Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2012, Article ID 780619, 11 pages
doi:10.1155/2012/780619

Research Article
Fractional Order Difference Equations

J. Jagan Mohan and G. V. S. R. Deekshitulu

Fluid Dynamics Division, School of Advanced Sciences, VIT University, Tamil Nadu, Vellore 632014, India

Correspondence should be addressed to J. Jagan Mohan, j.jaganmohan@hotmail.com

Received 11 May 2012; Accepted 17 October 2012

Academic Editor: Shaher Momani

A difference equation is a relation between the differences of a function at one or more general values of the independent variable. These equations usually describe the evolution of certain phenomena over the course of time. The present paper deals with the existence and uniqueness of solutions of fractional difference equations.

1. Introduction

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent.

Diaz and Osler [1] defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the nth difference, to be any real or complex number. Later, Hirota [2] defined the fractional order difference operator ∇^{α} where α is any real number, using Taylor's series. Nagai [3] adopted another definition for fractional order difference operator by modifying Hirota's [2] definition. Recently, Deekshitulu and Mohan [4] modified the definition of Nagai [3] for $0 < \alpha < 1$ in such a way that the expression for ∇^{α} does not involve any difference operator.

The study of theory of fractional differential equations was initiated and existence and uniqueness of solutions for different types of fractional differential equations have been established recently [5]. Much of literature is not available on fractional integrodifferential equations also, though theory of integrodifferential equations [6] has been almost all developed parallel to theory of differential equations. Very little progress has been made to develop the theory of fractional order difference equations.

The main aim of this paper is to establish theorems on existence and uniqueness of solutions of various classes of fractional order difference equations. Further we define autonomous and nonautonomous fractional order difference equations and find their solutions.

2. Preliminaries

Throughout the present paper, we use the following notations: $\mathbb{N}$ be the set of natural numbers including zero. $\mathbb{N}_a^+ = \{a, a+1, a+2, \ldots\}$ for $a \in \mathbb{Z}$. Let $u(n) : \mathbb{N}_0^+ \to \mathbb{R}$. Then for all $n_1, n_2 \in \mathbb{N}_0^+$ and $n_1 > n_2$, $\sum_{j=n_1}^{n_2} u(j) = 0$, and $\prod_{j=n_1}^{n_2} u(j) = 1$, that is, empty sums and products are taken to be 0 and 1, respectively. If n and $n+1$ are in $\mathbb{N}_0^+$, then for this function $u(n)$, the backward difference operator ∇ is defined as $\nabla u(n) = u(n) - u(n-1)$. Now we introduce some basic definitions and results concerning nabla discrete fractional calculus.

Definition 2.1. The extended binomial coefficient $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ is defined by

$$\binom{a}{n} = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0, \\ 1 & n = 0, \\ 0 & n < 0. \end{cases} \tag{2.1}$$

Definition 2.2 (see [7]). For any complex numbers α and β, let $(\alpha)_\beta$ be defined as follows:

$$(\alpha)_\beta = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha \text{ and } \alpha+\beta \text{ are neither zero nor negative integers}, \\ 1 & \text{when } \alpha = \beta = 0, \\ 0 & \text{when } \alpha = 0,\ \beta \text{ is neither zero nor negative integer}, \\ \text{undefined} & \text{otherwise}. \end{cases} \tag{2.2}$$

Remark 2.3. For any complex numbers α and β, when α, β and $\alpha+\beta$ are neither zero nor negative integers,

$$(\alpha+\beta)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_{n-k} (\beta)_k \tag{2.3}$$

for any positive integer n.

In 2003, Nagai [3] gave the following definition for fractional order difference operator.

Definition 2.4. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator ∇ of order α, with step length ε, is defined as

$$\nabla^{\alpha} u(n) = \begin{cases} \nabla^{\alpha-m}[\nabla^{m} u(n)] = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^{j} \nabla^{m} u(n-j) & \alpha > 0, \\ u(n) & \alpha = 0, \\ \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^{j} u(n-j) & \alpha < 0. \end{cases} \tag{2.4}$$

The above definition of $\nabla^{\alpha} u(n)$ given by Nagai [3] contains ∇ operator and the term $(-1)^{j}$ inside the summation index and hence it becomes difficult to study the properties of solution. To avoid this, Deekshitulu and Mohan [4] gave the following definition, for $\varepsilon = m = 1$. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$.

Definition 2.5. The fractional sum operator of order α is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j). \tag{2.5}$$

And the fractional order difference operator of order α is defined as

$$\nabla^{\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0). \tag{2.6}$$

Throughout this paper we assume that $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$ unless until specified.

