $\langle u_i^*, u_i \rangle \to \langle u^*, u \rangle$.

Proposition 6. Let X be real reflexive Banach space with dual X^* and let T_1, T_2 be two pseudomonotone mappings from X into 2^{X^*}. Then $T_1 + T_2$ is pseudomonotone.

Proposition 7. Let X be real reflexive Banach space with dual X^* and let $T : X \to 2^{X^*}$ be a pseudomonotone mapping from X into 2^{X^*}. Then T is a generalized pseudomonotone.

Proposition 8. Let X be real reflexive Banach space with dual X^* and let T be a bounded, generalized pseudomonotone mapping from X into 2^{X^*}. Assume that, for each $u \in X$, Tu is a nonempty closed convex subset of X^*. Then T is pseudomonotone.

Definition 9. Let X be real reflexive Banach space with dual X^*. The operator $T : X \to 2^{X^*}$ is said to be as follows:

(1) monotone if for all (u, u^*), (v, v^*) lying in the graph $G(T)$ of T, one has

$$\langle u^* - v^*, u - v \rangle \ge 0. \tag{12}$$

(2) maximal monotone if it is monotone and if $(u, u^*) \in X \times X^*$ is such that

$$\langle u^* - v^*, u - v \rangle \ge 0, \quad \forall (v, v^*) \in G(T); \tag{13}$$

then $(u, u^*) \in G(T)$.

(3) quasibounded if for each $M > 0$ there exists $K(M) > 0$ such that, whenever $(u, u^*) \in G(T)$, $\langle u^*, u \rangle \le M\|u\|$, and $\|u\| \le M$; then

$$\|u^*\| \le K(M). \tag{14}$$

(4) strongly quasibounded if for each $M > 0$ there exists $K(M) > 0$ such that for all $(u, u^*) \in G(T)$ with $\langle u^*, u \rangle \le M$ and $\|u\| \le M$, one has

$$\|u^*\| \le K(M). \tag{15}$$

Definition 10. Let X be real reflexive Banach space with dual X^*. A mapping T from X into 2^{X^*} is said to be as follows:

(1) coercive if there exists a real-valued function c on R^+ with $\lim_{r \to \infty} c(r) = \infty$ such that for all $(u, u^*) \in G(T)$, one has

$$\langle u^*, u \rangle \geq c\left(\|u\|\right) \|u\|, \tag{16}$$

(2) coercive with constant $\alpha > 0$ if

$$\langle u^*, u \rangle \geq \alpha \|u\|^2, \tag{17}$$

(3) u_0-coercive if there exists a real-valued function c on R^+ with $\lim_{r \to \infty} c(r) = \infty$ such that for some $u_0 \in X$ and for all $(u, u^*) \in G(T)$, one has

$$\langle u^*, u - u_0 \rangle \geq c\left(\|u\|\right) \|u\|. \tag{18}$$

The following theorem is a surjectivity theorem for the sum of a pseudomonotone, coercive operator, and a maximal monotone operator, which is important to the proof of our main results.

Theorem 11 (see [3]). *Let X be a real reflexive Banach space with dual X^*, let $\widehat{T}$ be a maximal monotone mapping from X into 2^{X^*} with $u_0 \in D(T) = \{x : Tx \neq \emptyset\}$, and let T be a u_0-coercive, pseudomonotone operator from X into 2^{X^*}. Suppose further that either $T_{u_0} : X \to 2^{X^*}$ is quasibounded or $\widehat{T}_{u_0} : X \to 2^{X^*}$ is strongly quasibounded, where $T_{u_0}(v) = T(u_0 + v)$ and the same for $\widehat{T}_{u_0}$. Then $R(T + \widehat{T}) = X^*$.*

3. Main Results

In this section, we first give an existence theorem for the solution to the problem (P') of a system of two generalized variational-hemivariational inequalities. And then, as a natural generalization, an existence theorem for the solution to the problem (P), a system of generalized variational-hemivariational inequalities concerning set-valued mappings is also obtained.

