Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 386785, 6 pages
doi:10.1155/2012/386785

Research Article

Bounded Approximate Identities in Ternary Banach Algebras

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Received 16 November 2011; Accepted 30 December 2011

Academic Editor: Marcia Federson

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Let A be a ternary Banach algebra. We prove that if A has a left-bounded approximating set, then A has a left-bounded approximate identity. Moreover, we show that if A has bounded left and right approximate identities, then A has a bounded approximate identity. Hence, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] who introduced the notion of cubic matrix which in turn was generalized by Kapranov et al. in [2]. The comments on physical applications of ternary structures can be found in [3–7].

A nonempty set G with a ternary operation $[\cdot,\cdot,\cdot] : G \times G \times G \to G$ is called a ternary groupoid and denoted by $(G, [\cdot,\cdot,\cdot])$. The ternary groupoid $(G, [\cdot,\cdot,\cdot])$ is called a ternary semigroup if the operation $[\cdot,\cdot,\cdot]$ is associative, that is, if

$$[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]] \tag{1.1}$$

holds for all $x, y, z, u, v \in G$. A ternary semigroup $(G, [\cdot,\cdot,\cdot]))$ is a ternary group if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[x, a, b] = [a, y, b] = [a, b, z] = c, \tag{1.2}$$

where the elements x, y, z are uniquely determined (see [8]).

A ternary Banach algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [y, z, u], v]$ and satisfies $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$. An element $e \in A$ is an identity of A if $x = [x, e, e] = [e, e, x]$ for all $x \in A$.

Assume that A is a ternary Banach algebra a bounded net (e_α, f_α) is a left-bounded approximate identity for A if $\lim_\alpha [e_\alpha, f_\alpha, a] = a$ for all $a \in A$. Similarly, a bounded net (e_α, f_α) is a right-bounded approximate identity for A if $\lim_\alpha [a, e_\alpha, f_\alpha] = a$ for all $a \in A$. Also, (e_α, f_α) is a middle-bounded approximate identity for A if $\lim_\alpha [e_\alpha, a, f_\alpha] = a$ for all $a \in A$. A net (e_α, f_α) is a bounded approximate identity for A if (e_α, f_α) is a left-, right-,and middle-bounded approximate identity for A.

For ternary Banach algebra A, a set $U \times V$ is said to be an approximating set for A (U and V are bounded subsets of A) if for every $\epsilon > 0$, and every $a \in A$, there exist $u \in U$, $v \in V$ such that $\|[u, v, a] - a\| < \epsilon$, $\|[u, a, v] - a\| < \epsilon$, $\|[a, u, v] - a\| < \epsilon$.

Existence of bounded approximating set for binary Banach algebras guarantees existing of bounded approximate identity (Altman's Theorem [9, Proposition 2, page 58] or [10]) and also this notion generalized for commutative Fréchet algebras [11]. For normed algebra A with left-bounded approximate identity and right-bounded approximate identity, Dixon [12] proved that A has a bounded approximate identity [13, Proposition 2.9.3].

In this paper, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras. By "$\circ$", we mean the quasiproduct between elements x, y of binary algebra A which are defined by $x \circ y = x + y - xy$.

2. Main Results

We start our work with the following theorem which can be regarded as a version of Altman's Theorem for ternary Banach algebras.

Theorem 2.1. *Let A be a ternary Banach algebra and U, V be bounded subsets of A such that for given $a \in A$ and $\epsilon > 0$ there are $u \in U$ and $v \in V$, $\|[u, v, a] - a\| < \epsilon$. Then A possess a left-bounded approximate identity.*

Proof. Let $\epsilon > 0$, and set

$$W = UV \circ UV = \{(u_1 v_1) \circ (u_2 v_2) : u_1, u_2 \in U,\ v_1 v_2 \in V\}. \tag{2.1}$$

For proof of theorem, it is sufficient to show that for every finite subset $F \subset A$, there exists $w = uv \circ st \in W$ such that $\|[uv \circ st, a] - a\| < \epsilon$ for every $a \in F$.

Step 1. Let $F = \{a\}$ be singleton. Then, there are $u \in U$ and $v \in V$ such that $\|uv\| < M$, and

$$\|[u, v, a] - a\| < \frac{\epsilon}{(M+1)}. \tag{2.2}$$

Letting $w = uv \circ uv$, then

$$\|[uv \circ uv, a] - a\| = \|[u, v, [u, v, a] - a] - ([u, v, a] - a)\| < \epsilon. \tag{2.3}$$

Step 2. Let $F = \{a_1, a_2\}$. There is a $(u_1, v_1) \in U \times V$ such that $\|[u_1, v_1, a_1] - a_1\| < \epsilon/(1 + M)$, and for $[u_1, v_1, a_2] - a_2 \in A$ there is a $(u_2, v_2) \in U \times V$ such that

$$\|[u_2, v_2, [u_1, v_1, a_2] - a_2] - ([u_1, v_1, a_2] - a_2)\| < \epsilon. \tag{2.4}$$

Put $w_1 = u_1 v_1$ and $w_2 = u_2 v_2$. Then

$$\begin{aligned} \|[w_2 \circ w_1, a_i] - a_i\| &= \|[u_2, v_2, a_i] + [u_1, v_1, a_i] - [u_2, v_2, [u_1, v_1, a_i]] - a_i\| \\ &= \|[u_2, v_2, a_i - [u_1, v_1, a_i]] - (a_i - [u_1, v_1, a_i])\| < \epsilon, \end{aligned} \tag{2.5}$$

for $i = 1, 2$.

