

Research Article

A Study of Some General Problems of Dieudonné-Rashevski Type

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We use a method of investigation based on employing adequate variational calculus techniques in the study of some generalized Dieudonné-Rashevski problems. This approach allows us to state and prove optimality conditions for such kind of vector multitime variational problems, with mixed isoperimetric constraints. We state and prove efficiency conditions and develop a duality theory.

1. Introduction and Preliminaries

Applied sciences and technology ranging from Economics (processes control), Psychology (impulse control disorders), Medicine (bladder control) to Engineering (robotics and automation) and Biology (population ecosystems), lead to traditional control problems; see [1]. Such kind of problems heavily rely on the temporal dependence of these applications. That is why multitime control problems have been intensively studied in the last few years both from theoretical and applied viewpoints [2, 3], and the references therein. Motivated by the work on this topic reported in [2, 3], this paper aims to establish some results of efficiency and duality for multitime control problems of generalized Dieudonné-Rashevski type, thought as variational problems with isoperimetric constraints, mainly arising when we talk about resources. The current paper may be viewed as a natural continuation and extension of some recent works [4–10].

In the following, for two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$, the relations $v = w$, $v < w$, $v \leq w$ and $v \leq w$, are defined as

$$\begin{aligned} v = w &\iff v_i = w_i, & i = \overline{1, n}, & & v < w &\iff v_i < w_i, & i = \overline{1, n}, \\ v \leq w &\iff v_i \leq w_i, & i = \overline{1, n}, & & v \leq w &\iff v \leq w, & v \neq w. \end{aligned} \quad (1.1)$$

Important Note

To simplify the notations, in our subsequent theory, we will set

$$\begin{aligned}\pi_x(t) &= (t, x(t), x_Y(t)), & \pi_u(t) &= (t, u(t), u_Y(t)), \\ \pi_{x^*}(t) &= (t, x^*(t), x_Y^*(t)), & \pi_{u^*}(t) &= (t, u^*(t), u_Y^*(t)).\end{aligned}\tag{1.2}$$

Let be given the functional of multiple integral type

$$I(x, u) = \int_{\Omega} X(\pi_x(t), \pi_u(t)) dv.\tag{1.3}$$

Consider the functions $Y_{\beta}(\pi_x(t), \pi_u(t))$, $\beta = \overline{1, q}$, of C^1 -class. We introduce the following problem with mixed isoperimetric constraints, within the class of generalized Dieudonné-Rashevski type problems [2, 3]

$$\begin{aligned}\text{Minimize } & I(x, u) \\ & \text{subject to } \int_{\Omega} X_{\alpha}^i(\pi_x(t), \pi_u(t)) dv = 0, \quad i = \overline{1, n}, \quad \alpha = \overline{1, m}, \\ & \int_{\Omega} Y_{\beta}(\pi_x(t), \pi_u(t)) dv \leq 0, \quad \beta = \overline{1, q}, \\ & x(0) = x_0, \quad x(t_0) = x_1.\end{aligned}\tag{SCP}$$

Here $t = (t^{\alpha}) \in \mathbb{R}_+^m$; $dv = dt^1 \cdots dt^m$ is the volume element in \mathbb{R}_+^m ; Ω is the parallelepiped in \mathbb{R}_+^m defined by the closed interval $[0, t_0] = \{t \in \mathbb{R}_+^m \mid 0 \leq t \leq t_0\}$, where $0 = (0, \dots, 0)$ and $t_0 = (t_0^1, \dots, t_0^m)$ are points in \mathbb{R}_+^m ; $x(t) = (x^i(t))$, $i = \overline{1, n}$, is a state vector of C^2 -class; $u(t) = (u^a(t))$, $a = \overline{1, \ell}$, is a C^2 -class control vector; the running cost $X(\pi_x(t), \pi_u(t))$ is a C^1 -class function; $X_{\alpha}^i(\pi_x(t), \pi_u(t))$ are C^1 -class functions.

Remark that the adjective *multitime* was introduced in physics, by Dirac, since 1932, and later it was used in mathematics. For up to date information concerning this notion, see [2, 3, 11, 12].

We also introduce our vector problem. In this respect, let the vector function (X_1, \dots, X_p) , producing the running costs be given

$$X_1(\pi_x(t), \pi_u(t)), \dots, X_p(\pi_x(t), \pi_u(t)).\tag{1.4}$$

We denote

$$I_k(x, u) = \int_{\Omega} X_k(\pi_x(t), \pi_u(t)) dv, \quad k = \overline{1, p},\tag{1.5}$$

and we consider the vector functional $I(x, u) = (I_1(x, u), \dots, I_p(x, u))$.