Before we present the main existence theorem, for the simplicity of writing, we define some useful symbols and give a crucial lemma in advance, which establishes the relationship between the problem (P') of a system of two variational-hemivariational inequalities and a generalized vector variational-hemivariational inequality. Let $V = V_1 \times V_2$. Endowed with the norm defined by

$$\|u\|_V = \|u_1\|_{V_1} + \|u_2\|_{V_2}, \quad \forall u = (u_1, u_2) \in V, \tag{19}$$

V is a reflexive Banach space with dual V^*. The duality pairing between V and V^* is given by

$$\langle u^*, u \rangle_{V^* \times V} = \langle u_1^*, u_1 \rangle_{V_1^* \times V_1} + \langle u_2^*, u_2 \rangle_{V_2^* \times V_2},$$
$$\forall u^* = (u_1^*, u_2^*) \in V^*, \ u = (u_1, u_2) \in V. \tag{20}$$

On the Banach space V defined above, we further define a set-valued mapping $A : V \to 2^{V^*}$, an operator $T : V \to X_1 \times X_2$, and a functional $G : V \to R \cup \{+\infty\}$, which are specified as follows. For all $u = (u_1, u_2) \in V$, one has

$$A(u) = \left(A_1(u), A_2(u)\right),$$
$$T(u) = \left(T_1 u_1, T_2 u_2\right), \tag{21}$$
$$G(u) = G_1(u_1) + G_2(u_2).$$

Lemma 12. *Assume that $G_i : V_i \to R \cup \{+\infty\}$, $i = 1, 2$ are proper, convex, and lower semicontinuous functionals. The functional G defined above is also a proper, convex, and lower semicontinuous functional on V. Moreover, $\widehat{\partial} G(u) = \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$.*

Proof. Since the functionals $G_i : V_i \to R \cup \{+\infty\}$, $i = 1, 2$ are proper and convex, it is easy to show that G is also proper and convex by the inequality $G(\lambda u + (1 - \lambda v)) = G_1(\lambda u_1 + (1 - \lambda)v_1) + G_2(\lambda u_2 + (1 - \lambda)v_2) \leq \lambda G(u) + (1 - \lambda)G(v)$ for all $u, v \in V$ and $\lambda \in [0, 1]$. As for the lower semicontinuity of the functional G, by assuming that $u^n \to u$ in V, which implies $u_1^n \to u_1$ in V_1 and $u_2^n \to u_2$ in V_2, we can get from the lower semicontinuity of G_1 and G_2 that

$$\liminf G(u^n) = \liminf \left(G_1(u_1^n) + G_2(u_2^n)\right)$$
$$\geq \liminf G_1(u_1^n) + \liminf G_2(u_2^n) \tag{22}$$
$$\geq G_1(u_1) + G_2(u_2) = G(u),$$

which means that G is lower semicontinuous on V.

Now, we prove the equality $\widehat{\partial} G(u) = \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$. Assume that $\mu \in \widehat{\partial} G(u) \subset V^*$, which says that

$$G(v) - G(u) \geq \langle \mu, v - u \rangle_{V^* \times V}, \quad \forall v \in V. \tag{23}$$

In particular, for any $v_1 \in V_1$, let $v = (v_1, u_2)$ in (23), and then we can get that

$$G_1(v_1) - G_1(u_1) \geq \langle \mu_1, v_1 - u_1 \rangle_{V_1^* \times V_1}, \quad \forall v_1 \in V_1. \tag{24}$$

Similarly, by letting $v = (u_1, v_2)$ in (23) for any $v_2 \in V_2$, we can obtain

$$G_2(v_2) - G_2(u_2) \geq \langle \mu_2, v_2 - u_2 \rangle_{V_2^* \times V_2}, \quad \forall v_2 \in V_2, \tag{25}$$

which together with (24) implies that $\mu \in \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$; that is, $\widehat{\partial} G(u) \subseteq \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$.

Conversely, let $\mu = (\mu_1, \mu_2) \in \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$. For all $v_i \in V_i$, $i = 1, 2$, it follows from $\mu_i \in \widehat{\partial} G_i(u_i)$ that

$$G_1(v_1) - G_1(u_1) \geq \langle \mu_1, v_1 - u_1 \rangle_{V_1^* \times V_1}, \quad \forall v_1 \in V_1,$$
$$G_2(v_2) - G_2(u_2) \geq \langle \mu_2, v_2 - u_2 \rangle_{V_2^* \times V_2}, \quad \forall v_2 \in V_2. \tag{26}$$

