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Research Article

Identities Involving q-Bernoulli and q-Euler Numbers

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We give some identities on the q-Bernoulli and q-Euler numbers by using p-adic integral equations on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic norm $|\cdot|_p$ is normally defined by $|p|_p = 1/p$.

As it is well known, the Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.1}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–14]). In the special case, $x=0$, $E_n(0)=E_n$ is called the nth Euler number.

The ordinary Bernoulli polynomials are also defined by

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.2}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [1–14]). In the special case, $x = 0$, $B_n(0) = B_n$ is called the nth Bernoulli number.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu\left(x + p^N\mathbb{Z}_p\right) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.3}$$

(see [1, 7]). Let f_1 be the translation of f with $f_1(x) = f(x+1)$. From (1.3) we have

$$I(f_1) - I(f) = f'(0), \tag{1.4}$$

(see [1, 7]).

The fermionic p-adic integral on $\mathbb{Z}_p$ is also defined by T. Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}\left(x + p^N\mathbb{Z}_p\right) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1.5}$$

(see [6, 15, 16]). By (1.5), we get

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.6}$$

(see [6, 8]).

Let $f(x) = e^{xt} \in UD(\mathbb{Z}_p)$ with $|t|_p < |p|_p^{1/(p-1)}$ and $|x|_p \le 1$. From (1.4), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.7}$$

Thus, by (1.7), we see that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \qquad n \in \mathbb{Z}_+. \tag{1.8}$$

By (1.8), we get

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} y^l d\mu(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l. \tag{1.9}$$

As an indeterminate, let us assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

From (1.1) and (1.6), we note that the q-Euler polynomials are given by

$$\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \tag{1.10}$$

where $E_{n,q}(x)$ are called the nth q-Euler polynomials (see [1, 3, 6, 8]).

Thus, by (1.10), we get

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y), \quad n \in \mathbb{Z}_+. \tag{1.11}$$

In the special case, $x = 0$, $E_{n,q}(0) = E_{n,q}$ is called the nth q-Euler number (see [8, 9]). By (1.10) and (1.11), we get the recurrence formula for the q-Euler numbers as follows:

$$E_{0,q} = \frac{2}{[2]_q}, \quad q(E_q + 1)^n + E_{n,q} = 2\delta_{0,n}, \tag{1.12}$$

with the usual convention about replacing E_q^n by $E_{n,q}$. Here $[x]_q = (1 - q^x)/(1 - q)$ is the q-number of x and $\delta_{k,n}$ is the Kronecker symbol (see [10, 11]).

From (1.2), (1.7), and (1.8), we have

$$B_0 = 1, \quad (B + 1)^n - B_n = \delta_{1,n}, \tag{1.13}$$

with the usual convention about replacing B^n by B_n (see [1, 3, 14]).

From (1.11), we easily see that

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}, \tag{1.14}$$

(see [14]).

In this paper we give some interesting properties of p-adic integrals on $\mathbb{Z}_p$ associated with the q-Bernoulli and the q-Euler numbers. From those properties, we derive new identities involving the q-Bernoulli and the q-Euler numbers arising from p-adic integrals of polynomial identities.

2. Identities on q-Bernoulli and q-Euler Numbers

Let C_{p^n} be the cyclic group of order p^n with $C_{p^n} = \{\zeta \in \mathbb{C}_p \mid \zeta^{p^n} = 1\}$. Then T_p is defined by the direct limit as $T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty}$. In this section, we assume that $q(\neq 1) \in T_p$, then $|q - 1|_p < 1$. From (1.4), we can derive the following equation (2.1):

$$\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu(y) = \frac{t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \tag{2.1}$$

where $B_{n,q}(x)$ is called the nth q-Bernoulli polynomial (see [7]). In the special case, $x = 0$, $B_{n,q}(0) = B_{n,q}$ is called the nth q-Bernoulli number.

By (2.1), we get

$$B_{0,q} = 0, \quad q(B_q + 1)^n - B_{n,q} = \delta_{1,n}, \tag{2.2}$$

with the usual convention about replacing B_q^n by $B_{n,q}$ (see [7, 14]).

