Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 548612, 9 pages
doi:10.1155/2012/548612

Research Article

Cocompact Open Sets and Continuity

Samer Al Ghour and Salti Samarah

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Correspondence should be addressed to Samer Al Ghour, algore@just.edu.jo

Received 10 October 2011; Accepted 5 December 2011

Academic Editor: Stefan Siegmund

Copyright © 2012 S. Al Ghour and S. Samarah. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Compact subsets of a topological space are used to define coc-open sets as new generalized open sets, and then coc-open sets are used to define (coc)*-open sets as another type of generalized open sets. Several results and examples related to them are obtained; particularly a decomposition of open sets is given. Also, coc-open sets and (coc)*-open sets are used to introduce coc-continuity and (coc)*-continuity, respectively. As a main result, a decomposition theorem of continuity is obtained.

1. Introduction

Throughout this paper by a space we mean a topological space. Let (X,τ) be a space, and let A be a subset of X. A point $x \in X$ is called a condensation point of A if, for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows: A is called ω-closed [1] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. In 1989, Hdeib [2] introduced ω-continuity as a generalization of continuity as follows: A function $f : (X,\tau) \to (Y,\sigma)$ is called ω-continuous if the inverse image of each open set is ω-open. The authors in [3] proved that the family of all ω-open sets in a space (X,τ) forms a topology on X finer than τ and that the collection $\{U - C : U \in \tau$ and C is a countable subset of $X\}$ forms a base for that topology. Recently, in [4], the authors introduced a class of generalized open sets which is stronger ω-open as follows: a subset A of a space (X,τ) is called N-open if for each $x \in A$, there exists $U \in \tau$ such that $x \in U$ and $U - A$ is finite. Throughout this paper, the family of all N-open sets in a space (X,τ) will be denoted by τ^n. The authors in [4] proved that τ^n is a topology on X that is finer than τ and they introduced several concepts related to N-open sets; in particular, they introduced N-continuity as a generalization of continuity and as a stronger form of ω-continuity as follows: a function $f : (X,\tau) \to (Y,\sigma)$ is called N-continuous if the inverse image of each open set is N-open. The authors in [5] continued the study of topological concepts via N-open sets. In the present work, we will generalize N-open sets by coc-open sets. Several results and concepts related to them will be introduced.

Throughout this paper, we use $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ to denote the set of real numbers, the set of rationals, and the set of natural numbers, respectively. For a subset A of a space (X, τ), The closure of A and the interior of A will be denoted by $\mathrm{Cl}_\tau(A)$ and $\mathrm{Int}_\tau(A)$, respectively. Also, we write $\tau|_A$ to denote the relative topology on A when A is nonempty. For a nonempty set X, $\tau_{\mathrm{disc}}, \tau_{\mathrm{ind}}, \tau_{\mathrm{cof}}$, and τ_{coc} will denote, respectively, the discrete topology on X, the indiscrete topology on X, the cofinite topology on X, and the cocountable topology on X. For a subset A of $\mathbb{R}$ we write (A, τ_u) to denote the subspace topology on A relative to the usual topology and we use τ_{lr} to denote the left ray topology on $\mathbb{R}$. For any two spaces (X, τ) and (Y, σ) we use τ_{prod} to denote the product topology on $X \times Y$. The family of all compact subsets of a space (X, τ) will be denoted by $C(X, \tau)$.

2. Cocompact Open Sets

Definition 2.1. A subset A of a space (X, τ) is called co-compact open set (notation: coc-open) if, for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset $K \in C(X, \tau)$ such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed.

The family of all coc-open subsets of a space (X, τ) will be denoted by τ^k and the family $\{U - K : U \in \tau \text{ and } K \in C(X, \tau)\}$ of coc-open sets will be denoted by $\mathcal{B}^k(\tau)$.

Theorem 2.2. *Let (X, τ) be a space. Then the collection τ^k forms a topology on X.*

Proof. By the definition one has directly that $\emptyset \in \tau^k$. To see that $X \in \tau^k$, let $x \in X$, take $U = X$ and $K = \emptyset$. Then $x \in U - K \subseteq X$.