Remark 2.6. Let $u(n), v(n) : \mathbb{N}_0^{+} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha, \beta,\ \alpha + \beta < 1$ and c, d are constants. Then

(1) $\nabla^{\alpha} \nabla^{\beta} u(n) = \nabla^{\alpha+\beta} u(n)$.

(2) $\nabla^{\alpha}[c u(n) + d v(n)] = c \nabla^{\alpha} u(n) + d \nabla^{\alpha} v(n)$.

(3) $\nabla^{-\alpha} \nabla^{\alpha} u(n) = u(n) - u(0)$.

(4) $\nabla^{\alpha} \nabla^{-\alpha} u(n) = u(n)$.

(5) $\nabla^{\alpha} u(0) = 0$ and $\nabla^{\alpha} u(1) = u(1) - u(0) = \nabla u(1)$.

3. Existence and Uniqueness of Solutions of Fractional Order Difference Equations

In the present section, we establish theorems on existence and uniqueness of solutions for various classes of fractional order difference equations.

Definition 3.1. Let $f(n,r)$ be any function defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$. Then a nonlinear difference equation of order α together with an initial condition is of the form

$$\nabla^\alpha u(n+1) = f(n,u(n)), \quad u(0) = u_0. \tag{3.1}$$

The existence of solutions to difference equations is trivial as the solutions are expressed as recurrence relations involving the values of the unknown function at the previous arguments.

Now we consider (2.5) and replace $u(n)$ by $\nabla^\alpha u(n)$, we have

$$\begin{aligned}
\nabla^{-\alpha}[\nabla^\alpha u(n)] &= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^\alpha u(j)], \\
\text{or } u(n) - u(0) &= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^\alpha u(j)], \\
\text{or } u(n) &= u_0 + \sum_{j=0}^{n-1} \binom{n-j+\alpha-2}{n-j-1} [\nabla^\alpha u(j+1)], \\
\text{or } u(n) &= u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j) f(j,u(j)),
\end{aligned} \tag{3.2}$$

where $B(n,\alpha;j) = \binom{n-j+\alpha-1}{n-j}$ for $0 \leq j \leq n$. The above recurrence relation shows the existence of solution of (3.1).

Example 3.2. If $f(n,u(n)) = a(n)u(n) + b(n)$ where $a(n)$, $b(n) : \mathbb{N}_0^+ \to \mathbb{R}$ are any two nonnegative functions then

$$u(n) = u_0 \prod_{j=0}^{n-1} [1 + B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \prod_{k=j+1}^{n-1} [1 + B(n-1,\alpha;j)a(k)]. \tag{3.3}$$

Recently, the authors have established the following fractional order discrete Gronwall-Bellman type inequality [8].

Theorem 3.3. *Let $u(n)$, $a(n)$, and $b(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$. If, for $0 < \alpha < 1$,*

$$u(n) \leq u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j) [a(j)u(j) + b(j)] \tag{3.4}$$

then

$$u(n) \leq u_0 \prod_{j=0}^{n-1} [1 + B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \prod_{k=j+1}^{n-1} [1 + B(n-1,\alpha;j)a(k)] \tag{3.5}$$

for $n \in \mathbb{N}_0^+$.

Definition 3.4. Let $f(n,r)$ and $g(n,m,r)$ be any functions defined for n, $m \in \mathbb{N}_0^+$, $m \le n$ and $0 \le r < \infty$. Then a nonlinear difference equation of Volterra type of order α together with an initial condition is of the form

$$\nabla^\alpha u(n+1) = f(n,u(n)) + \sum_{m=0}^{n-1} g(n,m,u(m)), \quad u(0) = u_0. \tag{3.6}$$

From (3.2) and (3.6), we have

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[f(j,v(j)) + \sum_{m=0}^{j-1} g(j,m,v(m))\right]. \tag{3.7}$$

The above recurrence relation shows the existence of solution of (3.6). Now we establish the uniqueness of solutions for the fractional order difference equations (3.1) and (3.3).