From (1.3), we have

$$\int_{\mathbb{Z}_p} f(-y)d\mu(y) = \int_{\mathbb{Z}_p} f(y+1)d\mu(y). \tag{2.3}$$

By (2.3), we get

$$q\int_{\mathbb{Z}_p} (1-x+y)^n q^y d\mu(y) = (-1)^n \int_{\mathbb{Z}_p} q^{-y}(x+y)^n d\mu(y). \tag{2.4}$$

From (2.1) and (2.4), we have

$$qB_{n,q}(1-x) = (-1)^n B_{n,q^{-1}}(x), \quad n \in \mathbb{Z}_+. \tag{2.5}$$

By using (1.4), we see that

$$q\int_{\mathbb{Z}_p} (1+x+y)^n q^y d\mu(y) = \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu(y) + nx^{n-1}. \tag{2.6}$$

Thus, by (2.1) and (2.6), we get

$$qB_{n,q}(1+x) = B_{n,q}(x) + nx^{n-1}, \quad n \in \mathbb{Z}_+. \tag{2.7}$$

Therefore, by (2.5) and (2.7), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$(-1)^n B_{n,q^{-1}}(-x) = qB_{n,q}(1+x) = B_{n,q}(x) + nx^{n-1}. \tag{2.8}$$

From (1.5) and (1.6), we note that

$$\int_{\mathbb{Z}_p} (1-x+y)^n q^y d\mu_{-1}(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n q^{-(y+1)} d\mu_{-1}(y). \tag{2.9}$$

Therefore, by (1.11) and (2.9), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$qE_{n,q}(1-x) = (-1)^n E_{n,q^{-1}}(x). \tag{2.10}$$

By (1.6), we get

$$q\int_{\mathbb{Z}_p} q^y (x+1+y)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = 2x^n. \tag{2.11}$$

Thus, by (1.11) and (2.11), we have

$$qE_{n,q}(x+1) = -E_{n,q}(x) + 2x^n, \quad n \in \mathbb{Z}_+. \tag{2.12}$$

Therefore, by Theorem 2.2 and (2.12), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$(-1)^n E_{n,q^{-1}}(-x) = qE_{n,q}(1+x) = -E_{n,q}(x) + 2x^n. \tag{2.13}$$

By using the p-adic integrals on $\mathbb{Z}_p$, we have the following equation (2.14):

$$\begin{aligned}
\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{(x+y)} e^{(x+y)t} d\mu(x) d\mu_{-1}(y) &= \int_{\mathbb{Z}_p} q^x e^{xt} d\mu(x) \int_{\mathbb{Z}_p} q^y e^{yt} d\mu_{-1}(y) \\
&= \left(\frac{t}{qe^t - 1}\right)\left(\frac{2}{qe^t + 1}\right) = \frac{2t}{q^2 e^{2t} - 1} \\
&= \int_{\mathbb{Z}_p} q^{2x} e^{2xt} d\mu(x).
\end{aligned} \tag{2.14}$$

By (2.14), we get

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{(x+y)} (x+y)^n d\mu(x) d\mu_{-1}(y) = 2^n \int_{\mathbb{Z}_p} q^{2x} x^n d\mu(x). \tag{2.15}$$

It is not difficult to show that

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{(x+y)} (x+y)^n d\mu(x) d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^x x^{n-l} q^y y^l d\mu(x) d\mu_{-1}(y). \tag{2.16}$$

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} B_{n-l,q} E_{l,q} = 2^n B_{n,q^2}. \tag{2.17}$$

By (2.5), (2.7), (2.12), Theorems 2.1, and 2.3, we get

$$qB_{n,q}(x) = (-1)^n B_{n,q^{-1}}(1-x) = B_{n,q}(x-1) + n(x-1)^{n-1}, \tag{2.18}$$

$$qE_{n,q}(x) = (-1)^n E_{n,q^{-1}}(1-x) = -E_{n,q}(x-1) + 2(x-1)^n. \tag{2.19}$$

From (2.1), we have

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu(y) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q} x^{n-l}, \quad n \in \mathbb{Z}_+. \tag{2.20}$$

Let

$$I_1 = q\int_{\mathbb{Z}_p} q^x B_{n,q}(x) d\mu(x) = q\sum_{l=0}^{n} \binom{n}{l} B_{n-l,q} B_{l,q}. \tag{2.21}$$