Let $U_1, U_2 \in \tau^k$, and let $x \in U_1 \cap U_2$. For each $i = 1, 2$, we find an open set V_i and a compact subset K_i such that $x \in V_i - K_i \subseteq U_i$. Take $V = V_1 \cap V_2$ and $K = K_1 \cup K_2$. Then V is open, K is compact, and $x \in V - K \subseteq U_1 \cap U_2$. It follows that $U_1 \cap U_2$ is coc-open.

Let $\{U_\alpha : \alpha \in \Delta\}$ be a collection of coc-open subsets of (X, τ) and $x \in \bigcup_{\alpha \in \Delta} U_\alpha$. Then there exists $\alpha_0 \in \Delta$ such that $x \in U_{\alpha_\circ}$. Since $U_{\alpha_\circ}$ is coc-open, then there exists an open set V and a compact subset K, such that $x \in V - K \subseteq U_{\alpha_\circ}$. Therefore, we have $x \in V - K \subseteq U_{\alpha_\circ} \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Hence, $\bigcup_{\alpha \in \Delta} U_\alpha$ is coc-open. □

The following result follows directly from Definition 2.1.

Proposition 2.3. *Let (X, τ) be a space. Then the collection $\mathcal{B}^k(\tau)$ forms a base for τ^k.*

Corollary 2.4. *Let (X, τ) be a space. Then the collection $\tau \cup \{X - K : K \in C(X, \tau)\}$ forms a subbase for τ^k.*

For a space (X, τ), the following example shows that the collection $\mathcal{B}^k(\tau)$ is not a topology on X in general.

Example 2.5. Let $X = \mathbb{R}$, and let $\tau = \tau_{\mathrm{lr}}$. Consider the collection of elements of $\mathcal{B}^k(\tau), G_n = (-\infty, n + 1/2) - \{1, 2, \ldots, n\}, n \in \mathbb{N}$. Then $\cup\{G_n : n \in \mathbb{N}\} = \mathbb{R} - \mathbb{N}$ which is not in $\mathcal{B}^k(\tau)$.

Theorem 2.6. *Let (X, τ) be a space. Then $\tau \subseteq \tau^n \subseteq \tau^k$.*

Proof. Obvious. □

Remark 2.7. Each of the two inclusions in Theorem 2.6 is not equality in general; to see this, let $X = \mathbb{N}$ and $\tau = \tau_{\text{ind}}$. Then $\tau^n = \tau_{\text{cof}}$ and $\tau^k = \tau_{\text{disc}}$, and therefore $\tau \neq \tau^n$ and $\tau^n \neq \tau^k$.

In Remark 2.7, the space (X, τ) is an example on a compact space (X, τ) for which (X, τ^k) is not compact.

Definition 2.8 (see [6]). A space (X, τ) is called CC if every compact set in X is closed.

It is well known that every T_2 space is CC, but not conversely.

Theorem 2.9. *Let (X, τ) be a space. Then the following are equivalent:*

(a) (X, τ) *is CC,*

(b) $\tau = \mathcal{B}^k(\tau)$,

(c) $\tau = \tau^n = \tau^k$.

Proof. (a) $\Rightarrow$ (b) As $\emptyset$ is a compact subset of X, then, for every $U \in \tau$, $U - \emptyset = U \in \mathcal{B}^k(\tau)$. Hence, we have $\tau \subseteq \mathcal{B}^k(\tau)$. Now let $U - K \in \mathcal{B}^k(\tau)$, where $U \in \tau$ and K is a compact subset of X. As (X, τ) is CC, then K is closed and hence $U - K \in \tau$. Therefore, we have $\mathcal{B}^k(\tau) \subseteq \tau$.

