

Research Article

A Spectral Deferred Correction Method for Fractional Differential Equations

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A spectral deferred correction method is presented for the initial value problems of fractional differential equations (FDEs) with Caputo derivative. This method is constructed based on the residual function and the error equation deduced from Volterra integral equations equivalent to the FDEs. The proposed method allows that one can use a relatively few nodes to obtain the high accuracy numerical solutions of FDEs without the penalty of a huge computational cost due to the nonlocality of Caputo derivative. Finally, preliminary numerical experiments are given to verify the efficiency and accuracy of this method.

1. Introduction

Fractional calculus, almost as old as the familiar integer-order calculus, has recently gained considerable popularity and importance due to its attractive applications in widespread fields of science and engineering (e.g., [1–6]), such as control theory [5], viscoelasticity [7], image processing [8], electromagnetism [9], anomalous diffusion [10, 11], and hydrology [12, 13]. The overwhelming advantage is that fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. These applications greatly highlight distinct superiorities and unsubstitutability of fractional calculus.

Similar to the integer-order differential equations, it is usually difficult to obtain the analytical solution for fractional differential equations (FDEs) (e.g., [14, 15]), and one has to resort to numerical methods. Thus, there has been a growing interest to develop numerical approaches in solving FDEs, and many and varied methods have been considered, for example [15–24]. In 2000, Podlubny [16] suggested a matrix form representation of discrete analogues of various forms of fractional differentiation and fractional integration. Based on the so-called triangular strip matrices, an approach was presented to significant simplification of the numerical solution of fractional integral and differential equations.

Diethelm et al. [17, 18] discussed an Adams-type predictor-corrector method for the numerical solution of FDEs and gave a detailed error analysis for this algorithm, including error bounds under various types of assumptions on the equation and asymptotic expansions for the error. This Adams-type method is convergent with order at least one if the analytical solution $y(t)$ is twice continuously differentiable. In 2007, Lin and Liu [19] developed a kind of linear multistep methods for fractional initial value problems based on Lubich's high-order approximations [20] to fractional derivatives and integrals. And they proved the consistence, convergence, and stability of these methods. In 2006, Kumar and Agrawal [21] utilized the equivalent Volterra integral equation and extended a block-by-block method to some kinds of FDEs. Numerical examples have shown the efficiency and stability of this scheme. However, it is a pity that the error estimate and convergence order analysis of this scheme was neglected. Thus Huang et al. [22] derived error estimate and precise convergence order of the block-by-block method under certain assumptions and tested the order via numerical experiments. It is shown that this block-by-block method is convergent with order at least 3 for any fractional order index $\alpha > 0$. Li [23] derived a Chebyshev wavelet operational matrix of the fractional integration and used it to solve a nonlinear FDEs. In 2011, Scherer et al. [24] discussed finite

difference schemes for the approximation of FDEs based on the Grünwald-Letnikov definition of fractional derivatives. Then, the asymptotic stability, the absolute stability, error representations, and estimates of the proposed explicit and implicit methods were obtained.

As for fractional partial differential equations (FPDEs), maybe the most famous model is the fractional diffusion equation used to describe the anomalous diffusion in porous medium. Further applications and numerical methods for fractional diffusion equation were discussed (e.g., [25–29] and references therein).

It is well known that the key difficulty in numerically solving FDEs or FPDEs is to discretize the nonlocal fractional operators, due to the requirement of using the values of the numerical solution for all the previous times at which the solution is calculated. This makes numerical methods slow and hugely memory demanding when the number of nodes n is large. The problem is so acute that a great deal of effort has been devoted to overcoming it. In [30], Yuste and Quintana-Murillo presented an implicit finite difference method with nonuniform timesteps for discretizing Caputo derivative. This method allows one to build adaptive methods where the size of the timesteps is adjusted to the behavior of the solution. In this paper, a spectral deferred correction method for classical ordinary differential equations [31, 32] is extended and reconstructed to solve FDEs based on accelerating the convergence of lower-order schemes. For the new method, a relatively few Gauss-Legendre-Lobatto points are needed, and the high accuracy numerical solutions of FDEs can be easily obtained without a huge computational cost.