From (2.18), (2.20), and (2.21), we note that

$$\begin{aligned} I_1 &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l,q} \int_{\mathbb{Z}_p} q^x (x-1)^l d\mu(x) + n\int_{\mathbb{Z}_p} q^x (x-1)^{n-1} d\mu(x) \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l,q} B_{l,q}(-1) + nB_{n-1,q}(-1) \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{l,q} B_{n-l,q}(-1) + nB_{n-1,q}(-1). \end{aligned} \tag{2.22}$$

By (2.5), we get

$$qB_{n,q}(-1) = (-1)^n B_{n,q^{-1}}(2), \quad n \in \mathbb{Z}_+. \tag{2.23}$$

By (2.3), we easily see that

$$q^2 B_{n,q}(2) = nq + qB_{n,q}(1) = nq + B_{n,q} + \delta_{1,n}, \tag{2.24}$$

where $\delta_{1,n}$ is the Kronecker symbol.

Thus, by (2.23) and (2.24), we get

$$B_{n,q}(-1) = (-1)^n \left(n + qB_{n,q^{-1}} + q\delta_{1,n}\right). \tag{2.25}$$

By (2.22) and (2.25), we get

$$\begin{aligned} I_1 &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{l,q} (-1)^{n-l} \left(n - l + qB_{n-l,q^{-1}} + q\delta_{1,n-l}\right) + nB_{n-1,q}(-1) \\ &= n\sum_{l=0}^{n-1} \binom{n-1}{l} B_{l,q} (-1)^l + q\sum_{l=0}^{n} \binom{n}{l} (-1)^l B_{l,q} B_{n-l,q^{-1}} + q(-1)^{n-1} nB_{n-1,q} + nB_{n-1,q}(-1) \\ &= q\sum_{l=0}^{n} \binom{n}{l} (-1)^l B_{l,q} B_{n-l,q^{-1}} + q(-1)^{n-1} nB_{n-1,q} + \left(1 + (-1)^{n-1}\right) nB_{n-1,q}(-1). \end{aligned} \tag{2.26}$$

Therefore, by (2.21) and (2.26), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{2n}\binom{2n}{l}B_{2n-l,q}B_{l,q}-\sum_{l=0}^{2n}\binom{2n}{l}(-1)^l B_{l,q}B_{2n-l,q^{-1}}=-2nB_{2n-1,q}. \tag{2.27}$$

Let us consider the following integral:

$$I_2=\int_{\mathbb{Z}_p}q^{x+1}E_{n,q}(x)d\mu_{-1}(x)=q\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q}E_{l,q}. \tag{2.28}$$

By (2.19), we get

$$\begin{aligned}I_2&=-\int_{\mathbb{Z}_p}q^xE_{n,q}(x-1)d\mu_{-1}(x)+2\int_{\mathbb{Z}_p}q^x(x-1)^nd\mu_{-1}(x)\\&=-\sum_{l=0}^{n}\binom{n}{l}E_{l,q}E_{n-l,q}(-1)+2E_{n,q}(-1)\\&=-\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q}E_{l,q}(-1)+2E_{n,q}(-1).\end{aligned} \tag{2.29}$$

From Theorem 2.2, we note that

$$qE_{n,q}(-1)=(-1)^nE_{n,q^{-1}}(2),\quad n\in\mathbb{Z}_+. \tag{2.30}$$

By (1.12), we get

$$q^2E_{n,q}(2)=2q+E_{n,q}-2\delta_{0,n}. \tag{2.31}$$

Thus, by (2.30) and (2.31), we get

$$E_{n,q}(-1)=(-1)^n\left(2+qE_{n,q^{-1}}-2q\delta_{0,n}\right). \tag{2.32}$$

From (2.29) and (2.32), we note that

$$\begin{aligned}I_2&=-\sum_{l=0}^{n}\binom{n}{l}E_{l,q}(-1)^{n-l}\left(2+qE_{n-l,q^{-1}}-2q\delta_{0,n-l}\right)+2(-1)^n\left(2+qE_{n,q^{-1}}-2q\delta_{0,n}\right)\\&=-q\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}E_{l,q}E_{n-l,q^{-1}}+2qE_{n,q}.\end{aligned} \tag{2.33}$$