It is the aim of our work to study the multitime control vector problem with isoperimetric constraints

$$\begin{aligned}
 & \text{Minimize (Pareto) } I(x, u) \\
 & \text{subject to } \int_{\Omega} X_{\alpha}^i(\pi_x(t), \pi_u(t)) dv = 0, \quad i = \overline{1, n}, \quad \alpha = \overline{1, m}, \\
 & \int_{\Omega} Y_{\beta}(\pi_x(t), \pi_u(t)) dv \leq 0, \quad \beta = \overline{1, q}, \\
 & x(0) = x_0, \quad x(t_0) = x_1,
 \end{aligned} \tag{VCP}$$

with Δ the domain of problem (VCP).

This kind of problems appears when we describe the torsion of prismatic bars in the elastic case [11], as well as in the elastic-plastic case [12]. Also, they arise when we think of optimization problems for convex bodies and the geometrical constraints, that is maximization of the surface for given width and diameter. These lead us again to generalized Dieudonné-Rashevski type problems for support functions in spherical coordinates [13, 14].

The first problem has a scalar objective function and is a necessary tool for pointing out our main results concerning a vectorial multitime multiobjective problem.

Our method of investigation is based on employing adequate variational calculus techniques in the study of the problems of optimal control. This fact is possible since the optimal control problems can be changed in variational problems. Moreover, the solutions of these problems belong to the interior of the problems domain.

In the following, we state necessary optimality conditions for the scalar problem (SCP).

Theorem 1.1 (necessary conditions). *Let (x, u) be an optimal solution of (SCP). Then there are $\varphi \in \mathbb{R}$, $\lambda = (\lambda_i^{\alpha}) \in \mathbb{R}^{mn}$, and $\mu \in \mathbb{R}^q$, which satisfy the conditions*

$$\begin{aligned}
 \varphi \frac{\partial X}{\partial x^i} + \lambda_j^{\alpha} \frac{\partial X_{\alpha}^j}{\partial x^i} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial x^i} - D_{\gamma} \left(\varphi \frac{\partial X}{\partial x_{\gamma}^i} + \lambda_j^{\alpha} \frac{\partial X_{\alpha}^j}{\partial x_{\gamma}^i} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial x_{\gamma}^i} \right) &= 0, \\
 \varphi \frac{\partial X}{\partial u^a} + \lambda_i^{\alpha} \frac{\partial X_{\alpha}^i}{\partial u^a} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial u^a} - D_{\gamma} \left(\varphi \frac{\partial X}{\partial u_{\gamma}^a} + \lambda_i^{\alpha} \frac{\partial X_{\alpha}^i}{\partial u_{\gamma}^a} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial u_{\gamma}^a} \right) &= 0, \\
 \mu^{\beta} Y_{\beta}(\pi_x(t), \pi_u(t)) &= 0 \quad (\text{no summation}), \\
 \varphi \geq 0, \quad \mu^{\beta} &\geq 0.
 \end{aligned} \tag{SFJ}$$

The proof of this theorem essentially uses Fritz-John conditions and the fundamental lemma of variational calculus. This result will be later used for finding and proving necessary optimality conditions for our multitime multiobjective vector problem.

2. Main Results

In this section, we establish necessary efficiency conditions for our main problem. We develop a duality theory by stating weak, direct and converse theorems, using essentially the notion

of invexity [15, 16]. Moreover, we give sufficient conditions for the efficiency of a feasible solution.

Definition 2.1. A point $(x, u) \in \Delta$ is called *efficient solution* (Pareto minimum) for (VCP) if there is no $(\bar{x}, \bar{u}) \in \Delta$ such that $I(\bar{x}, \bar{u}) \leq I(x, u)$.

The following Lemma shows the equivalence between the efficient solutions of (VCP) and the optimal solution of p scalar problems. This connection is needed in order to find necessary efficiency conditions.