Lemma 3.5. *For $n \in \mathbb{N}_0^+$,*

$$\sum_{j=0}^{n} B(n,\alpha;j) = \binom{n+\alpha}{n}. \tag{3.8}$$

Proof. Consider

$$\begin{aligned}
\sum_{j=0}^{n} B(n,\alpha;j) &= \sum_{j=0}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(\alpha)\Gamma(n-j+1)} \\
&= \frac{1}{\Gamma(n+1)} \sum_{j=0}^{n} \binom{n}{j} (1)_j (\alpha)_{n-j} \quad (\text{using Definition } 2.2) \\
&= \frac{1}{\Gamma(n+1)} (1+\alpha)_n \quad (\text{using Remark } 2.3) \\
&= \frac{1}{\Gamma(n+1)} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} = \binom{n+\alpha}{n}.
\end{aligned} \tag{3.9}$$

□

Theorem 3.6. *Let $f(n,r)$ be any function defined for $n \in \mathbb{N}_0^+$, $0 \le r < \infty$. Suppose that f satisfies*

$$\left|f(n,v(n)) - f(n,w(n))\right| \le L|v(n) - w(n)|, \tag{3.10}$$

where $v(n), w(n) : \mathbb{N}_0^+ \to \mathbb{R}$ are any two nonnegative functions, and L is a nonnegative constant. Then there exists unique solution for the initial value problem (3.1).

Proof. Let $v(n)$ and $w(n)$ be any two solutions of the initial value problem (3.1) such that $v(0) = w(0)$. Then, using (3.2) and (3.10), we get

$$\begin{aligned}
|v(n+1) - w(n+1)| &= \left|\sum_{j=0}^{n} B(n,\alpha;j)\left[f(j,v(j)) - f(j,w(j))\right]\right| \\
&\le \sum_{j=0}^{n} B(n,\alpha;j)\left|f(j,v(j)) - f(j,w(j))\right| \\
&\le L\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right| \\
&< \epsilon + L\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right|,
\end{aligned} \tag{3.11}$$

where $\epsilon > 0$ be any small quantity. Let $y(n) = |v(n) - w(n)|$. Then

$$y(n+1) < \epsilon + L\sum_{j=0}^{n} B(n,\alpha;j)y(j). \tag{3.12}$$

Using Discrete Fractional Gronwall's inequality (i.e., Theorem 3.3) we get

$$y(n) < \epsilon\prod_{j=0}^{n-1}\left[1 + LB(n-1,\alpha;j)\right] < \epsilon\exp\left[L\sum_{j=0}^{n-1} B(n-1,\alpha;j)\right]. \tag{3.13}$$

Using Lemma 3.5, we get

$$y(n) < \epsilon\exp\left[L\binom{n+\alpha-1}{n-1}\right]. \tag{3.14}$$

Since ϵ is arbitrary, inequality (3.14) implies that $y(n)$ tends to zero. Thus $v(n) = w(n)$. □

Theorem 3.7. *Consider a nonlinear fractional difference equation of Volterra type* (3.6)*. Suppose that f and g satisfies*

$$\begin{gathered}
\left|f(n,v(n)) - f(n,w(n))\right| \le L|v(n) - w(n)|, \\
\sum_{m=0}^{n-1}\left|g(n,m,v(m)) - g(n,m,w(m))\right| \le L^2|v(n) - w(n)|,
\end{gathered} \tag{3.15}$$

where $v(n), w(n) : \mathbb{N}_0^+ \to \mathbb{R}$ are any two nonnegative functions and L is a nonnegative constant. Then there exists a unique solution for the initial value problem (3.6)*.*

Proof. Let $v(n)$ and $w(n)$ be any two solutions of the initial value problem (3.6) such that $v(0) = w(0)$. Then, using (3.2), we get

$$
\begin{aligned}
|v(n+1) - w(n+1)| &= \Bigg| \sum_{j=0}^{n} B(n,\alpha;j)\left[f(j,v(j)) - f(j,w(j))\right] \\
&\qquad + \sum_{j=0}^{n} B(n,\alpha;j) \sum_{m=0}^{j-1}\left[g(j,m,v(m)) - g(j,m,w(m))\right] \Bigg| \\
&\le \sum_{j=0}^{n} B(n,\alpha;j)\left|f(j,v(j)) - f(j,w(j))\right| \\
&\quad + \sum_{j=0}^{n} B(n,\alpha;j) \sum_{m=0}^{j-1}\left|g(j,m,v(m)) - g(j,m,w(m))\right| \\
&\le L\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right| + L^2\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right| \\
&\le \left(L + L^2\right)\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right| \\
&< \epsilon + \left(L + L^2\right)\sum_{j=0}^{n} B(n,\alpha;j)\left|v(j) - w(j)\right|,
\end{aligned}
\tag{3.16}
$$

where $\epsilon > 0$ be any small quantity. Let $y(n) = |v(n) - w(n)|$. Then

$$
y(n+1) < \epsilon + \left(L + L^2\right)\sum_{j=0}^{n} B(n,\alpha;j)y(j). \tag{3.17}
$$