(b) $\Rightarrow$ (c) By Theorem 2.6, it is sufficient to see that $\tau^k \subseteq \tau$. Since by (b) $\tau = \mathcal{B}^k(\tau)$ and as $\mathcal{B}^k(\tau)$ is a base for τ^k, then $\tau^k \subseteq \tau$.

(c) $\Rightarrow$ (a) Let $K \in C(X, \tau)$. Then $X - K \in \tau^k$, and by (c), $X - K \in \tau$. Therefore, K is closed in X. □

Corollary 2.10. *If (X, τ) is a T_2-space, then $\tau = \tau^n = \tau^k$.*

Theorem 2.11. *For any space (X, τ), (X, τ^k) is CC.*

Proof. Let $K \in C(X, \tau^k)$. As $\tau \subseteq \tau^k$, then $C(X, \tau^k) \subseteq C(X, \tau)$ and hence $K \in C(X, \tau)$. Thus, we have $X - K \in \tau^k$, and hence K is closed in the space (X, τ^k). □

Corollary 2.12. *For any space (X, τ), $(\tau^k)^k = \tau^k$.*

Proof. Theorems 2.9 and 2.11. □

Theorem 2.13. *If (X, τ) is a hereditarily compact space, then $\tau^k = \tau_{\text{disc}}$.*

Proof. For every $x \in X$, $X - \{x\}$ is compact and so $\{x\} = X - (X - \{x\}) \in \mathcal{B}^k(\tau) \subseteq \tau^k$. Therefore, $\tau^k = \tau_{\text{disc}}$. □

Each of the following three examples shows that the converse of Theorem 2.13 is not true in general.

Example 2.14. Let $X = \mathbb{R}$ and $\tau = \tau_{\text{lr}}$. For every $x \in X$, take $K = (-\infty, x+1] - \{x\}$ and $U = (-\infty, x+1)$. Then $K \in C(X, \tau)$, $U = (-\infty, x+1) \in \tau$, and $\{x\} = U - K$. This shows that $\tau^k = \tau_{\text{disc}}$. On the other hand, it is well known that (X, τ) is not hereditarily compact.

Example 2.15. Let $X = \mathbb{N}$ and $\tau = \{\emptyset, \mathbb{N}\} \cup \{U_n : n \in \mathbb{N}\}$, where $U_n = \{1, 2, \ldots, n\}$. Then the compact subsets of (X, τ) are the finite sets. For every $n \in \mathbb{N}$, $U_n \in \tau$, U_{n-1} is compact, and $\{n\} = U_n - U_{n-1}$. Therefore, $\tau^k = \tau_{\text{disc}}$.

Example 2.16. Let $X = \mathbb{N}$ and τ be the topology on $\mathbb{N}$ having the family $\{\{2n-1, 2n\} : n \in \mathbb{N}\}$ as a base. Then the compact subsets of (X, τ) are the finite sets. If $x \in \mathbb{N}$ with x is odd, then $\{x\} = \{x, x+1\} - \{x+1\}$ and as $\{x, x+1\} \in \tau$ and $\{x+1\}$ is compact, then $\{x\} \in \tau^k$. Similarly, if x is even then $\{x\} \in \tau^k$. Therefore, $\tau^k = \tau_{\text{disc}}$.

The following question is natural: Is there a space (X, τ) for which $\tau^k \neq \tau$ and $\tau^k \neq \tau_{\text{disc}}$? The following example shows that the answer of the above question is yes.

Example 2.17. Let $X = \mathbb{R}$ and $\tau = \{X\} \cup \{U \subseteq X : 1 \notin U\}$. Then $C(X, \tau) = \{K \subseteq X : 1 \in K\} \cup \{K \subseteq X : 1 \notin K \text{ and } K \text{ is finite}\}$, hence $\tau^k = \tau \cup \{U \subseteq X : 1 \in U \text{ and } X - U \text{ is finite}\}$. Note that $\tau^k \neq \tau$ and $\tau^k \neq \tau_{\text{disc}}$.

