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Research Article

Analytical Study of Nonlinear Fractional-Order Integrodifferential Equation: Revisit Volterra's Population Model

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This paper suggests two component homotopy method to solve nonlinear fractional integrodifferential equations, namely, Volterra's population model. Padé approximation was effectively used in this method to capture the essential behavior of solutions for the mathematical model of accumulated effect of toxins on a population living in a closed system. The behavior of the solutions and the effects of different values of fractional-order α are indicated graphically. The study outlines significant features of this method as well as sheds some light on advantages of the method over the other. The results show that this method is very efficient, convenient, and can be adapted to fit a larger class of problems.

1. Introduction

Ecology is the study of different species in relation to their surroundings, competition for resources within and among the species, and predator-prey [1] relations among them. At times the surroundings may be infected by metabolic actions of the crowd [2, 3]. In all these situations, since time rates of changes of population sizes are concerned, it is natural that the mathematical modeling be given by differential equations or integrodifferential equations. Integrodifferential equations are usually difficult to solve especially analytically, so an effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Also, fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of

memory and hereditary properties of various materials and processes due to the existence of a "memory" term in the model. This memory term insures the history and its impact to the present and future.

Problems of this type have gained escalating importance in recent years and many interesting outcomes have been accumulated. In the living thing population, the accumulation of metabolic products may cause inconvenience to the whole population and may ultimately result in a fall of the birth rate while the death rate is increased. We assumed that the total toxic effect on birth and death rates be expressed by the following nonlinear fractional-order integrodifferential equation [4]:

$$\frac{d^{\alpha}p\left(\bar{t}\right)}{d\bar{t}^{\alpha}} = p\left(\bar{t}\right)\left(a - bp\left(\bar{t}\right) - c\int_{0}^{\bar{t}} p(x)dx\right), \qquad p(0) = k, 0 < \alpha \leq 1, \tag{1.1}$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term, k is the initial population, $p = p(\bar{t})$ denotes the population at time $\bar{t}$, and the term $p(\bar{t})\int_0^{\bar{t}} p(x)dx$ represents the effect of toxin accumulation on the species. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor α. We apply the scale time and population by introducing the nondimensional variables $t = c\bar{t}/b$, $v = bp/a$ to obtain the following nondimensional problem:

$$\kappa\frac{d^{\alpha}v(t)}{dt^{\alpha}} - v(t) + v^{2}(t) + v(t)\int_{0}^{t} v(x)dx = 0, \qquad v(0) = v_{0}, 0 < \alpha \leq 1, \tag{1.2}$$

where $v(t)$ is the scaled population of identical individuals at time t, and $\kappa = c/ab$ is a prescribed nondimensional parameter. The only equilibrium solution of (1.2) is the trivial solution $v(t) = 0$.

Several numerical and analytical methods for approximating the ordinary nonlinear Volterra's integrodifferential equation are known (see [5–12]).

In the present paper, we introduce a two component homotopy perturbation method [13–15] for solving nonlinear fractional-order integrodifferential population model. It is interesting to note that the method depends only on two components of the homotopy series. The method is an improvement of the classical homotopy perturbation method [16].

2. Approximations of the Nonlinear Fractional-Order Integrodifferential Equation

To obtain the solution of (1.2) by two components HPM, we construct the following homotopy:

$$(1-\varepsilon)\left(D_{t}^{\alpha}V(t) - v_{0}(t)\right) + \varepsilon\left(D_{t}^{\alpha}V(t) - \frac{1}{\kappa}\left(V(t) - V^{2}(t) - V(t)\int_{0}^{t} V(x)dx\right)\right) = 0. \tag{2.1}$$

Applying the inverse operator, J_t^α to both sides of (2.1), we obtain

$$V(t) = V(0) + J_t^\alpha v_0(t) - \varepsilon J_t^\alpha \left(v_0(t) - \frac{1}{\kappa} \left(V(t) - V^2(t) - V(t) \int_0^t V(t)dt \right) \right). \tag{2.2}$$

The solution of (2.2) to have the following form

$$V(t) = V_0(t) + \varepsilon V_1(t). \tag{2.3}$$

Substituting (2.3) in (2.2) and equating the coefficients of like powers of ε, we get

$$V_0(t) = V(0) + J_t^\alpha v_0(t), \tag{2.4}$$

$$V_1(t) = J_t^\alpha \left(-v_0(t) + \frac{1}{\kappa} \left(V_0(t) - V_0^2(t) - V_0(t) \int_0^t V_0(x)dx \right) \right). \tag{2.5}$$