Lemma 2.2. *The point $(x^0, u^0) \in \Delta$ is an efficient solution of problem (VCP) if and only if (x^0, u^0) is an optimal solution of each scalar problem $(SCP)_k$, $k = \overline{1, p}$, where*

$$\begin{aligned} & \underset{x, u}{\text{Minimize}} && I_k(x, u) \\ & \text{subject to} && \int_{\Omega} X_{\alpha}^i(\pi_x(t), \pi_u(t)) dv = 0, \quad i = \overline{1, n}, \quad \alpha = \overline{1, m}, \\ & && \int_{\Omega} Y_{\beta}(\pi_x(t), \pi_u(t)) dv \leq 0, \quad \beta = \overline{1, q}, \\ & && I_j(x, u) \leq I_j(x^0, u^0), \quad j = \overline{1, p}, \quad j \neq k, \\ & && x(0) = x_0, \quad x(t_0) = x_1. \end{aligned} \tag{SCP}_k$$

Proof. We will prove both implications.

Necessity. To prove the direct implication, we suppose that $(x^0, u^0) \in \Delta$ is an efficient solution of problem (VCP) and there is $k \in \{1, \dots, p\}$ such that (x^0, u^0) is not an optimal solution of the scalar problem $(SCP)_k$. Then there exists (y, w) such that

$$I_j(y, w) \leq I_j(x^0, u^0), \quad j = \overline{1, p}, \quad j \neq k; \quad I_k(y, w) < I_k(x^0, u^0). \tag{2.1}$$

These relations contradict the efficiency of the pair (x^0, u^0) for problem (VCP). Consequently, (x^0, u^0) is an optimal solution for each program $(SCP)_k$, $k = \overline{1, p}$.

Sufficiency. To prove the converse, let us consider that the pair (x^0, u^0) is an optimal solution of all problems $(SCP)_k$, $k = \overline{1, p}$. Suppose that (x^0, u^0) is not an efficient solution of problem (VCP). Then there exists a pair $(y, w) \in \Delta$ which satisfies $I_j(y, w) \leq I_j(x^0, u^0)$, $j = \overline{1, p}$, and there is $k \in \{1, \dots, p\}$ such that $I_k(y, w) < I_k(x^0, u^0)$. This is a contradiction to the assumption that the pair (x^0, u^0) minimizes the functional I_k on the set of all feasible solutions of problem $(SCP)_k$. Therefore, the pair (x^0, u^0) is an efficient solution of the problem (VCP). \square

This lemma plays a role of paramount importance in suggesting the study of the efficient solutions of problem (VCP). It allows us to state and prove the following necessary efficiency conditions, too.

Theorem 2.3. Let $(x, u) \in \Delta$ be an efficient solution of program (VCP). Then there are $\tau \in \mathbb{R}^p$, $\lambda_i^\alpha \in \mathbb{R}$ and $\mu \in \mathbb{R}^q$, such that

$$\begin{aligned} \tau^k \frac{\partial X_k}{\partial x^i} + \lambda_j^\alpha \frac{\partial X_\alpha^j}{\partial x^i} + \mu^\beta \frac{\partial Y_\beta}{\partial x^i} - D_\gamma \left(\tau^k \frac{\partial X_k}{\partial x_\gamma^i} + \lambda_j^\alpha \frac{\partial X_\alpha^j}{\partial x_\gamma^i} + \mu^\beta \frac{\partial Y_\beta}{\partial x_\gamma^i} \right) &= 0, \\ \tau^k \frac{\partial X_k}{\partial u^a} + \lambda_j^\alpha \frac{\partial X_\alpha^j}{\partial u^a} + \mu^\beta \frac{\partial Y_\beta}{\partial u^a} - D_\gamma \left(\tau^k \frac{\partial X_k}{\partial u_\gamma^a} + \lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial u_\gamma^a} + \mu^\beta \frac{\partial Y_\beta}{\partial u_\gamma^a} \right) &= 0, \quad (\text{VFJ}) \\ \mu^\beta Y_\beta(\pi_x(t), \pi_u(t)) &= 0, \quad \beta = \overline{1, q}, \\ \tau &= (\tau^k) \geq 0, \quad \mu = (\mu^\beta) \geq 0. \end{aligned}$$