Theorem 2.18. *Let (X, τ) be a space and A a nonempty subset of X. Then $(\tau|_A)^k \subseteq \tau^k|_A$.*

Proof. Let $B \in (\tau|_A)^k$ and $x \in B$. Then there exists $V \in \tau|_A$ and a compact subset $K \subseteq A$ such that $x \in V - K \subseteq B$. Since $V \in \tau|_A$, then we can write $V = U \cap A$, where U is open in X. Since $U - K \in \tau^k$, $(U - K) \cap A \in \tau^k|_A$. Hence, $B \in \tau^k|_A$. □

Question 1. Let (X, τ) be a space and A a nonempty subset of X. Is it true that $(\tau|_A)^k = \tau^k|_A$?

The following result is a partial answer for Question 1.

Theorem 2.19. *Let (X, τ) be a space and A be a nonempty closed set in (X, τ). Then $(\tau|_A)^k = \tau^k|_A$.*

Proof. By Theorem 2.18, $(\tau|_A)^k \subseteq \tau^k|_A$. Conversely, let $B \in \tau^k|_A$ and $x \in B$. Choose $H \in \tau^k$ such that $B = H \cap A$. As $H \in \tau^k$, there exists $U \in \tau$ and $K \in C(X, \tau)$ such that $x \in U - K \subseteq H$. Thus, we have $x \in (U \cap A) - (K \cap A) \subseteq B$, $U \cap A \in \tau|_A$, and $K \cap A \in C(A, \tau|_A)$. It follows that $B \in (\tau|_A)^k$. □

Definition 2.20. Let (X, τ) be a space, and let $A \subseteq X$. The coc-closure of A in (X, τ) is denoted by coc-$\text{Cl}_\tau(A)$ and defined as follows:

$$\text{coc-Cl}_\tau(A) = \cap\{B : B \text{ is coc-closed in } (X, \tau) \text{ and } A \subseteq B\}. \tag{2.1}$$

Remark 2.21. Let (X, τ) be a space, and let $A \subseteq X$. Then $\text{coc} - \text{Cl}_\tau(A) = \text{Cl}_{\tau^k}(A)$ and coc-$\text{Cl}_\tau(A) \subseteq \text{Cl}_\tau(A)$.

Definition 2.22. A space (X, τ) is called antilocally compact if any compact subset of X has empty interior.

For any infinite set X, (X, τ_{coc}) is an anti-locally compact space. Also, $(\mathbb{Q}, \tau_u)$ is an example of an anti-locally compact space.

Theorem 2.23. *Let (X, τ) be an anti-locally compact space. If $A \in \tau$ then coc-$Cl_\tau(A) = Cl_\tau(A)$.*

Proof. According to Remark 2.21, only we need to show that $\text{Cl}_\tau(A) \subseteq \text{Cl}_{\tau^k}(A)$. Suppose to the contrary that there is $x \notin \text{Cl}_\tau(A) - \text{Cl}_{\tau^k}(A)$. As $x \notin \text{Cl}_{\tau^k}(A)$, there exists $G \supseteq A$ coc-closed such that $x \in G$ and $G \cap A = \varnothing$. Take $U \in \tau$ and $K \in C(X, \tau)$ such that $x \in U - K \subseteq G$. Thus we have $U \cap A \subseteq K$. Since $x \in \text{Cl}_\tau(A)$, it follows that $U \cap A \neq \varnothing$ and hence $\text{Int}(K) \neq \varnothing$. This contradicts the assumption that (X, τ) is anti-locally compact. □

In Theorem 2.23 the assumption "anti-locally compact" on the space cannot be dropped. As an example let $X = \mathbb{R}$ and $\tau = \{\emptyset, X, \{0\}\}$, then $\{0\} \in \tau$, coc-$\mathrm{Cl}_\tau(\{0\}) = \{0\}$ while $\mathrm{Cl}_\tau(\{0\}) = \mathbb{R}$.