By adding the two inequalities (26), we obtain

$$G(v) - G(u) \geq \langle \mu, v - u \rangle_{V^* \times V}, \quad \forall v \in V, \tag{27}$$

which implies that $\mu \in \widehat{\partial} G(u)$; that is, $\widehat{\partial} G(u) \supseteq \widehat{\partial} G_1(u_1) \times \widehat{\partial} G_2(u_2)$. This completes the proof of Lemma 12. $\square$

Now, we consider the following generalized vector variational-hemivariational inequality. Find $u = (u_1, u_2) \in V$ and $\mu = (\mu_1, \mu_2) \in A(u)$ such that

$$(\mathrm{P}'') \quad \langle \mu, v - u \rangle_{V^* \times V} + J^\circ (Tu, Tv - Tu) + G(v)$$
$$-G(u) \geq 0, \quad \forall v \in V. \tag{28}$$

We first give a crucial lemma which establishes the relationship between the problem (P') of a system of two variational-hemivariational inequalities and the problem (P'') of a generalized vector variational-hemivariational inequality.

Lemma 13. *Assume that the locally Lipschitz functional $J : X_1 \times X_2 \to R$ is regular on $X_1 \times X_2$. Then any solution $u = (u_1, u_2) \in V$ to the problem (P'') is always a solution to the problem (P').*

Proof. Assume that $u = (u_1, u_2)$ solves the problem (P''), which says that there exists an $\mu = (\mu_1, \mu_2) \in A(u)$ such that for all $v \in V$, one has

$$\langle \mu, v - u \rangle_{V^* \times V} + J^\circ (Tu, Tv - Tu) + G(v) - G(u) \geq 0. \tag{29}$$

Specially, for any $v_1 \in V_1$, let $v = (v_1, u_2) \in V$ in (29), and then we can get from Proposition 3 that

$$0 \leq \langle \mu_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J^\circ (Tu, (T_1 v_1 - T_1 u_1, 0))$$
$$+ G_1 (v_1) - G_1 (u_1)$$
$$\leq \langle \mu_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ (Tu, T_1 v_1 - T_1 u_1)$$
$$+ G_1 (v_1) - G_1 (u_1), \quad \forall v_1 \in V_1. \tag{30}$$

Similarly, by letting $v = (u_1, v_2) \in V$ in (29) for any $v_2 \in V_2$, we can obtain that

$$\langle \mu_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ (Tu, T_2 v_2 - T_2 u_2)$$
$$+ G_2 (v_2) - G_2 (u_2) \geq 0, \quad \forall v_2 \in V_2, \tag{31}$$

which together with the inequality (30) implies that $u = (u_1, u_2)$ is a solution to the problem (P'). This completes the proof of Lemma 13. $\square$

Remark 14. It follows from Proposition 3 that, just under regularity condition of the functional J, $J^\circ(u, v) = J_1^\circ(u, v_1) + J_2^\circ(u, v_2)$ does not hold in general, while the inequality $J^\circ(u, v) \leq J_1^\circ(u, v_1) + J_2^\circ(u, v_2)$ is true. Therefore, without other much stronger conditions on functional J, the inverse of the Lemma 13 is not true in general.

We give some assumptions on the operators A_i and J in the system (3) of two generalized variational-hemivariational inequalities.

The assumption (HA) is as follows.

(1) $A_1 : V_1 \times V_2 \to 2^{V_1^*}$ is bounded on $V_1 \times V_2$ and pseudomonotone with respect to the first

argument; that is, for all $u_2 \in V_2$, the operator $A_1(\cdot, u_2) : V_1 \to 2^{V_1^*}$ is pseudomonotone on V_1.

(2) $A_2 : V_1 \times V_2 \to 2^{V_2^*}$ is bounded on $V_1 \times V_2$ and pseudomonotone with respect to the second argument; that is, for all $u_1 \in V_1$, the operator $A_2(u_1, \cdot) : V_2 \to 2^{V_2^*}$ is pseudomonotone on V_2.