Proof. Since (x, u) is an efficient solution of problem (VCP), (x, u) is an optimal solution of each problem (SCP) $_k$, $k = \overline{1, p}$. Let k be fixed between 1 and p . According to Theorem 1.1, there are real scalars s_k , $\lambda_{i,k}^\alpha$ and μ_k^β , which satisfy the following conditions:

$$\begin{aligned} s_k \frac{\partial X_k}{\partial x^i} + \lambda_{j,k}^\alpha \frac{\partial X_\alpha^j}{\partial x^i} + \mu_k^\beta \frac{\partial Y_\beta}{\partial x^i} - D_\gamma \left(s_k \frac{\partial X_k}{\partial x_\gamma^i} + \lambda_{j,k}^\alpha \frac{\partial X_\alpha^j}{\partial x_\gamma^i} + \mu_k^\beta \frac{\partial Y_\beta}{\partial x_\gamma^i} \right) &= 0, \\ s_k \frac{\partial X_k}{\partial u^a} + \lambda_{i,k}^\alpha \frac{\partial X_\alpha^i}{\partial u^a} + \mu_k^\beta \frac{\partial Y_\beta}{\partial u^a} - D_\gamma \left(s_k \frac{\partial X_k}{\partial u_\gamma^a} + \lambda_{i,k}^\alpha \frac{\partial X_\alpha^i}{\partial u_\gamma^a} + \mu_k^\beta \frac{\partial Y_\beta}{\partial u_\gamma^a} \right) &= 0, \quad (2.2) \\ \mu_k^\beta Y_\beta(\pi_x(t), \pi_u(t)) &= 0, \quad \beta = \overline{1, q}, \\ s_k &\geq 0, \quad \mu_k^\beta \geq 0. \end{aligned}$$

Denoting $\tau = (\tau^k)$, $\tau^k = s_k$, $k = \overline{1, p}$, $\lambda_i^\alpha = \sum_{r=1}^p \lambda_{i,r}^\alpha$, $\mu^\beta = \sum_{r=1}^p \mu_r^\beta$, and summing in (2.2) over $k = \overline{1, p}$, we obtain relations (VFJ). \square

A nontrivial situation arises when each component of vector τ is positive. In this case, dividing relations (2.2) by a positive constant, we can consider $\tau^k = 1$, for each $k = \overline{1, p}$, therefore we can introduce.

Definition 2.4. The efficient solution (x^0, u^0) of (VCP) is called *normal* if $\tau^k = 1$ for each $k = \overline{1, p}$.

Given program (VCP) and its dual, in the following we will develop our dual program theory, stating weak, direct, and converse duality theorems. The base of our research is the notion of ρ -invexity, [15, 16].

Let $f(\pi_x(t), \pi_u(t))$ be a scalar function of C^1 -class. Consider the functional

$$F(x, u) = \int_{\Omega} f(\pi_x(t), \pi_u(t)) dv. \quad (2.3)$$

Definition 2.5. The function $F(x, u)$ is called ρ -invex (strictly ρ -invex) at the point (x^*, u^*) if there exist the vector function $\eta(t) \in \mathbb{R}^n$ of C^1 -class, with $\eta|_{\partial\Omega} = 0$, $\xi(t) \in \mathbb{R}^k$ of C^0 -class and the bounded vector function $\theta(x, u) \in \mathbb{R}^n$ such that

$$\forall(x, u), \quad [(x, u) \neq (x^*, u^*)], \quad F(x, u) - F(x^*, u^*) \geq [\>],$$

$$\int_{\Omega} \left(\eta^i \frac{\partial f}{\partial x^i}(t, x^*, u^*) + (D_{\gamma} \eta^i) \frac{\partial f}{\partial x_{\gamma}^i} + \xi^a \frac{\partial f}{\partial u^a}(t, x^*, u^*) + (D_{\gamma} \xi^a) \frac{\partial f}{\partial u_{\gamma}^a} \right) dv + \rho \|\theta(x, u)\|^2.$$
(2.4)

To develop our dual program theory, we consider the Lagrangian functions

$$L_k(\pi_x(t), \pi_u(t), \lambda, \mu) = X_k(\pi_x(t), \pi_u(t)) + \frac{1}{p} \left[\lambda_i^{\alpha} X_{\alpha}^i(\pi_x(t), \pi_u(t)) + \mu^{\beta} Y_{\beta}(\pi_x(t), \pi_u(t)) \right],$$
(2.5)

where $k = \overline{1, p}$, which determine the vector function $L = (L_1, \dots, L_p)$.