Theorem 2.24. *If $f : (X,\tau) \to (Y,\sigma)$ is injective, open, and continuous, then $f : (X,\tau^k) \to (Y,\sigma^k)$ is open.*

Proof. Let $G = U - K$ where $U \in \tau$ and $K \in C(X,\tau)$ be a basic element for τ^k. As f is injective, $f(G) = f(U) - f(K)$. Also, as $f : (X,\tau) \to (Y,\sigma)$ is open, $f(U) \in \sigma$. And as $f : (X,\tau) \to (Y,\sigma)$ is continuous, $f(K) \in C(Y,\sigma)$. This ends the proof. □

Remark 2.25. In Theorem 2.24, the continuity condition cannot be dropped. Take $f : (\mathbb{R}, \tau_{\text{ind}}) \to (\mathbb{R}, \tau_u)$, where $f(x) = \tan^{-1} x$. Then f is injective and open. On the other hand, as $(\mathbb{R}, \tau_{\text{ind}})$ is hereditarily compact we have $(\tau_{\text{ind}})^k = \tau_{\text{disc}}$, and as $(\mathbb{R}, \tau_u)$ is T_2 we have $(\tau_u)^k = \tau_u$. Thus, $f : (\mathbb{R}, (\tau_{\text{ind}})^k) \to (\mathbb{R}, (\tau_u)^k)$ is not open.

3. $(\text{coc})^*$-Open Sets

Definition 3.1. A subset A of a space (X,τ) is called $(\text{coc})^*$-open if $\mathrm{Int}_{\tau^k}(A) = \mathrm{Int}_\tau(A)$.

The family of all $(\text{coc})^*$-open subsets of a space (X,τ) will be denoted by $\mathcal{B}^{k^*}(\tau)$.

Theorem 3.2. *Let (X,τ) be a space. Then $\tau \subseteq \mathcal{B}^{k^*}(\tau)$.*

Proof. Let $A \in \tau$. Then $A = \mathrm{Int}_\tau(A) \subseteq \mathrm{Int}_{\tau^k}(A) \subseteq A$. Thus $\mathrm{Int}_\tau(A) = \mathrm{Int}_{\tau^k}(A)$ and hence $A \in \mathcal{B}^{k^*}(\tau)$. □

Theorem 3.3. *If (X,τ) is a CC space, then every subset of X is $(\text{coc})^*$-open.*

Proof. Let $A \subseteq X$. Since (X,τ) is CC, then $\tau = \tau^k$, and so $\mathrm{Int}_{\tau^k}(A) = \mathrm{Int}_\tau(A)$. Therefore, A is $(\text{coc})^*$-open. □

Corollary 3.4. *If (X,τ) is a T_2 space, then every subset of X is $(\text{coc})^*$-open.*

According to Corollary 3.4, the inclusion in Theorem 3.2 is not equality in any T_2 space that is not discrete for example, in $(\mathbb{R}, \tau_u)$ for the set $A = (0,1) \cup \{2\}$, we have $\mathrm{Int}_{(\tau_u)^k}(A) = \mathrm{Int}_{\tau_u}(A) = (0,1)$ and thus $A \in \mathcal{B}^{k^*}(\tau_u) - \tau_u$.

Theorem 3.5. *If (X,τ) is a hereditarily compact space, then $\tau = \mathcal{B}^{k^*}(\tau)$.*

Proof. By Theorem 3.2, we need only to show that $\mathcal{B}^{k^*}(\tau) \subseteq \tau$. Let $A \in \mathcal{B}^{k^*}(\tau)$. Then $\mathrm{Int}_{\tau^k}(A) = \mathrm{Int}_\tau(A)$. Since (X,τ) is hereditarily compact, then, by Theorem 2.13, $\tau^k = \tau_{\text{disc}}$ and thus $\mathrm{Int}_\tau(A) = \mathrm{Int}_{\tau^k}(A) = A$. Therefore, $A \in \tau$. □

The following result is a new decomposition of open sets in a space.