Therefore, by (2.28) and (2.33), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q}E_{l,q}+\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}E_{l,q}E_{n-l,q^{-1}}=2E_{n,q}. \tag{2.34}$$

Now we consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the nth q-Euler polynomials as follows:

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p} q^x E_{n,q}(x)d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\int_{\mathbb{Z}_p} q^x x^{n-l}d\mu_{-1}(x) \\ &= \sum_{l=0}^{n}\binom{n}{l}E_{l,q}E_{n-l,q}, \quad n\in\mathbb{Z}_+. \end{aligned} \tag{2.35}$$

On the other hand, by Theorem 2.2, we get

$$\begin{aligned} I_3 &= (-1)^n q^{-1}\int_{\mathbb{Z}_p} E_{n,q^{-1}}(1-x)q^x d\mu_{-1}(x) \\ &= (-1)^n q^{-1}\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q^{-1}}\int_{\mathbb{Z}_p} q^x(1-x)^l d\mu_{-1}(x) \\ &= q^{-1}\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q^{-1}}(-1)^{n-l}E_{l,q}(-1). \end{aligned} \tag{2.36}$$

From (2.32) and (2.36), we note that

$$\begin{aligned} I_3 &= (-1)^n q^{-1}\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q^{-1}}\left(2+qE_{l,q^{-1}}-2q\delta_{0,l}\right) \\ &= 2(-1)^n q^{-1}E_{n,q^{-1}}(1)+(-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q^{-1}}E_{l,q^{-1}}-2(-1)^n E_{n,q^{-1}} \\ &= -2(-1)^n E_{n,q^{-1}}+4(-1)^n\delta_{0,n}+(-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q^{-1}}E_{l,q^{-1}}-2(-1)^n E_{n,q^{-1}}. \end{aligned} \tag{2.37}$$

Therefore, by (2.35) and (2.37), we obtain the following theorem.

Theorem 2.7. *For $n\in\mathbb{Z}_+$, one has*

$$\sum_{l=0}^{2n+1}\binom{2n+1}{l}E_{l,q}E_{2n+1-l,q}+\sum_{l=0}^{2n+1}\binom{2n+1}{l}E_{l,q^{-1}}E_{2n+1-l,q^{-1}}=4E_{2n+1,q^{-1}}. \tag{2.38}$$

From (2.1) and (2.7), we note that

$$x^n=\frac{q}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}B_{l,q}(x)+\frac{q-1}{n+1}B_{n+1,q}(x). \tag{2.39}$$

Let us consider the following fermionic p-adic integral on $\mathbb{Z}_p$:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x) &= \frac{q}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\int_{\mathbb{Z}_p} q^x B_{l,q}(x)d\mu_{-1}(x) + \frac{q-1}{n+1}\int_{\mathbb{Z}_p} q^x B_{n+1,q}(x)d\mu_{-1}(x)\\ &= \frac{q}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k,q}E_{k,q} + \frac{q-1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_{n+1-l,q}E_{l,q}.\end{aligned} \tag{2.40}$$

Therefore, by (2.40), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$, one has*

$$E_{n,q} = \frac{q}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k,q}E_{k,q} + \frac{q-1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_{n+1-l,q}E_{l,q}. \tag{2.41}$$

From (1.10) and (2.12), we note that

$$x^n = \frac{[2]_q}{2}E_{n,q}(x) + \frac{q}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_{l,q}(x). \tag{2.42}$$

Thus, by (2.42), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x) &= \frac{[2]_q}{2}\int_{\mathbb{Z}_p} E_{n,q}(x)q^x d\mu_{-1}(x) + \frac{q}{2}\sum_{l=0}^{n-1}\binom{n}{l}\int_{\mathbb{Z}_p} q^x E_{l,q}(x)d\mu_{-1}(x)\\ &= \frac{[2]_q}{2}\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q}E_{l,q} + \frac{q}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k,q}E_{k,q}.\end{aligned} \tag{2.43}$$

Thus, by (2.43), we have

$$E_{n,q} = \frac{[2]_q}{2}\sum_{l=0}^{n}\binom{n}{l}E_{n-l,q}E_{l,q} + \frac{q}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k,q}E_{k,q}. \tag{2.44}$$

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