Let us introduce the following vector of multiple integrals

$$J(x, u, \lambda, \mu) = \int_{\Omega} L(\pi_x(t), \pi_u(t), \lambda, \mu) dv.$$
(2.6)

To problem (VCP), we associate the next dual vector multitime control problem:

Maximize (Pareto) $J(x(t), u(t), \lambda, \mu)$

subject to

$$\frac{\partial X_k}{\partial x^i} + \lambda_j^{\alpha} \frac{\partial X_{\alpha}^j}{\partial x^i} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial x^i} - D_{\gamma} \left(\frac{\partial X_k}{\partial x_{\gamma}^i} + \lambda_j^{\alpha} \frac{\partial X_{\alpha}^j}{\partial x_{\gamma}^i} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial x_{\gamma}^i} \right) = 0,$$

$$\frac{\partial X_k}{\partial u^a} + \lambda_j^{\alpha} \frac{\partial X_{\alpha}^j}{\partial u^a} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial u^a} - D_{\gamma} \left(\frac{\partial X_k}{\partial u_{\gamma}^a} + \lambda_i^{\alpha} \frac{\partial X_{\alpha}^i}{\partial u_{\gamma}^a} + \mu^{\beta} \frac{\partial Y_{\beta}}{\partial u_{\gamma}^a} \right) = 0, \quad (\text{VCD})$$

$$\mu^{\beta}(t) Y_{\beta}(\pi_x(t), \pi_u(t)) = 0, \quad \beta = \overline{1, q},$$

$$\mu = \left(\mu^{\beta} \right) \geq 0, \quad x(0) = x_0, \quad x(t_0) = x_1.$$

Denote by \mathfrak{D} the domain of dual program (VCD) and by $(x, x_{\gamma}, u, u_{\gamma}, \lambda, \mu) = (x, x_{\gamma}, u, u_{\gamma}, \lambda_i^{\alpha}, \mu^{\beta})$ the current point of \mathfrak{D} .

Now we can state and prove our duality theorems, as in the following.

Theorem 2.6 (weak duality). *Let $(x^*, u^*) \in \Delta$ and $(x, x_{\gamma}, u, u_{\gamma}, \lambda, \mu) \in \mathfrak{D}$ be two feasible solutions of problems (VCP) and (VCD). Consider the functions λ_i^{α} and μ^{β} as in Theorem 2.3 and suppose that the following conditions are satisfied:*

- (a) for each index $k \in \{1, \dots, p\}$, the integral $\int_{\Omega} X_k(\pi_x(t), \pi_u(t)) dv$ is ρ_k -invex at (x, u) ;
- (b) $\int_{\Omega} \lambda_i^{\alpha} X_{\alpha}^i(\pi_x(t), \pi_u(t)) dv$ is ρ' -invex at (x, u) ;

- (c) $\int_{\Omega} \mu^\beta Y_\beta(\pi_x(t), \pi_u(t)) dv$ is ρ'' -invex at (x, u) ;
all with respect to η and ξ , as in Definition 2.5;
- (d) at least one of the functionals from (a), (b), and (c) is strictly ρ -invex;
- (e) $\sum_{k=1}^p \rho_k + \rho' + \rho'' \geq 0$.

Then $I(x^*, u^*) \leq J(x, u, \lambda, \mu)$ is false.

Proof. We have

$$I(x^*, u^*) - I(x, u) = \left(\int_{\Omega} [X_k(\pi_{x^*}(t), \pi_{u^*}(t)) - X_k(\pi_x(t), \pi_u(t))] dv \right), \quad k = \overline{1, p}. \quad (2.7)$$

According to (a) and Definition 2.5, we get

$$\begin{aligned} & \int_{\Omega} [X_k(\pi_{x^*}(t), \pi_{u^*}(t)) - X_k(\pi_x(t), \pi_u(t))] dv \\ & \geq \int_{\Omega} \left(\eta^i \frac{\partial X_k}{\partial x^i} + (D_\gamma \eta^i) \frac{\partial X_k}{\partial x_\gamma^i} + \xi^a \frac{\partial X_k}{\partial u^a} + (D_\gamma \xi^a) \frac{\partial X_k}{\partial u_\gamma^a} \right) dv + \rho_k \|\theta\|^2. \end{aligned} \quad (2.8)$$

After calculations, using Theorem 8.2 in [17], the previous inequality becomes

$$\begin{aligned} & \int_{\Omega} [X_k(\pi_{x^*}(t), \pi_{u^*}(t)) - X_k(\pi_x(t), \pi_u(t))] dv \\ & \geq \int_{\Omega} \left(\eta^i \frac{\partial X_k}{\partial x^i} + D_\gamma \left(\eta^i \frac{\partial X_k}{\partial x_\gamma^i} \right) - \eta^i D_\gamma \left(\frac{\partial X_k}{\partial x_\gamma^i} \right) + \xi^a \frac{\partial X_k}{\partial u^a} + D_\gamma \left(\xi^a \frac{\partial X_k}{\partial u_\gamma^a} \right) - \xi^a D_\gamma \left(\frac{\partial X_k}{\partial u_\gamma^a} \right) \right) dv \\ & \quad + \rho_k \|\theta\|^2. \end{aligned} \quad (2.9)$$