(3) For all $u_2 \in V_2$, there exist an element $w_1 \in D(\widehat{\partial} G_1) \subset V_1$ and a constant $\alpha_1 > 0$ such that

$$\langle u_1^*, u_1 - w_1 \rangle_{V_1^* \times V_1} \geq \alpha_1 \|u_1\|_{V_1}^2,$$
$$\forall u_1 \in V_1, u_1^* \in A_1 (u_1, u_2). \tag{32}$$

(4) For all $u_1 \in V_1$, there exist an element $w_2 \in D(\widehat{\partial} G_2) \subset V_2$ and a constant $\alpha_2 > 0$ such that

$$\langle u_2^*, u_2 - w_2 \rangle_{V_2^* \times V_2} \geq \alpha_2 \|u_2\|_{V_2}^2,$$
$$\forall u_2 \in V_2, u_2^* \in A_2 (u_1, u_2). \tag{33}$$

Remark 15. It is clear that the hypotheses (1) and (2) in the assumption (HA) imply that the operator A defined in (21) is also bounded on V. The hypotheses (3) on the operator A_1 and (4) on the operator A_2 in the assumption (HA) imply the w_1-coercivity of A_1 with respect to the first argument and w_2-coercivity of A_2 with respect to the second argument, respectively. Moreover, for $w = (w_1, w_2) \in D(\widehat{\partial} G)$, the operator A defined in (21) is also w-coercive with constant $\beta = \min\{\alpha_1, \alpha_2\}/2$. In fact, for all $\mu \in A(u)$, one has

$$\langle \mu, u - w \rangle_{V^* \times V} = \langle \mu_1, u_1 - w_1 \rangle_{V_1^* \times V_1}$$
$$+ \langle \mu_2, u_2 - w_2 \rangle_{V_2^* \times V_2}$$
$$\geq \alpha_1 \|u_1\|_{V_1}^2 + \alpha_2 \|u_2\|_{V_2}^2$$
$$\geq \frac{\min \{\alpha_1, \alpha_2\}}{2} \|u\|_V^2,$$
$$= \beta \|u\|_V^2, \tag{34}$$

which implies the w-coercivity with constant β of operator A on V.

The assumption (HJ) is as follows.

(1) For all $x_2 \in X_2$, there exist constants $c_1, d_1 \geq 0$ such that

$$\|\eta_1\|_{X_1^*} \leq c_1 + d_1 \|x_1\|_{X_1}, \quad \forall \eta_1 \in \partial_1 J (x_1, x_2). \tag{35}$$

(2) For all $x_1 \in X_1$, there exist constants $c_2, d_2 \geq 0$ such that

$$\|\eta_2\|_{X_2^*} \leq c_2 + d_2 \|x_2\|_{X_2}, \quad \forall \eta_2 \in \partial_2 J (x_1, x_2). \tag{36}$$

Remark 16. It is clear that the hypotheses in assumption (HJ) imply that $\partial_1 J(\cdot, x_2)$ and $\partial_2 J(x_1, \cdot)$ are bounded on X_1 and X_2, respectively. Moreover, if J is regular on X, then ∂J is also bounded on X. (In the following, let $X = X_1 \times X_2$ and $X^* = X_1^* \times X_2^*$ for simplicity of writing.) In fact, since J is regular on X, the inclusion $\partial J(x_1, x_2) \subseteq \partial_1 J(x_1, x_2) \times \partial_2 J(x_1, x_2)$ holds. It follows from (35) and (36) that

$$\|\eta\|_{X^*} \le c + d\|x\|_X, \quad \forall \eta \in \partial J(x), \tag{37}$$

with $c = c_1 + c_2 \ge 0$ and $d = \max\{d_1, d_2\} \ge 0$. This also means that ∂J is bounded on X.

We are now in a position to give our main result on the existence of solution to the problem (P'), a system of two generalized variational-hemivariational inequalities.

Theorem 17. *Suppose that the set-valued mappings $A_i : V_1 \times V_2 \to 2^{V_i^*}$, $i = 1, 2$, which satisfy the assumption (HA), is such that the operator A defined in* (21) *is pseudomonotone on V. Let $T_i : V_i \to X_i$ be linear continuous and compact operators, let $J : X \to R$ be a regular, locally Lipschitz functional which satisfies the hypothesis (HJ), let and $G_i : V_i \to R \cup \{+\infty\}$, $i = 1, 2$ be proper, convex, and lower semicontinuous functionals. Then the problem (P') admits at least one solution under the condition*

$$\beta > d\|T\|^2, \tag{38}$$

where $\|T\|$ is the norm of the operator T defined by (21).