This paper is organized as follows. In Section 2, the equivalent Volterra integral equation is given; then a residual function and an error equation are defined. In Section 3, based on Gauss-Legendre-Lobatto points, a spectral approximation of the residual function is obtained. In Section 4, a fractional Adams method is used as a preconditioner to solve the error equation; then we describe the new spectral deferred correction method for FDEs. In Section 5, preliminary numerical experiments are given to verify the efficiency and accuracy of the proposed method.

2. Residual Function and Error Equation

In this paper, we discuss the numerical solution of the following FDE initial value problem (IVP):

$$\begin{aligned} D_*^\alpha y(t) &= f(t, y(t)), \quad 0 \leq t \leq T, \quad 0 < \alpha \leq 1, \\ y(0) &= y_0, \end{aligned} \quad (1)$$

where D_*^α is fractional Caputo derivative of order α and defined as

$$D_*^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dy(\tau)}{d\tau} d\tau. \quad (2)$$

Assume that the right-hand side function $f(t, y)$ is continuous with respect to two variables t and y . Then, according

to Lemma 1 of [33], IVP (1) is equivalent to the following Volterra integral equation of the second kind:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (3)$$

Obviously, if $0 < \alpha < 1$, the kernel in the integral of (3) is singular, and if $\alpha = 1$, then (3) becomes the classical Picard integral equation formulation.

Suppose now that an approximate solution $y^0(t)$ to (3) has been obtained by a low-order method. A measure of the quality of the approximation is given by the residual function

$$\begin{aligned} \varepsilon(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^t (t-\tau)^{\alpha-1} f(\tau, y^0(\tau)) d\tau - y^0(t). \end{aligned} \quad (4)$$

And the error can be defined by

$$\delta(t) = y(t) - y^0(t). \quad (5)$$

However, in (5), $y(t)$ is the exact solution of IVP (1), and we often cannot know it beforehand. Next, a relation between the residual function and the error will be deduced.

Substituting (5) into (3), (3) can be rewritten as

$$\begin{aligned} &y^0(t) + \delta(t) \\ &= y_0 + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^t (t-\tau)^{\alpha-1} f(\tau, y^0(\tau) + \delta(\tau)) d\tau; \end{aligned} \quad (6)$$

then after some algebraic manipulation, it becomes

$$\begin{aligned} &\delta(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ &\times [f(\tau, y^0(\tau) + \delta(\tau)) - f(\tau, y^0(\tau))] d\tau + \varepsilon(t) \\ &\triangleq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} G(\tau, y^0(\tau), \delta(\tau)) d\tau + \varepsilon(t). \end{aligned} \quad (7)$$

In (7), once the residual function is fixed, then a low-order method, such as fractional Adams method presented in [18], will be used to solve (7). In the following, we specify how the values $\varepsilon(t_i)$ are actually computed. For this, some stable and high-order accurate methods for interpolation and integration are required.

3. Spectral Approximation of the Residual Function

In this section, a spectral method is designed for obtaining high-order numerical approximation of the residual function.

Suppose that $s_0 < s_1 < \dots < s_n$ are the $n + 1$ Gauss-Legendre-Lobatto nodes on $[-1, 1]$, where $s_0 = -1$ and $s_n = 1$. Noticing that the interval of IVP (1) is $[0, T]$, we can obtain $n + 1$ Gauss nodes on $[0, T]$ through the following linear transformation:

$$t_k = \frac{T}{2}s_k + \frac{T}{2}, \quad k = 0, 1, \dots, n. \tag{8}$$

Now, a low-order method is used to solve IVP (1) and get the initial approximation $y^0(t)$. Here, we take the following fractional explicit Euler method; namely, for $k = 0, 1, \dots, n - 1$,

$$y_{k+1} = y_0 + \sum_{j=0}^k b_{j,k+1} f(t_j, y_j), \tag{9}$$

where y_{k+1} is the approximations of solution function $y(t)$ at nodes t_{k+1} ($k = 0, 1, \dots, n - 1$), the weight coefficients $b_{j,k+1}$ are computed by

$$b_{j,k+1} = \frac{1}{\Gamma(\alpha + 1)} \left((t_{k+1} - t_j)^\alpha - (t_{k+1} - t_{j+1})^\alpha \right), \tag{10}$$