Making the sum over $k = \overline{1, p}$, and using the constraints of (VCD), we obtain

$$\begin{aligned} & \int_{\Omega} [X_k(\pi_{x^*}(t), \pi_{u^*}(t)) - X_k(\pi_x(t), \pi_u(t))] dv \\ & \geq - \int_{\Omega} \left[\eta^i \left(\lambda_j^\alpha \frac{\partial X_\alpha^j}{\partial x^i} + \mu^\beta \frac{\partial Y_\beta}{\partial x^i} - D_\gamma \left(\lambda_j^\alpha \frac{\partial X_\alpha^j}{\partial x_\gamma^i} + \mu^\beta \frac{\partial Y_\beta}{\partial x_\gamma^i} \right) \right) \right. \\ & \quad \left. + \xi^a \left(\lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial u^a} + \mu^\beta \frac{\partial Y_\beta}{\partial u^a} - D_\gamma \left(\lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial u_\gamma^a} + \mu^\beta \frac{\partial Y_\beta}{\partial u_\gamma^a} \right) \right) \right] dv + \|\theta\|^2 \sum_{k=1}^p \rho_k \quad (2.10) \\ & = - \int_{\Omega} \left(\eta^i \lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial x^i} + \xi^a \lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial u^a} - D_\gamma \left(\eta^i \lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial x_\gamma^i} + \xi^a \lambda_i^\alpha \frac{\partial X_\alpha^i}{\partial u_\gamma^a} \right) \right) dv \\ & \quad - \int_{\Omega} \left(\eta^i \mu^\beta \frac{\partial Y_\beta}{\partial x^i} + \xi^a \mu^\beta \frac{\partial Y_\beta}{\partial u^a} - D_\gamma \left(\eta^i \mu^\beta \frac{\partial Y_\beta}{\partial x_\gamma^i} + \xi^a \mu^\beta \frac{\partial Y_\beta}{\partial u_\gamma^a} \right) \right) dv + \|\theta\|^2 \sum_{k=1}^p \rho_k. \end{aligned}$$

Taking into account hypothesis (b), we have

$$\begin{aligned} & \int_{\Omega} \lambda_i^\alpha \left(X_\alpha^i(\pi_{x^*}(t), \pi_{u^*}(t)) - X_\alpha^i(\pi_x(t), \pi_u(t)) \right) dv \\ & \geq \int_{\Omega} \lambda_j^\alpha \left(\eta^i \frac{\partial X_\alpha^j}{\partial x^i} + (D_\gamma \eta^i) \frac{\partial X_\alpha^j}{\partial x_\gamma^i} + \xi^a \frac{\partial X_\alpha^j}{\partial u^a} + (D_\gamma \xi^a) \frac{\partial X_\alpha^j}{\partial u_\gamma^a} \right) dv + \rho' \|\theta\|^2, \end{aligned} \quad (2.11)$$

that is,

$$\begin{aligned} & \int_{\Omega} \lambda_i^\alpha \left(X_\alpha^i(\pi_{x^*}(t), \pi_{u^*}(t)) - X_\alpha^i(\pi_x(t), \pi_u(t)) \right) dv \\ & \geq \int_{\Omega} \lambda_j^\alpha \left(\eta^i \frac{\partial X_\alpha^j}{\partial x^i} - \eta^i D_\gamma \left(\frac{\partial X_\alpha^j}{\partial x_\gamma^i} \right) + \xi^a \frac{\partial X_\alpha^j}{\partial u^a} - \xi^a D_\gamma \left(\frac{\partial X_\alpha^j}{\partial u_\gamma^a} \right) \right) dv + \rho' \|\theta\|^2. \end{aligned} \quad (2.12)$$