Proof. By Lemma 13, the existence of solution to the problem (P') of a system of two generalized variational-hemivariational inequalities can be proved as long as the problem (P'') of a generalized vector variational-hemivariational inequality is solvable. Therefore, we consider the following inclusion problem. Find $u \in V$ such that

$$F(u) + \widehat{\partial}G(u) \ni 0, \tag{39}$$

where $F : V \to 2^{V^*}$ with $F(u) = A(u) + T^* \circ \partial J \circ T(u)$ for all $u \in V$. We will prove the existence of solution to the inclusion problem (39) by the surjectivity theorem (Theorem 11), which implies that the problem (P'') is solvable.

Claim 1 (F is bounded on V). Since the operator A is bounded on V under assumption (HA) by Remark 15, ∂J is also bounded on X under the assumption (HJ) by Remark 16, and T is linear continuous by the linearity and continuity of the operators T_i, $i = 1, 2$, and it is easy to check that F is bounded on V, which implies that $F_{u_0} : V \to 2^{V^*}$ with $F_{u_0}(u) = F(u_0 + u)$ is quasibounded for any $u_0 \in V$.

Claim 2 (F is pseudomonotone on V). Since $F = A + T^* \circ \partial J \circ T : V \to 2^{V^*}$ and the operator A is pseudomonotone, we only need to prove that $T^* \circ \partial J \circ T : V \to 2^{V^*}$ is pseudomonotone. To this end, firstly, we prove that $T^* \circ \partial J \circ T$ is generalized pseudomonotone. Let $u^n \to u$ weakly in V, $\xi^n \in T^*(\partial J(Tu^n))$ with $\xi^n \to \xi$ weakly in V^* and $\limsup \langle \xi^n, u^n - u \rangle_{V^* \times V} \le 0$.

There exist $\eta^n \in \partial J(Tu^n)$ such that $\xi^n = T^* \eta^n$. Since ∂J is bounded on X by Remark 16, $Tu^n \to Tu$ in X by the compactness of the operators T_i, $i = 1, 2$, and $\eta^n \in \partial J(Tu^n)$, we have the fact that η^n is bounded in X^*. Thus there exists a subsequence, which is also denoted by η^n, such that $\eta^n \to \eta$ weakly in X^* with some $\eta \in X^*$. By using the equality $\xi^n = T^* \eta^n$, it is easy to get that $\xi = T^* \eta$. Since $\eta^n \in \partial J(Tu^n)$ with $\eta^n \to \eta$ weakly in X^* and $Tu^n \to Tu$ in X, we get by the closedness of ∂J with $X \times (w^* - X^*)$ topology and the reflexivity of X that $\eta \in \partial J(Tu)$, and thus $\xi = T^* \eta \in T^*(\partial J(Tu))$. Moreover, it follows from $\eta^n \to \eta$ weakly in X^* and $Tu^n \to Tu$ in X that

$$\langle \xi^n, u^n \rangle_{V^* \times V} = \langle \eta^n, Tu^n \rangle_{X^* \times X} \longrightarrow \langle \eta, Tu \rangle_{X^* \times X}$$
$$= \langle T^* \eta, u \rangle_{V^* \times V} = \langle \xi, u \rangle_{V^* \times V}, \tag{40}$$

which together with $\xi \in T^*(\partial J(Tu))$ implies that $T^* \circ \partial J \circ T$ is generalized pseudomonotone on V. Secondly, it is easy to check that $T^*(\partial J(Tu))$ is nonempty, convex, and closed in V^* for all $u \in V$ since $\partial J(x)$ is a nonempty, convex, and closed subset in X^* for all $x \in X$ and T is linear and continuous on V. Thirdly, the operator $T^* \circ \partial J \circ T$ is bounded on V, which has been proved in Claim 1. Consequently, it follows from the Proposition 8 that $T^* \circ \partial J \circ T$ is pseudomonotone on V.

Claim 3 (F is w-coercive on V). Let $u \in V$ and $\tau \in F(u)$, and then there exist $\mu \in A(u)$ and $\eta \in \partial J(Tu)$ such that $\tau = \mu + T^* \eta$. By Remarks 15 and 16, we have

$$\langle \tau, u - w \rangle_{V^* \times V} = \langle \mu, u - w \rangle_{V^* \times V} + \langle T^* \eta, u - w \rangle_{V^* \times V}$$

$$\ge \beta \|u\|_V^2 + \langle \eta, T(u - w) \rangle_{X^* \times X}$$

$$\ge \beta \|u\|_V^2 - \|\eta\|_{X^*} \|T\| (\|u\|_V + \|w\|_V)$$

$$\ge \beta \|u\|_V^2 - (c + d\|Tu\|_X) \|T\| (\|u\|_V + \|w\|_V)$$

$$\ge (\beta - d\|T\|^2) \|u\|_V^2$$

$$- \|T\| (c + d\|T\|\|w\|_V) \|u\|_V - c\|T\|\|w\|_V, \tag{41}$$

which together with the condition $\beta > d\|T\|^2$ means that F is w-coercive on V with function $c(t) = (\beta - d\|T\|^2)t - \|T\|(c + d\|T\|\|w\|_V) - c\|T\|\|w\|_V/t$.