Using Discrete Fractional Gronwall's inequality (i.e., Theorem 3.3) we get

$$
y(n) < \epsilon\prod_{j=0}^{n-1}\left[\left(L + L^2\right)B(n-1,\alpha;j)\right] < \epsilon\exp\left[\left(L + L^2\right)\sum_{j=0}^{n-1} B(n-1,\alpha;j)\right]. \tag{3.18}
$$

Using Lemma 3.5 we have

$$
y(n) < \epsilon\exp\left[\left(L + L^2\right)\binom{n+\alpha-1}{n-1}\right]. \tag{3.19}
$$

Since ϵ is arbitrary, inequality (3.19) implies that $y(n)$ tends to zero. Thus $v(n) = w(n)$. □

4. Solutions of Fractional Order Difference Equations

In this section, we define autonomous and nonautonomous fractional order difference equations and find their solutions.

Definition 4.1. Let $g(r)$ be any real-valued function defined on $\mathbb{R}$. Then an autonomous difference equation of order α together with an initial condition is of the form

$$\nabla^{\alpha} u(n+1) = g(u(n)), \quad u(0) = u_0. \tag{4.1}$$

If the function g in (4.1) is replaced by a function f of two variables, that is, $f : \mathbb{N}_0^+ \times \mathbb{R} \to \mathbb{R}$, then we have

$$\nabla^{\alpha} u(n+1) = f(n, u(n)), \quad u(0) = u_0. \tag{4.2}$$

Equation (4.2) is called a nonautonomous difference equation of order α.

Now we consider the simplest special cases of (4.1) and (4.2), namely, linear equations. The linear autonomous difference equation of order α has following general form:

$$\nabla^{\alpha} u(n+1) = a u(n) + b, \quad u(0) = u_0, \tag{4.3}$$

where $a \neq 0$ and b are known constants.

The general form of linear homogeneous nonautonomous difference equation of order α is given by

$$\nabla^{\alpha} u(n+1) = a(n) u(n), \quad u(0) = u_0 \tag{4.4}$$

and the associated nonhomogeneous equation is given by

$$\nabla^{\alpha} u(n+1) = a(n) u(n) + b(n), \quad u(0) = u_0, \tag{4.5}$$

where $a(n) \neq 0$ and $b(n)$ are real-valued functions defined for $n \in \mathbb{N}_0^+$.

Now we find the solutions of (4.3), (4.4), and (4.5).

Solution of (4.4)

Using (3.2) and (4.4), we obtain

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1, \alpha; j) a(j) u(j) \tag{4.6}$$

implies

$$u(n)=u_0\prod_{j=n_0}^{n-1}\left[1+B(n-1,\alpha;j)a(j)\right]. \tag{4.7}$$

Solution of (4.5)

Using (3.2) and (4.5), we obtain

$$u(n)=u_0+\sum_{j=0}^{n-1}B(n-1,\alpha;j)\left[a(j)u(j)+b(j)\right] \tag{4.8}$$

implies

$$u(n)=u_0\prod_{j=n_0}^{n-1}\left[1+B(n-1,\alpha;j)a(j)\right]+\sum_{j=n_0}^{n-1}B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}\left[1+B(n-1,\alpha;k)a(k)\right]. \tag{4.9}$$

Solution of (4.3)

Using (4.9), we obtain

$$\begin{aligned}u(n)&=u(0)\prod_{j=0}^{n-1}\left[1+aB(n-1,\alpha;j)\right]+b\sum_{j=0}^{n-1}B(n-1,\alpha;j)\prod_{k=j+1}^{n-1}\left[1+B(n-1,\alpha;k)a\right]\\&=u(0)\prod_{j=0}^{n-1}\left[1+aB(n-1,\alpha;j)\right]+\frac{b}{a}\sum_{j=0}^{n-1}aB(n-1,\alpha;j)\prod_{k=j+1}^{n-1}\left[1+aB(n-1,\alpha;k)\right]\\&=u(0)\prod_{j=0}^{n-1}\left[1+aB(n-1,\alpha;j)\right]+\frac{b}{a}\left[\prod_{j=0}^{n-1}\left[1+aB(n-1,\alpha;j)-1\right]\right]\\&=\left[u(0)+\frac{b}{a}\right]\prod_{j=0}^{n-1}\left[1+aB(n-1,\alpha;j)\right]-\frac{b}{a}.\end{aligned} \tag{4.10}$$

Example 4.2. Find the solution of

$$\nabla^{1/2}u(n+1)=u(n)+1,\quad u(0)=0. \tag{4.11}$$

Solution. Using (4.10), the solution of (4.11) is

$$u(n)=\prod_{j=0}^{n-1}\left[1+\binom{n-j-\frac{3}{2}}{n-j-1}\right]-1. \tag{4.12}$$