Taking into account hypothesis (c), we obtain

$$\begin{aligned} & \int_{\Omega} \mu^\beta [\gamma_\beta(\pi_{x^*}(t), \pi_{u^*}(t)) - \gamma_\beta(\pi_x(t), \pi_u(t))] dv \\ & \geq \int_{\Omega} \mu^\beta \left(\eta^i \frac{\partial \gamma_\beta}{\partial x^i} + (D_\gamma \eta^i) \frac{\partial \gamma_\beta}{\partial x_\gamma^i} + \xi^a \frac{\partial \gamma_\beta}{\partial u^a} + (D_\gamma \xi^a) \frac{\partial \gamma_\beta}{\partial u_\gamma^a} \right) dv + \rho'' \|\theta\|^2, \end{aligned} \quad (2.13)$$

which leads us to

$$\begin{aligned} & \int_{\Omega} \mu^\beta [\gamma_\beta(\pi_{x^*}(t), \pi_{u^*}(t)) - \gamma_\beta(\pi_x(t), \pi_u(t))] dv \\ & \geq \int_{\Omega} \mu^\beta \left(\eta^i \frac{\partial \gamma_\beta}{\partial x^i} - \eta^i D_\gamma \left(\frac{\partial \gamma_\beta}{\partial x_\gamma^i} \right) + \xi^a \frac{\partial \gamma_\beta}{\partial u^a} - \xi^a D_\gamma \left(\frac{\partial \gamma_\beta}{\partial u_\gamma^a} \right) \right) dv + \rho'' \|\theta\|^2. \end{aligned} \quad (2.14)$$

Multiplying (2.12) and (2.14) by -1 , and summing side by side, we obtain

$$\begin{aligned} & - \int_{\Omega} \lambda_j^\alpha \left(\eta^i \frac{\partial X_\alpha^j}{\partial x^i} - \eta^i D_\gamma \left(\frac{\partial X_\alpha^j}{\partial x_\gamma^i} \right) + \xi^a \frac{\partial X_\alpha^j}{\partial u^a} - \xi^a D_\gamma \left(\frac{\partial X_\alpha^j}{\partial u_\gamma^a} \right) \right) dv \\ & - \int_{\Omega} \mu^\beta \left(\eta^i \frac{\partial \gamma_\beta}{\partial x^i} - \eta^i D_\gamma \left(\frac{\partial \gamma_\beta}{\partial x_\gamma^i} \right) + \xi^a \frac{\partial \gamma_\beta}{\partial u^a} - \xi^a D_\gamma \left(\frac{\partial \gamma_\beta}{\partial u_\gamma^a} \right) \right) dv \\ & \geq \int_{\Omega} \lambda_i^\alpha X_\alpha^i(\pi_x(t), \pi_u(t)) dv + \int_{\Omega} \mu^\beta \gamma_\beta(\pi_x(t), \pi_u(t)) dv + (\rho' + \rho'') \|\theta\|^2. \end{aligned} \quad (2.15)$$

Then, from (2.10) and (2.15), it follows

$$\begin{aligned} & \int_{\Omega} [X_k(\pi_{x^*}(t), \pi_{u^*}(t)) - X_k(\pi_x(t), \pi_u(t))] dv \\ & \geq \int_{\Omega} \lambda_i^\alpha X_\alpha^i(\pi_x(t), \pi_u(t)) dv + \int_{\Omega} \mu^\beta Y_\beta(\pi_x(t), \pi_u(t)) dv + \left(\sum_{k=1}^p \rho_k + \rho' + \rho'' \right) \|\theta\|^2 \end{aligned} \quad (2.16)$$

and taking into account hypotheses (d) and (e) of the theorem, we infer

$$\begin{aligned} & \int_{\Omega} X_k(\pi_{x^*}(t), \pi_{u^*}(t)) dv \\ & > \int_{\Omega} \left(X_k(\pi_x(t), \pi_u(t)) + \lambda_i^\alpha X_\alpha^i(\pi_x(t), \pi_u(t)) + \mu^\beta Y_\beta(\pi_x(t), \pi_u(t)) \right) dv, \end{aligned} \quad (2.17)$$

that is the inequality

$$\sum_{k=1}^p I_k(x^*, u^*) > \sum_{k=1}^p J_k(x, u, \lambda, \mu) \quad (2.18)$$

is true. Consequently,

$$\int_{\Omega} X(\pi_{x^*}(t), \pi_{u^*}(t)) dv \leq \int_{\Omega} L(\pi_x(t), \pi_u(t), \lambda, \mu) dv, \quad (2.19)$$

is not true. Therefore, the inequality $I(x^*, u^*) \leq J(x, u, \lambda, \mu)$ is false. \square

We would like to continue our study stating and proving a direct duality theorem. In this respect, let us consider (x^0, u^0) a normal efficient solution of problem (VCP). According to Theorem 2.3, there are the real scalars $(\lambda_i^\alpha)^0$ and $(\mu^\beta)^0$ such that conditions (VFJ) are satisfied.