It is well known that $\widehat{\partial}G$ is a maximal monotone operator on V since, by Lemma 12, the functional G is proper, convex, and lower semicontinuous on V (see [18]). We are now in a position to apply Theorem 11 to the set-valued operators F and $\widehat{\partial}G$. We deduce that $F + \widehat{\partial}G$ is surjective, which implies that there exist $u \in V$ such that

$$0 \in F(u) + \widehat{\partial}G(u). \tag{42}$$

By the definition of the operator F, there exist $\mu \in A(u)$, $\eta \in \partial J(Tu)$, and $\xi \in \widehat{\partial}G(u)$ such that

$$\mu + T^* \eta + \xi = 0. \tag{43}$$

By multiplying the equality (43) by $v - u$ for all $v \in V$, we obtain from the definition of Clarke's generalized subgradient of the functional J and subgradient in the sense of convex analysis of the functional G that

$$
\begin{aligned}
0 &= \langle \mu, v - u \rangle_{V^* \times V} + \langle T^* \eta, v - u \rangle_{V^* \times V} + \langle \xi, v - u \rangle_{V^* \times V} \\
&= \langle \mu, v - u \rangle_{V^* \times V} + \langle \eta, Tv - Tu \rangle_{X^* \times X} + \langle \xi, v - u \rangle_{V^* \times V} \\
&\leq \langle \mu, v - u \rangle_{V^* \times V} + J^\circ (Tu, Tv - Tu) + G(v) - G(u),
\end{aligned}
\tag{44}
$$

which implies that u solves the problem (P″) of a generalized vector variational-hemivariational inequality. As stated at the beginning of our proof, u is also a solution to the problem (P′) of a system of two generalized variational-hemivariational inequalities by Lemma 13. This completes the proof of Theorem 17. □

Remark 18. The pseudomonotonicity of the operator A defined in (21) is necessary for the proof of the existence of solution to the problem (P′) by the surjectivity theorem since the pseudomonotonicity of operator A_1 with respect to the first argument and the pseudomonotonicity of operator A_2 with respect to the second argument, which are necessary to prove the existence of solution to each generalized variational-hemivariational inequality in problem (P′), cannot guarantee the pseudomonotonicity of the operator A defined in (21) in general. However, some special cases in which the pseudomonotonicity of operator A_1 with respect to the first argument and the pseudomonotonicity of operator A_2 with respect to the second argument imply the pseudomonotonicity of the operator A defined in (21) can be given under some stronger conditions (see [5]).

It is obvious that, by similar arguments as proof of Theorem 17, we have the following results for the existence of solution to each generalized variational-hemivariational inequality in the system (3).

Theorem 19. *Suppose that, for $i = 1, 2$, $A_i : V_1 \times V_2 \to 2^{V_i^*}$ are set-valued mappings satisfying the assumption (HA) and that $T_i : V_i \to X_i$ are linear continuous and compact operators. Let $J : X \to R$ be a regular, locally Lipschitz functional on X, which satisfies the hypothesis (HJ), and let $G_i : V_i \to R \cup \{+\infty\}$, $i = 1, 2$ be proper, convex, and lower semicontinuous functionals. Then, the ith $i = 1, 2$ generalized variational-hemivariational inequality in the system (3) admits at least one solution $u_i \in V_i$ for all $u_j \in V_j$, $j \neq i$ under the condition*

$$
\alpha_i > d_i \|T_i\|^2,
\tag{45}
$$

where $\|T_i\|$ is the norm of the operator T_i.

Remark 20. By comparing Theorems 17 with 19, we remark here that, in addition to the pseudomonotonicity of the operator A defined in (21), we need strongly condition (38) than (45) to obtain the existence of solution to the problem (P′) of a system of two generalized variational-hemivariational inequalities.