$$j = 0, 1, \dots, k.$$

After obtaining the numerical solutions y_1, y_2, \dots, y_n , using the nodes t_0, t_1, \dots, t_n as the nodes of Lagrange interpolation for the initial approximation $y^{[0]}(t)$ and the right-hand side function $f(t, y^{[0]}(t))$, we have

$$y^0(t) = \sum_{k=0}^n y_k L_k(t), \tag{11}$$

$$f(t, y^0(t)) = \sum_{k=0}^n f_k L_k(t), \tag{12}$$

where $L_k(t)$ is the Lagrange interpolation polynomial of order n , given as

$$L_k(t) = \prod_{i=0, i \neq k}^n \frac{t - t_i}{t_k - t_i}, \tag{13}$$

$$f_k = f(t_k, y^0(t_k)).$$

In specific computations, the fractional Riemann-Liouville integral of Lagrange interpolation polynomial is needed. To this end, $L_k(t)$ should be transformed into following expressions:

$$L_k(t) = \sum_{j=0}^n c_{j,k} t^j, \quad k = 0, 1, \dots, n, \tag{14}$$

where the coefficients $c_{j,k}$ can be computed by its matrix form, namely,

$$\begin{pmatrix} t_0^0 & t_0^1 & \dots & t_0^n \\ t_1^0 & t_1^1 & \dots & t_1^n \\ \dots & \dots & \dots & \dots \\ t_n^0 & t_n^1 & \dots & t_n^n \end{pmatrix} \begin{pmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,n} \\ c_{1,0} & c_{1,1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots \\ c_{n,0} & c_{n,1} & \dots & c_{n,n} \end{pmatrix} = I_{n+1}, \tag{15}$$

here I_{n+1} is the identity matrix of order $n + 1$.

By (11), the fractional integral of $L_k(t)$ is calculated as

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} L_k(\tau) d\tau \\ &= \sum_{j=0}^n c_{j,k} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \tau^j d\tau \\ &= \sum_{j=0}^n c_{j,k} \frac{\Gamma(j + 1)}{\Gamma(j + 1 + \alpha)} t^{j+\alpha}. \end{aligned} \tag{16}$$

Substituting (11) and (12) into (4) and using (16), the residual function (4) is approximated as follows:

$$\begin{aligned} \varepsilon(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t - \tau)^{\alpha-1} f(\tau, y^0(\tau)) d\tau - y^0(t) \\ &= y_0 + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t - \tau)^{\alpha-1} \sum_{k=0}^n f_k L_k(\tau) d\tau - y^0(t) \\ &= y_0 + \sum_{k=0}^n f_k \sum_{j=0}^n c_{j,k} \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t - \tau)^{\alpha-1} \tau^j d\tau - y^0(t) \\ &= y_0 + \sum_{k=0}^n f_k \sum_{j=0}^n c_{j,k} \frac{\Gamma(j + 1)}{\Gamma(j + 1 + \alpha)} \\ &\quad \times t^{j+\alpha} - y^0(t). \end{aligned} \tag{17}$$

4. Spectral Deferred Correction Method

Once the residual function is computed by (17), the error equation (7) can be numerically solved by the fractional Adams method [18]; that is, for $k = 0, 1, \dots, n - 1$

$$\begin{aligned} \delta_{k+1}^p &= \varepsilon(t_{k+1}) \\ &\quad + \sum_{j=0}^k b_{j,k+1} G(t_j, y^0(t_j), \delta_j), \\ \delta_{k+1} &= \varepsilon(t_{k+1}) \\ &\quad + \left(\sum_{j=0}^k a_{j,k+1} G(t_j, y^0(t_j), \delta_j) \right. \\ &\quad \left. + a_{k+1,k+1} G(t_{k+1}, y^0(t_{k+1}), \delta_{k+1}^p) \right), \end{aligned} \tag{18}$$

where δ_{k+1}^p is the so-called predictor, and $a_{j,k+1}$ are defined as

$$\begin{aligned} a_{0,k+1} &= \omega_{0,k+1}, \\ a_{k+1,k+1} &= \gamma_{k+1,k+1}, \\ a_{j,k+1} &= \gamma_{j,k+1} + \omega_{j,k+1}, \\ j &= 1, 2, \dots, k, \end{aligned} \tag{19}$$