Assuming $v_0(t) = \sum_{n=0}^{10} a_n t^n$, $V(0) = v(0)$ then (2.4) becomes

$$V_0(t) = \frac{a_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{a_1 t^{1+\alpha}}{\Gamma(\alpha+2)} + \frac{2a_2 t^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{6a_3 t^{3+\alpha}}{\Gamma(\alpha+4)} + \cdots + \frac{3629800 a_{10} t^{10+\alpha}}{\Gamma(\alpha+11)}. \tag{2.6}$$

Utilizing (2.6) in (2.5), we obtain

$$V_1(t) = \frac{4a_0 t}{5\kappa} + \frac{4a_1 t^{2+\alpha}}{5\kappa\Gamma(\alpha+3)} - \frac{2a_2 t^{2+\alpha}}{\Gamma(\alpha+3)} - \frac{3a_0 t^{2+\alpha}}{10\kappa\Gamma(\alpha+3)} + \cdots. \tag{2.7}$$

Vanishing $V_1(t)$ with $\alpha = 1$ and comparing the coefficients of t, t^2, t^3, t^4,..., we obtain the coefficients as follows:

$$\begin{aligned}
t^1 &: -a_0 + \frac{9}{100\kappa} + \frac{4a_0}{5\kappa} = 0, \\
t^2 &: -\frac{1}{200\kappa} - \frac{4a_1}{2} = 0, \\
t^3 &: -\frac{a_2}{3} - \frac{a_0}{20\kappa} - \frac{a_0^2}{3\kappa} + \frac{2a_1}{15\kappa} = 0, \\
t^4 &: -\frac{a_3}{4} - \frac{a_0^2}{8\kappa} - \frac{a_1}{60\kappa} + \frac{a_0 a_1}{4\kappa} + \frac{a_2}{15\kappa} = 0, \\
&\vdots \qquad \vdots \qquad \vdots
\end{aligned} \tag{2.8}$$

After solving the system of (2.8), we obtain the solutions of (1.2) as follows:

$$v(t) = \frac{1}{10} + \frac{9t^{\alpha}}{20(5\kappa-4)\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{100\kappa\Gamma(\alpha+2)} + \frac{\left(-128+455\kappa-875\kappa^2\right)t^{\alpha+2}}{1000\kappa^2(5\kappa-4)^2\Gamma(\alpha+3)} + \frac{\left(-1024+3200\kappa-13325\kappa^2+1000\kappa^3\right)t^{\alpha+3}}{10000\kappa^3(5\kappa-4)^2\Gamma(\alpha+4)} + \cdots. \tag{2.9}$$

When $\alpha = 1$, we recovered [5]. Our objective is to find the behavior of the solutions for different values of the parameter α. In order to find the behavior of the solution for fractional values of α, we use Padé approximants which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $v(t)$. It is well known that Padé approximants will converge on the entire real axis if the solution of the problem is free of singularities on the real axis. Here, we consider the following two special cases.

Case I. We inspect the case $\alpha = 1/2$, we reproduce the approximate solution obtained in (2.9) given by the Taylor expansion of $V(t)$ at $t = 0$ which gives

$$V(t) = \frac{1}{10} + \frac{9\sqrt{t}}{10(5\kappa-4)\sqrt{\pi}} - \frac{t^{3/2}}{75\kappa\sqrt{\pi}} + \frac{\left(-128+455\kappa-875\kappa^2\right)t^{5/2}}{1875\kappa^2(5\kappa-4)^2\sqrt{\pi}} + \cdots. \tag{2.10}$$

For simplicity, let $t^{1/2} = z$, then

$$V(z) = \frac{1}{10} + \frac{9z}{10(5\kappa-4)\sqrt{\pi}} - \frac{z^3}{75\kappa\sqrt{\pi}} + \frac{\left(-128+455\kappa-875\kappa^2\right)z^5}{1875\kappa^2(5\kappa-4)^2\sqrt{\pi}} + \cdots + \frac{-8\left(-68719476736+2601139568640\kappa+\cdots+55821093750000\kappa^{10}\right)z^{21}}{671353055419921875\kappa^2(5\kappa-4)^6\sqrt{\pi}}. \tag{2.11}$$

Calculating the [10/10] Padé approximants and recalling that $z = t^{1/2}$, we get

$$V_{[10/10]}(t) = \frac{0.1 + m_1t^{1/2} + m_2t + m_2t^{3/2}\cdots + m_rt^5}{n_0 + n_1t + n_2t^2 + n_3t^3\cdots + n_st^5}, \tag{2.12}$$

where m_i and n_i, $i = 0, 1, 2$ are constants.