Table 1: Numerical solution of (4.17).

n	α 0.2	0.4	0.6	0.8	1.0
0	1.0000	1.0000	1.0000	1.0000	1.0000
1	2.0000	2.0000	2.0000	2.0000	2.0000
2	2.4000	2.8000	3.2000	3.6000	4.0000
3	2.6880	3.5840	4.7360	6.1920	8.0000
4	2.9245	4.3868	6.7062	10.3530	16.0000
5	3.1304	5.2221	9.2170	16.9624	32.0000
6	3.3156	6.0970	12.3917	27.3580	64.0000
7	3.4855	7.0164	16.3755	43.5659	128.0000
8	3.6437	7.9838	21.3391	68.6384	256.0000
9	3.7926	9.0020	27.4839	107.1529	512.0000
10	3.9337	10.0735	35.0464	165.9425	1024.0000

Example 4.3. Find the solution of

$$\nabla^{1/2}u(n+1) = u(n) + n, \quad u(0) = 0. \tag{4.13}$$

Solution. Using (4.9), the solution of (4.13) is

$$u(n) = \sum_{j=0}^{n-1} j \binom{n-j-\frac{3}{2}}{n-j-1} \prod_{k=j+1}^{n-1} \left[1 + \binom{n-k-\frac{3}{2}}{n-k-1}\right]. \tag{4.14}$$

Example 4.4. Find the solution of

$$\nabla^{1/2}u(n+1) = u(n), \quad u(0) = 1. \tag{4.15}$$

Solution. Using (4.7), the solution of (4.15) is

$$u(n) = \prod_{j=0}^{n-1} \left[1 + \binom{n-j-\frac{3}{2}}{n-j-1}\right]. \tag{4.16}$$

Evaluating fractional differences of various functions numerically is quite cumbersome. Now we find the numerical the solution of the following fractional order difference equation together with an initial condition

$$\nabla^{\alpha}u(n+1) = u(n), \quad u(0) = 1 \tag{4.17}$$

using a simple program in MATLAB for various values of α.

The numerical solution for various values of n and α are tabulated in Table 1. The graphical representation of the numerical solution for various values of n and α are shown in Figure 1.

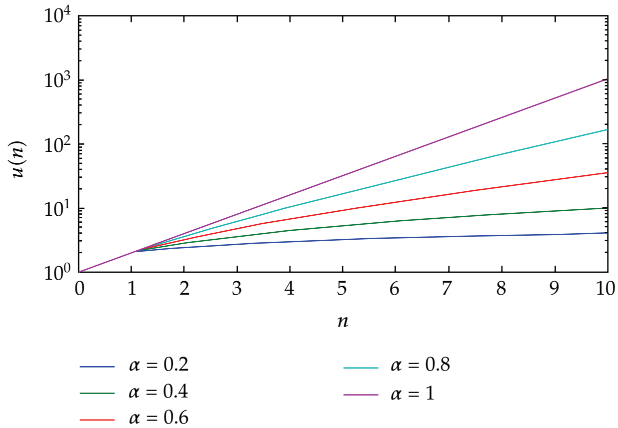

Figure 1: Numerical solution of (4.17).

References

[1] J. B. Diaz and T. J. Osler, "Differences of fractional order," *Mathematics of Computation*, vol. 28, pp. 185–202, 1974.

[2] R. Hirota, *Lectures on Difference Equations*, Science-sha, Tokyo, Japan, 2000.

[3] A. Nagai, "Discrete Mittag-Leffler function and its applications, new developments in the research of integrable systems: continuous, discrete, ultra-discrete," *RIMS Kokyuroku*, no. 1302, pp. 1–20, 2003.

[4] G. V. S. R. Deekshitulu and J. J. Mohan, "Fractional difference inequalities," *Communications in Applied Analysis*, vol. 14, no. 1, pp. 89–97, 2010.

[5] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, CSP, Cambridge, UK, 2009.

[6] V. Lakshmikantham and M. R. M. Rao, *Theory of Integro-Differential Equations*, vol. 1, Gordon and Breach Science Publishers, Lausanne, The Switzerland, 1995.

[7] H. L. Gray and N. F. Zhang, "On a new definition of the fractional difference," *Mathematics of Computation*, vol. 50, no. 182, pp. 513–529, 1988.

[8] G. V. S. R. Deekshitulu and J. J. Mohan, "Some new fractional difference inequalities," in *Proceedings of the International Conference on Mathematical Modelling and Scientific Computation (ICMMSC '12)*, vol. 283 of *Communications in Computer and Information Science*, pp. 403–412, Springer-Verlag, Berlin, Heidelberg, 2012.