As a natural generalization of Theorem 17 for the existence of solution to the problem (P′) of a system of two generalized variational-hemivariational inequalities, we can obtain the following theorem for the existence of solution to the problem (P) of a system of generalized variational-hemivariational inequalities concerning set-valued mappings.

Theorem 21. *Suppose that the following assumptions on the operators in the problem (P) of a system of generalized variational-hemivariational inequalities hold.*

(1) *For $i = 1, 2, \ldots, n$, $A_i : \prod_{k=1}^n V_k \to 2^{V_i^*}$ are set-valued mappings satisfying the following.*

    (a) *A_i are bounded on $\prod_{k=1}^n V_k$ and pseudomonotone with respect to the ith argument.*

    (b) *the operator $A : V = \prod_{k=1}^n V_k \to 2^{V^*}$, which is defined by $A(u) = (A_1(u), \ldots, A_n(u))$, is pseudomonotone on V.*

    (c) *For all $u_j \in V_j$, $j \neq i$, there exist an element $w_i \in D(\hat{\partial} G_i) \subset V_i$ and a constant $\alpha_i > 0$ such that*

$$
\begin{aligned}
&\langle u_i^*, u_i - w_i \rangle_{V_i^* \times V_i} \geq \alpha_i \|u_i\|_{V_i}^2, \\
&\forall u_i \in V_i, \ u_i^* \in A_i(u_1, \ldots, u_n).
\end{aligned}
\tag{46}
$$

(2) *$J : \prod_{k=1}^n X_k \to R$ is a regular and locally Lipschitz functional which satisfies that for all $i = 1, 2, \ldots, n$ and $x_j \in X_j$, $j \neq i$, there exist constants $c_i, d_i \geq 0$ such that*

$$
\begin{aligned}
&\|\eta_i\|_{X_i^*} \leq c_i + d_i \|x_i\|_{X_i}, \\
&\forall x_i \in X_i, \ \eta_i \in \partial_i J(x_1, \ldots, x_n).
\end{aligned}
\tag{47}
$$

(3) *For $i = 1, 2, \ldots, n$, $T_i : V_i \to X_i$ are linear continuous and compact operators and $G_i : V_i \to R \cup \{+\infty\}$ are proper, convex, and lower semicontinuous functionals.*

Then the problem (P) admits at least one solution under the condition

$$
\tilde{\beta} > \tilde{d} \|\tilde{T}\|^2,
\tag{48}
$$

where $\tilde{\beta} = \min\{\alpha_i\}/n$, $\tilde{d} = \max\{d_i\}$ and $\|\tilde{T}\|$ is the norm of the operator $\tilde{T} : \prod_{k=1}^n V_k \to \prod_{k=1}^n X_k$ defined by $\tilde{T}(u_1, \ldots, u_n) = (T_1 u_1, \ldots, T_n u_n)$.

4. An Application

In this section, we are concerned with an application of our results to a system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives.

Let $\Omega \subset R^n$ be a bounded and open set in R^n, let $V_1, V_2, \ldots, V_n$ be real, separable, and reflexive Banach spaces with dual spaces $V_1^*, V_2^*, \ldots, V_n^*$. For $i = 1, 2, \ldots, n$, $A_i : \prod_{k=1}^n V_k \to 2^{V_i^*}$ are set-valued mappings, $T_i : V_i \to L^2(\Omega)$

are linear continuous and compact operators on V_i, and $G_i : V_i \to R \cup \{+\infty\}$ are proper, convex, and lower semicontinuous functionals. We consider the following system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives. For all $i = 1, 2, \ldots, n$, find $u_i \in V_i$ and $\mu_i \in A_i(u)$ such that

$$
\langle \mu_i, v_i - u_i \rangle_{V_i^* \times V_i} + \int_\Omega j_i^\circ \left(x, \widehat{\mathbf{u}}(x); \widehat{v}_i(x) - \widehat{u}_i(x) \right) dx \\
+ G_i(v_i) - G_i(u_i) \geq 0, \quad \forall v_i \in V_i,
\tag{49}
$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \prod_{k=1}^n V_k$, $\widehat{\mathbf{u}} = (\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n) = (T_1 u_1, T_2 u_2, \ldots, T_n u_n) \in L^2(\Omega; R^n)$, $\widehat{v}_i = T_i v_i$ and $j(x, y) : \Omega \times R^n \to R$ is a function satisfying the following assumption:

(Hj) is as follows.