Case II. Now, we observe the case $\alpha = 3/4$, we reproduce the approximate solution obtained in (2.9) given by the Taylor expansion of $V(t)$ at $t = 0$ which gives

$$V(t) = \frac{1}{10} + \frac{9t^{3/4}}{20(5\kappa-4)\Gamma(7/4)} - \frac{t^{7/4}}{100\kappa\Gamma(11/4)} + \frac{\left(-128+455\kappa-875\kappa^2\right)t^{11/4}}{1000\kappa^2(5\kappa-4)^2\Gamma(15/4)} + \cdots. \tag{2.13}$$

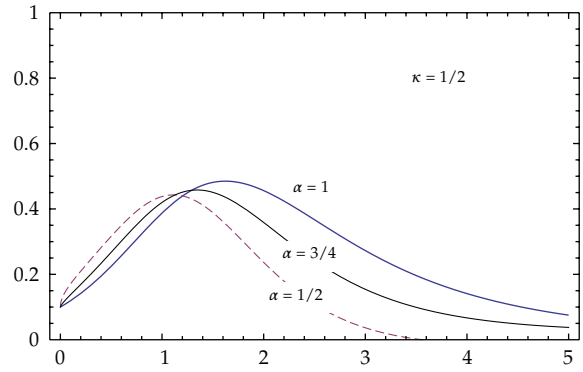


Figure 1: Approximate curves of fractional-order integrodifferential equation t versus v.

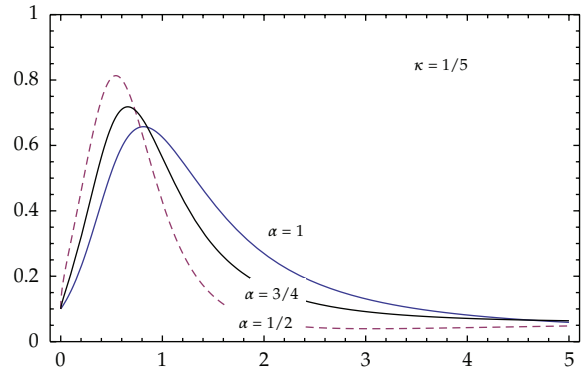


Figure 2: Approximate curves of fractional-order integrodifferential equation t versus v.

For simplicity, let $t^{1/4} = z$, then

$$V(z) = \frac{1}{10} + \frac{9z^3}{20(5\kappa-4)\Gamma(7/4)} - \frac{z^7}{100\kappa\Gamma(11/4)} + \frac{\left(-128+455\kappa-875\kappa^2\right)z^{11}}{1000\kappa^2(5\kappa-4)^2\Gamma(15/4)} + \cdots + \frac{-8\left(-68719476736+2601139568640\kappa+\cdots+55821093750000\kappa^{10}\right)z^{43}}{12500000000\kappa^{10}(5\kappa-4)^6\Gamma(47/4)}. \tag{2.14}$$

Calculating the [21/21] Padé approximants and recalling that $z = t^{1/4}$, we get

$$v_{[21/21]}(t) = \frac{0.1 + M_1 t^{1/4} + M_2 t^{3/4} + M_2 t^{5/4} \cdots + M_r t^{43/4}}{N_0 + N_1 t + N_2 t^2 + N_3 t^3 \cdots + N_s t^{14}}. \tag{2.15}$$

Figure 1 shows the numerical solutions generated by the two component HPM and Padé approximation for various values of α and $\kappa = 1/2$. It should be noted that solution falls slowly when value of α decreases. One trend that can be observed in Figures 2, 3, 4, and 5 shows the two component HPM combining with Padé approximants shows the quick rise along the logistic curve followed by the slow exponential decay later than

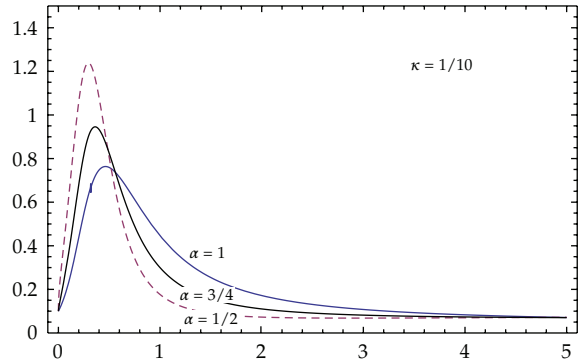


Figure 3: Approximate curves of fractional-order integrodifferential equation t versus v.