Theorem 3.6. *Let (X,τ) be a space. Then $\tau = \tau^k \cap \mathcal{B}^{k^*}(\tau)$.*

Proof. By Theorems 2.6 and 3.2, it follows that $\tau \subseteq \tau^k \cap \mathcal{B}^{k^*}(\tau)$. Conversely, let $A \in \tau^k \cap \mathcal{B}^{k^*}(\tau)$. As $A \in \tau^k$, then $\mathrm{Int}_{\tau^k}(A) = A$. Also, since $A \in \mathcal{B}^{k^*}(\tau)$, then $\mathrm{Int}_{\tau^k}(A) = \mathrm{Int}_\tau(A)$. It follows that $\mathrm{Int}_\tau(A) = A$ and hence $A \in \tau$. □

Theorem 3.7. *For a space (X, τ), one has the following:*

(a) $\emptyset, X \in \mathcal{B}^{k^*}(\tau)$,

(b) *if* $A, B \in \mathcal{B}^{k^*}(\tau)$, *then* $A \cap B \in \mathcal{B}^{k^*}(\tau)$.

Proof. (a) The proof follows directly from Theorem 3.2.

(b) Let $A, B \in \mathcal{B}^{k^*}(\tau)$. Then $\mathrm{Int}_{\tau^k}(A) = \mathrm{Int}_{\tau}(A)$ and $\mathrm{Int}_{\tau^k}(B) = \mathrm{Int}_{\tau}(B)$. Thus we have

$$\begin{aligned} \mathrm{Int}_{\tau}(A \cap B) &= \mathrm{Int}_{\tau}(A) \cap \mathrm{Int}_{\tau}(B) \\ &= \mathrm{Int}_{\tau^k}(A) \cap \mathrm{Int}_{\tau^k}(B) \\ &= \mathrm{Int}_{\tau^k}(A \cap B). \end{aligned} \tag{3.1}$$

It follows that $A \cap B \in \mathcal{B}^{k^*}(\tau)$. □

The following example shows that arbitrary union of k^*-open sets need not to be k^*-open in general.

Example 3.8. Consider the space defined in Example 2.17. For every natural number $n \geq 3$, put $A_n = \mathbb{R} - \{n, n+1, n+2, \ldots\}$. Then, for each $n \geq 3$, $\mathrm{Int}_{\tau^k}(A_n) = \mathrm{Int}_{\tau}(A_n) = A_n - \{1\}$, and thus A_n is k^*-open. On the other hand, $\mathrm{Int}_{\tau^k}(\bigcup_{n\geq 3} A_n) = \bigcup_{n\geq 3} A_n$ while $\mathrm{Int}_{\tau}(\bigcup_{n\geq 3} A_n) = (\bigcup_{n\geq 3} A_n) - \{1\}$.

4. coc-Continuous Functions

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called coc-continuous at a point $x \in X$, if for every open set V containing $f(x)$ there is a coc-open set U containing x such that $f(U) \subseteq V$. If f is coc-continuous at each point of X, then f is said to be coc-continuous.

The following theorem follows directly from the definition.

Theorem 4.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is* coc-*continuous if and only if $f : (X, \tau^k) \to (Y, \sigma)$ is continuous.*

Theorem 4.3. *Every N-continuous function is* coc-*continuous.*

Proof. Straightforward. □

The identity function $I : (\mathbb{R}, \tau_{\text{ind}}) \to (\mathbb{R}, \tau_{\text{disc}})$ is a coc-continuous function that is not N-continuous.

The proof of the following result follows directly from Theorem 2.9.

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function for which (X, τ) is CC, then the following are equivalent.*

(a) *f is continuous.*

(b) *f is N-continuous.*

(c) *f is* coc-*continuous.*

The following example shows that the composition of two N-continuous functions need not to be even coc-continuous.