(1) $j(x, \cdot)$ is locally Lipschitz on R^n for a.e. $x \in \Omega$.

(2) $j(x, y) : \Omega \times R^n \to R$ is a Carathéodory function.

(3) Either $j(x, \cdot)$ or $-j(x, \cdot)$ is regular on R^n for a.e. $x \in \Omega$.

(4) For all $i = 1, 2, \ldots, n$ and a.e. $x \in \Omega$, there exists constant $c_i, d_i \geq 0$ such that

$$
|s| \leq c_i + d_i |y_i|, \quad \forall y \in R^n, s \in \partial_i j(x, y). \tag{50}
$$

Remark 22. The problem (49) we considered in this section includes the problem studied by Panagiotopoulos et al. [19] by using Brouwer's fixed point theorem as a special case where $n = 1$, A_i is single-valued and G_i is an indicator of a convex subset K.

We define a functional J on $L^2(\Omega, R^n)$ as follows:

$$
J(u) = \int_\Omega j(x, u(x)) dx, \quad \forall u \in L^2(\Omega, R^n). \tag{51}
$$

It follows from Theorem 3.47 in [2] that, under the assumption (Hj) on the function j, J defined by (51) is a locally Lipschitz functional on $L^2(\Omega, R^n)$, which satisfies

$$
J_i^\circ(u, v_i) = \int_\Omega j_i^\circ(x, u(x), v_i(x)) dx, \\
\forall u \in L^2(\Omega, R^n), v_i \in L^2(\Omega), \\
\|\eta\|_{L^2(\Omega)} \leq \widetilde{c}_i + \widetilde{d}_i \|u_i\|_{L^2(\Omega)}, \\
\forall u \in L^2(\Omega, R^n), \eta \in \partial_i J(u),
\tag{52}
$$

where $\widetilde{c}_i = \sqrt{2} c_i |\Omega| \geq 0$ and $\widetilde{d}_i = \sqrt{2} d_i \geq 0$.

Now, under the conditions (52), we are in a position to apply our result, Theorem 21, to the problem (49), a system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives. We conclude this section with the following theorem, which gives the existence of solution to the problem (49).

Theorem 23. *For the problem* (49), *a system of generalized variational-hemivariational inequalities involving integrals of Clarke's generalized directional derivatives, one assumes the following.*

(1) *For $i = 1, 2, \ldots, n$, $A_i : \prod_{k=1}^n V_k \to 2^{V_i^*}$ are set-valued mappings satisfying the following.*

(a) *A_i are bounded on $\prod_{k=1}^n V_k$ and pseudomonotone with respect to the ith argument.*

(b) *the operator $A : V = \prod_{k=1}^n V_k \to 2^{V^*}$, which is defined by $A(u) = (A_1(u), \ldots, A_n(u))$, is pseudomonotone on V.*

(c) *For all $u_j \in V_j$, $j \neq i$, there exist an element $w_i \in D(\widehat{\partial} G_i) \subset V_i$ and a constant $\alpha_i > 0$ such that*

$$
\langle u_i^*, u_i - w_i \rangle_{V_i^* \times V_i} \geq \alpha_i \|u_i\|_{V_i}^2, \\
\forall u_i \in V_i, u_i^* \in A_i(u_1, \ldots, u_n).
\tag{53}
$$

(2) *$j(x, y) : \Omega \times R^n \to R$ is a function satisfying the assumption (Hj).*

(3) *For $i = 1, 2, \ldots, n$, $T_i : V_i \to L^2(\Omega)$ are linear continuous and compact operators and $G_i : V_i \to R \cup \{+\infty\}$ are proper, convex, and lower semicontinuous functionals.*

Then the problem (49) *admits at least one solution under the condition*

$$
\widetilde{\beta} > \widetilde{d} \|\widetilde{T}\|^2, \tag{54}
$$

where $\widetilde{\beta} = \min\{\alpha_i\}/n$, $\widetilde{d} = \sqrt{2} \max\{d_i\}$, and $\|\widetilde{T}\|$ is the norm of the operator $\widetilde{T} : \prod_{k=1}^n V_k \to L^2(\Omega; R^n)$ defined by $\widetilde{T}(u_1, \ldots, u_n) = (T_1 u_1, \ldots, T_n u_n)$.

Acknowledgments

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