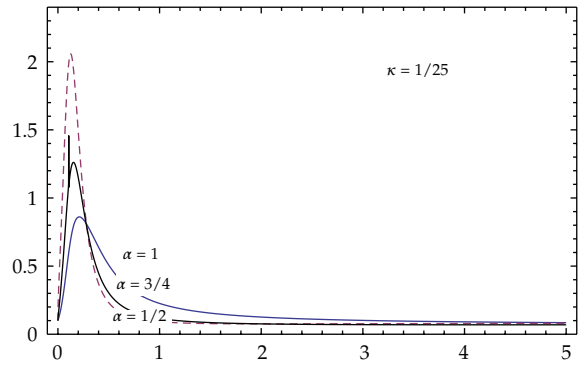


Figure 4: Approximate curves of fractional-order integrodifferential equation t versus v.

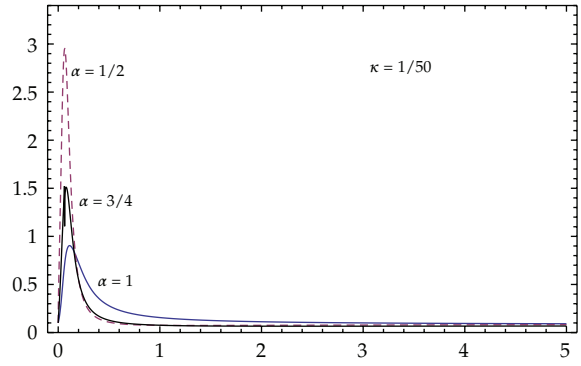


Figure 5: Approximate curves of fractional-order integrodifferential equation t versus v.

reaching the maximum point when κ increases, the amplitude of $v(t)$ decreases whereas the exponential decay increases, for $\kappa = 1/50, 1/25, 1/10, 1/5$. It should be noted that the solution quickly rises when value of α increasing. These figures give the two component HPM-Padé approximations for some values of α. From these figures, it is shown that the maximum of $v(t)$ deceases as κ increases. Figures 1–5 show the impact of fractional time parameter and κ on the behavior of population.

Table 1: A comparison of two component HPM-Padé with the exact values of $v_{\max}$ for different fractional-order.

κ	Exact $v_{\max}$	Reference [5] $\alpha = 1$	Present $\alpha = 3/4$	Present $\alpha = 1/2$
0.02	0.92342717	0.922942037	1.49200000	2.92000000
0.04	0.87371998	0.873725344	1.24600000	2.01600000
0.1	0.76974149	0.765113089	0.94150000	1.23200000
0.2	0.65905038	0.659050432	0.71690000	0.81130000
0.5	0.48519030	0.485190290	0.45740000	0.44320000

Table 2: A comparison of t_{Present} with the exact values and fractional values of critical t.

κ	t_{Exact} $\alpha = 1$	Reference [5] t $\alpha = 1$	t_{Present} $\alpha = 3/4$	t_{Present} $\alpha = 1/2$
0.02	0.11184543	0.11184543	0.07810000	0.06291000
0.04	0.21024644	0.21024644	0.15400000	0.12370000
0.1	0.46447670	0.46447670	0.35150000	0.29070000
0.2	0.81685810	0.81685810	0.67050000	0.54900000
0.5	1.62671100	1.62594670	1.33900000	1.12600000

3. Conclusions

In this paper, two component homotopy-Padé solutions have been presented for the fractional-order integrodifferential equation. We have calculated the solutions for two different cases $\alpha = 3/4, 1/2$. The solutions of the population problem will be helpful in past/future predictions due to memory term. This version of HPM is independent of solving functional-differential equations. Also, the procedure can be applied for the system of fractional-order integrodifferential equations. The obtained outcome indicates that this method is powerful and has an important effect for solving the nonlinear fractional-order equations. Tables 1 and 2 represents the numerical comparison of ordinary and fractional order results for maximum population and critical time. Finally, the solutions of the problem for $\alpha = 1$ have been reported in [5].

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