with

$$\begin{aligned} \gamma_{j+1,k+1} &= \frac{1}{t_{j+1} - t_j} \\ &\times \left(\frac{(t_{k+1} - t_j)^{\alpha+1}}{\Gamma(\alpha + 2)} - \frac{(t_{k+1} - t_{j+1})^{\alpha+1}}{\Gamma(\alpha + 2)} \right. \\ &\quad \left. + \frac{(t_{k+1} - t_{j+1})^\alpha (t_j - t_{j+1})}{\Gamma(\alpha + 1)} \right), \\ \omega_{j,k+1} &= \frac{1}{t_{j+1} - t_j} \\ &\times \left(\frac{(t_{k+1} - t_{j+1})^{\alpha+1}}{\Gamma(\alpha + 2)} - \frac{(t_{k+1} - t_j)^{\alpha+1}}{\Gamma(\alpha + 2)} \right. \\ &\quad \left. + \frac{(t_{k+1} - t_j)^\alpha (t_{j+1} - t_j)}{\Gamma(\alpha + 1)} \right). \end{aligned} \tag{20}$$

Let the initial approximation vector $\mathbf{Y}^{[0]} = (y_1, y_2, \dots, y_n)^T$, the error vector $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_n)^T$; then the initial approximation solution is updated with

$$\mathbf{Y}^{[1]} = \mathbf{Y}^{[0]} + \boldsymbol{\delta}. \tag{21}$$

Based on the new solution $\mathbf{Y}^{[1]}$, we can continue to compute the new residual function and the new error. And this procedure can be repeated until the high-accuracy solution is obtained. The pseudocode for spectral deferred correction method is given by the following.

Step 1 (initialization). Set a small parameter $etol > 0$. Use fractional explicit Euler method (9) to compute an initial solution $y_k^{[0]} \approx y(t_k)$ at nodes t_1, t_2, \dots, t_n on the interval $[0, T]$.

Step 2. Based on the initial approximation $y_k^{[0]}$, compute the approximate residual function ε_k ($k = 1, 2, \dots, n$) by using (17).

Step 3. Use fractional Adams method (18) to solve the error equation and obtain the approximation δ_k ($k = 1, 2, \dots, n$).

Step 4. Update the approximate solution $y_k^{[1]} = y_k^{[0]} + \delta_k$ ($k = 1, 2, \dots, n$). If $\|\boldsymbol{\delta}\|_\infty < etol$, then stop, and the approximation $y_k^{[1]}$ is the solution we get finally. If $\|\boldsymbol{\delta}\|_\infty > etol$, let $y_k^{[0]} = y_k^{[1]}$, and go to Step 2.

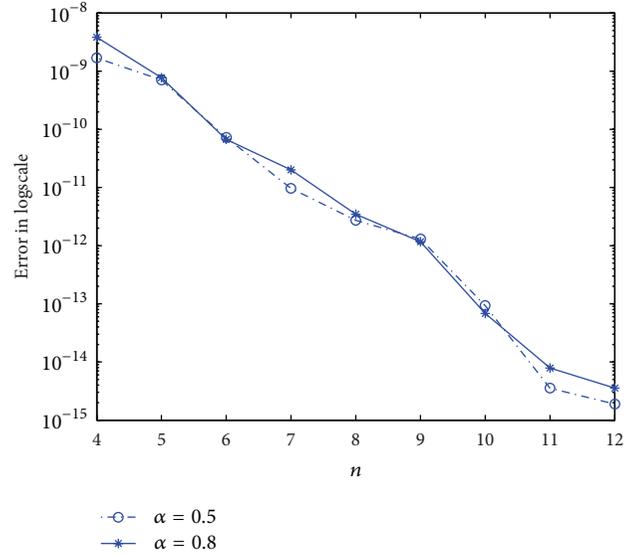


FIGURE 1: The errors for different n and α in Example 1.

It can be checked that each correction procedure in this algorithm can improve the accuracy of the method, as long as such improvement has not gone beyond the degree of the underlying interpolating polynomial and the quadrature rules.

5. Numerical Examples

In this section, we verify the performance and high accuracy of the proposed method by following two examples. In these computations, the small parameter $etol = 1e - 14$. All codes are written in Matlab 2010b and run on a personal computer with Intel(R) Core(TM)2 Duo CPU P7350 processor 2.00 GHz, 2 GB memory.