Theorem 2.7 (direct duality). *Suppose that the conditions of Theorem 2.6 are satisfied and $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$, above introduced, is an efficient solution of dual variational problem (VCD).*

Then $I(x^0, u^0) = J(x^0, u^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$, that is

$$\min(\text{VCP})(x^0, u^0) = \max(\text{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0). \quad (2.20)$$

Proof. By Theorem 2.6, the inequality $I(x^0, u^0) \leq J(x^0, u^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ is not true. Therefore,

$$\min(\text{VCP})(x^0, u^0) = \max(\text{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0), \quad (2.21)$$

\square

We finish this ongoing study with results concerning converse duality. These are introduced in the following two theorems.

Theorem 2.8 (converse duality). *Let $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ be an efficient solution of dual problem (VCD) which satisfies the conditions in Theorem 2.6 at the point (x^0, u^0) . Consider (\bar{x}, \bar{u}) a normal efficient solution of primal (VCP) such that $I(\bar{x}, \bar{u})$ is in relation with $I(x^0, u^0)$.*

Then, $(\bar{x}, \bar{u}) = (x^0, u^0)$ and

$$\min(\text{VCP})(\bar{x}, \bar{u}) = \max(\text{VCD})\left(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0\right). \quad (2.22)$$

Proof. By reductio ad absurdum, suppose that $(\bar{x}, \bar{u}) \neq (x^0, u^0)$. Applying Theorem 2.6, it follows that there are the real scalars $\bar{\lambda}_i^\alpha$ and $\bar{\mu}^\beta$ such that the conditions (VFJ) are satisfied at the point (\bar{x}, \bar{u}) . We obtain that $(\bar{x}, \bar{x}_\gamma, \bar{u}, \bar{u}_\gamma, \bar{\lambda}_i^\alpha, \bar{\mu}^\beta)$ is a point from \mathfrak{D} , the set of feasible solutions of dual (VCD) and the equality $\min(\text{VCP})(\bar{x}, \bar{u}) = \max(\text{VCD})(\bar{x}, \bar{x}_\gamma, \bar{u}, \bar{u}_\gamma, (\bar{\lambda}_i^\alpha), (\bar{\mu}^\beta))$ holds true. According to the weak duality theorem, $\min(\text{VCP})(\bar{x}, \bar{u}) \not\leq \max(\text{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$. This relation implies that

$$\max(\text{VCD})(\bar{x}, \bar{x}_\gamma, \bar{u}, \bar{u}_\gamma, (\bar{\lambda}_i^\alpha), (\bar{\mu}^\beta)) \not\leq \max(\text{VCD})\left(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0\right), \quad (2.23)$$

which contradicts the (Pareto) maximal efficiency of $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$. Therefore, we obtain $(\bar{x}, \bar{u}) = (x^0, u^0)$ and

$$\min(\text{VCP})(\bar{x}, \bar{u}) = \max(\text{VCD})\left(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0\right) \quad (2.24)$$

and this concludes the proof. \square

Following the same steps from the proof of the weak duality theorem, a sufficiency result follows. It states that the necessity conditions of problem (VCP) become sufficient, by adding several more conditions from Theorem 2.6.

Theorem 2.9. *Suppose that (x^0, u^0) is a feasible solution of problem (VCP) and $(\lambda_i^\alpha)^0, (\mu^\beta)^0$ are the multipliers from Theorem 2.3 and the conditions (VFJ) from Theorem 2.3. If the hypotheses (a)–(e) from Theorem 2.6 are satisfied, then (x^0, u^0) is an efficient solution of problem (VCP).*

3. Conclusion

In this work, we introduced and studied a new vector variational problem of generalized Dieudonné-Rashevski type. Employing isoperimetric constraints and a simplified scalar variational problem, we derived necessary efficiency conditions. The notion of invexity allowed us to develop a dual program theory and to obtain sufficient conditions of efficiency. The results of this paper are new and they complement previously known results. For other different viewpoints regarding the theory of efficiency and duality for optimum problems with constraints, we address the reader to the following research works: [2–10, 15–22].

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