Example 4.5. Let $X = \mathbb{R}$, $Y = \{0,1,2\}$, $Z = \{a,b\}$, τ be as in Example 2.17, $\sigma = \{\varnothing, Y, \{0\}, \{0,1\}\}$, and $\mu = \{\varnothing, Z, \{a\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(x) = 2$ if $x \in \{0,1\}$ and $f(x) = 1$ otherwise, and define the function $g : (Y,\sigma) \to (Z,\mu)$ by $g(0) = g(2) = a$ and $g(1) = b$. Then f and g are N-continuous functions, but $g \circ f$ is not coc-continuous since $(g \circ f)^{-1}(\{a\}) = \{0,1\} \notin \tau^k$.

Theorem 4.6. (a) *If $f : (X,\tau) \to (Y,\sigma)$ is N-continuous and if $g : (Y,\sigma) \to (Z,\mu)$ is continuous, then $g \circ f : (X,\tau) \to (Z,\mu)$ is N-continuous.*

(b) *If $f : (X,\tau) \to (Y,\sigma)$ is* coc*-continuous and if $g : (Y,\sigma) \to (Z,\mu)$ is continuous, then $g \circ f : (X,\tau) \to (Z,\mu)$ is* coc*-continuous.*

Proof. (a) It follows by noting that a function $f : (X,\tau) \to (Y,\sigma)$ is N-continuous if and only if $f : (X,\tau^n) \to (Y,\sigma)$.

(b) The proof follows directly from Theorem 4.2. □

Theorem 4.7. *If $f : (X,\tau) \to (Y,\sigma)$ is* coc*-continuous and A is a nonempty closed set in (X,τ), then the restriction of f to $A, f|_A : (A,\tau|_A) \to (Y,\sigma)$ is a coc-continuous function.*

Proof. Let V be any open set in Y. Since f is coc-continuous, then $f^{-1}(V)$ is coc-open in X and by Theorem 2.19, $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is coc-open in A. Therefore $f|_A$ is coc-continuous. □

Theorem 4.8. *If $f : (X,\tau) \to (Y,\sigma)$ is coc-continuous and $X = A \cup B$, where A and B are* coc*-closed subsets in (X,τ) and $f|_A : (A,\tau|_A) \to (Y,\sigma)$, $f|_B : (B,\tau|_B) \to (Y,\sigma)$ are* coc*-continuous functions, then f is* coc*-continuous.*

Proof. By Theorem 4.2 it is sufficient to show that $f : (X,\tau^k) \to (Y,\sigma)$ is continuous. Let C be a closed subset of (Y,σ). Then $f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (A \cup B) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$. Since $f|_A : (A,\tau|_A) \to (Y,\sigma)$ is coc-continuous, then $f^{-1}(C) \cap A = (f|_A)^{-1}(C)$ is coc-closed in $(A,\tau|_A)$, and as A is coc-closed in (X,τ), it follows that $f^{-1}(C) \cap A$ is coc-closed in (X,τ); similarly one can conclude that $f^{-1}(C) \cap B$ is coc-closed in (X,τ). It follows that $f^{-1}(C)$ is closed in (X,τ^k) and hence $f : (X,\tau^k) \to (Y,\sigma)$ is continuous. □

The following result follows directly from Theorem 4.2.

Theorem 4.9. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (X,\tau) \to (Z,\mu)$ be two functions. Then the function $h : (X,\tau) \to (Y \times Z, \tau_{\text{prod}})$ defined by $h(x) = (f(x), g(x))$ is* coc*-continuous if and only if f and g are* coc*-continuous.*

Corollary 4.10. *A function $w : (X,\tau) \to (Y,\sigma)$ is* coc*-continuous if and only if the graph function $h : (X,\tau) \to (X \times Y, \tau_{\text{prod}})$, given by $h(x) = (x, w(x))$ for every $x \in X$, is* coc*-continuous.*

Theorem 4.11. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. If there is a* coc*-open subset A of (X,τ) containing $x \in X$ such that the restriction of f to A, $f|_A : (A,\tau|_A) \to (Y,\sigma)$ is* coc*-continuous at x, then f is* coc*-continuous at x.*