Example 1. Consider the following FDEs:

$$D_*^\alpha y(t) = \frac{\Gamma(5 + \alpha)}{24} t^4 + t^{8+2\alpha} - y^2(t) \tag{22}$$

with initial condition $y(0) = 0$. The exact solution of this problem is given as

$$y(t) = t^{4+\alpha}. \tag{23}$$

In Table 1, we list the number of iterations denoted by “Iter.,” the CPU time in second “Time[s]” and the errors in the maximum norm for different nodes n and fractional order index α . It demonstrates that when the new spectral deferred correction method is used to solve Example 1, only a relatively few nodes are used, and the high-order accuracy numerical solution is obtained with small computation complexity and computing time. To illustrate this point, we use fractional block-by-block method [21] for comparison. To achieve the accuracy of 10^{-9} , fractional block-by-block method needs about 320 nodes and 20.031599 seconds. Figure 1 shows that the error fast converge to machine precision, and it almost shows an exponential decay with increasing n .

TABLE 1: The performance of spectral deferred correction method for solving Example 1.

n	$\alpha = 0.5$			$\alpha = 0.8$		
	Iter.	Time (s)	Error	Iter.	Time (s)	Error
4	11	0.120460	$1.684548e - 09$	9	0.099106	$3.829423e - 09$
6	11	0.144606	$7.243051e - 11$	9	0.120215	$6.735190e - 11$
8	11	0.177994	$2.727929e - 12$	10	0.148923	$3.474332e - 12$
10	11	0.219745	$9.281464e - 14$	10	0.181589	$6.827872e - 14$

TABLE 2: The performance of spectral deferred correction method for solving Example 2.

n	$\alpha = 0.5$			$\alpha = 0.8$		
	Iter.	Time (s)	Error	Iter.	Time (s)	Error
5	16	0.383436	$2.017025e - 3$	13	0.208751	$2.177103e - 3$
7	15	0.443425	$4.038208e - 6$	13	0.270258	$1.418501e - 6$
9	14	0.496825	$1.748804e - 7$	12	0.331811	$3.051062e - 7$
11	15	0.655070	$3.506529e - 8$	11	0.399406	$5.716511e - 8$

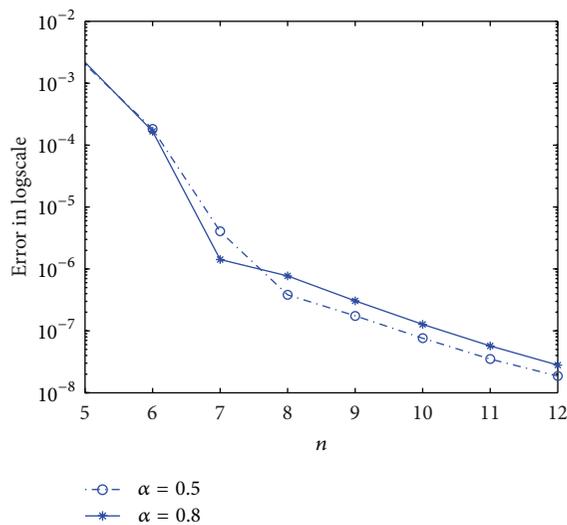


FIGURE 2: The errors for different n and α in Example 2.

Example 2. Let one consider the following equations:

$$D_*^\alpha y(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3 \frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4} \Gamma(\alpha+1) + \left(\frac{3}{2} t^{\alpha/2} - t^4\right)^3 - [y(t)]^{3/2} \quad (24)$$

subject to the initial condition $y(0) = 0$. The exact solution is

$$y(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4} t^\alpha. \quad (25)$$

For numerical experiments of Example 2, compared with the Example 1, the similar numerical results can be obtained. However, from Table 2 and Figure 2, one should notice that the error is larger than that of Example 1's, and its convergence speed is also relatively slow. We think the reason is that the exact solution in Example 1 is smoother than the solution in Example 2, because it is well known that the results of any

numerical method often depend on the smoothness of a given problem.

6. Conclusion

In this paper, a spectral deferred correction method is presented for fractional differential equations with initial value condition. Numerical experiments show it can obtain the high-order accuracy numerical solutions of FDEs without a huge computational cost caused by the nonlocality of fractional derivative. However, more efforts are needed to further the study such as detailed theoretical analysis and improve the algorithm for long-time simulation.

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