Proof. Let $V \in \sigma$ with $f(x) \in V$. Since $f|_A$ is coc-continuous at x, there is $U \in (\tau|_A)^k \subseteq \tau^k$ such that $x \in U$ and $(f|_A)(U) = f(U) \subseteq V$. □

Corollary 4.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, and let $\{A_\alpha : \alpha \in \Delta\} \subseteq \tau^k$ be a cover of X such that, for each $\alpha \in \Delta$, $f|_{A_\alpha} : (A_\alpha, \tau|_{A_\alpha}) \to (Y, \sigma)$ is* coc*-continuous, then f is* coc*-continuous.*

Proof. Let $x \in X$. We show that $f : (X, \tau) \to (Y, \sigma)$ is coc-continuous at x. Since $\{A_\alpha : \alpha \in \Delta\}$ is a cover of X, then there exists $\alpha_\circ \in \Delta$ such that $x \in A_{\alpha_\circ}$. Therefore, by Theorem 4.11, it follows that f is coc-continuous at x. □

Definition 4.13. A function $f : (X, \tau) \to (Y, \sigma)$ is called $(\text{coc})^*$-continuous if the inverse image of each open set is $(\text{coc})^*$-open.

Theorem 4.14. *Every continuous function is $(\text{coc})^*$-continuous.*

Proof. The proof follows directly from Theorem 3.2. □

Theorem 4.15. *If (X, τ) is CC, then every function $f : (X, \tau) \to (Y, \sigma)$ is $(\text{coc})^*$-continuous.*

Proof. The proof follows directly from Theorem 3.3. □

Corollary 4.16. *If (X, τ) is T_2, then every function $f : (X, \tau) \to (Y, \sigma)$ is $(\text{coc})^*$-continuous.*

By Corollary 4.16, it follows that the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ where $f(x) = 0$ for x is rational and $f(x) = 1$ for x is irrational is $(\text{coc})^*$-continuous. On the other hand, it is well known that this function is discontinuous every where. Also, by Theorem 4.4, f is not coc-continuous.

Theorem 4.17. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function with (X, τ) being a hereditarily compact space. Then f is continuous if and only if f is $(\text{coc})^*$-continuous.*

Proof. The proof follows directly from Theorem 3.5. □

By Theorem 4.17, it follows that the identity function $I : (\mathbb{R}, \tau_{\text{ind}}) \to (\mathbb{R}, \tau_{\text{disc}})$ is not $(\text{coc})^*$-continuous. Therefore, this is an example of a coc-continuous function that is not $(\text{coc})^*$-continuous.

We end this section by the following decomposition of continuity via coc-continuity and $(\text{coc})^*$-continuity.

Theorem 4.18. *A function $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is coc-continuous and $(\text{coc})^*$-continuous.*

Proof. The proof follows directly from Theorem 3.6. □

Acknowledgments

The authors are very grateful to the referees for their valuable comments and suggestions. Also, the authors would like to thank Jordan University of Science and Technology for the financial assistant of this paper.

References

[1] H. Z. Hdeib, "ω-closed mappings," *Revista Colombiana de Matemáticas*, vol. 16, no. 1-2, pp. 65–78, 1982.
[2] H. Z. Hdeib, "ω-continuous functions," *Dirasat Journal*, vol. 16, no. 2, pp. 136–153, 1989.

[3] K. Al-Zoubi and B. Al-Nashef, “The topology of ω-open subsets,” *Al-Manarah Journal*, vol. 9, no. 2, pp. 169–179, 2003.
[4] A. Al-Omari and M. Salmi, “New characterization of compact spaces,” in *Proceedings of the 5th Asian Mathematical Conference*, pp. 53–60, Kuala Lumpur, Malaysia, 2009.
[5] A. Al-Omari, T. Noiri, and M. S. Noorani, “Characterizations of strongly compact spaces,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 573038, 9 pages, 2009.
[6] N. Levine, “When are compact and closed equivalent?” *Mathematical Notes*, vol. 72, pp. 41–44, 1965.