Step 3. Now, suppose that obtained results in Steps 1 and 2 are true for $i = 1, 2, \ldots, n$. Let $F = \{a_1, a_2, \ldots, a_{n+1}\}$, and let $K = \max\{\|a_i\| : i = 1, \ldots, n\}$. There exist $w_1 \circ w_2 \in W$ such that $\|a_i - [w_2 \circ w_1, a_i]\| < \epsilon/3(M+1)^2$, for $i = 1, 2, \ldots, n$, where w_1 and w_2 are defined as in Step 2. Also, we can choose $\alpha_1 = \theta_1 \eta_1$ and $\alpha_2 = \theta_2 \eta_2$ such that $\alpha_1 \circ \alpha_2 \in W$,

$$\begin{aligned} &\|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \\ &\quad = \left\| \left[\theta_2, \eta_2, w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1]\right] - \left(w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1]\right) \right\| \\ &\quad < \frac{\epsilon}{3K}, \end{aligned} \tag{2.6}$$

and $\|[\alpha_2 \circ \alpha_1, a_{n+1}] - a_{n+1}\| < \epsilon$. Then for every $j = 1, 2, \ldots, n$ we have

$$\begin{aligned} \left\| [\alpha_2 \circ \alpha_1, a_j] - a_j \right\| &\le \left\| a_j - [w_2 \circ w_1, a_j] \right\| + \left\| [\alpha_2 \circ \alpha_1, a_j] - \left[\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]\right] \right\| \\ &\quad + \left\| \left[\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]\right] - [w_2 \circ w_1, a_j] \right\| \\ &\le \left\| a_{j - [w_2 \circ w_1, a_j]} \right\| + \|\alpha_2 \circ \alpha_1\| \left\| a_j - [w_2 \circ w_1, a_j] \right\| \\ &\quad + \|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \left\| a_j \right\| \\ &< \epsilon. \end{aligned} \tag{2.7}$$

Let $F(A)$ be the collection of all finite subsets of A and $\Lambda = \mathbb{N} \times \mathbb{N} \times F(A)$. Then Λ is a direct set with the following partial order:

$$(n_1, m_1, F_1) \leq (n_2, m_2, F_2) \quad \text{iff } F_1 \subseteq F_2,\ n_1 \leq n_2,\ m_1 \leq m_2. \tag{2.8}$$

Now, we can choose a bounded approximate identity $(e_\lambda, f_\lambda)_{\lambda \in \Lambda}$ for A. □

Now, we prove Dixon's Theorem for ternary Banach algebras. Hence, we prove that if a ternary Banach algebra has both left- and right-bounded approximate identities, then it has a bounded approximate identity.

Theorem 2.2. *Let A be a ternary Banach algebra and (e_α, f_α) and (e_β, f_β) be bounded left and right approximate identities of A, respectively. Then A has a bounded approximate identity.*

Proof. Consider $(c_{\alpha,\beta}, d_{\alpha,\beta}) = (e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha)$. We claim that $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a bounded approximate identity for A. Boundedness of mentioned net is clear. Therefore, we have to show that $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a right, left, and middle approximate identity for A.

Step 1. $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a left approximate identity. Because

$$\begin{aligned}
&\| [e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \| \\
&\quad = \| [e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha, a] + [e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \| \\
&\quad \leq \| [e_\alpha f_\alpha, e_\alpha f_\alpha, a] - a \| + \| [e_\alpha f_\alpha, e_\beta f_\beta, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\
&\qquad + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \| + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta, a] - [e_\beta f_\beta, e_\beta f_\beta, a] \| \\
&\qquad + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\
&\quad \leq \| [e_\alpha, f_\alpha, [e_\alpha, f_\alpha, a]] - [e_\alpha, f_\alpha, a] \| + \| [e_\alpha, f_\alpha, a] - a \| + \| e_\alpha f_\alpha \| \| e_\beta f_\beta \| \| a - [e_\alpha, f_\alpha, a] \| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\alpha f_\alpha, a]] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta, a]] - [e_\beta f_\beta, e_\beta f_\beta, a] \| \\
&\qquad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a]] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\
&\quad \leq \frac{5\epsilon}{MN+1} + MN \frac{\epsilon}{MN+1} < \epsilon,
\end{aligned} \tag{2.9}$$

where $\|e_\alpha f_\alpha\| \leq \|e_\alpha\| \|f_\alpha\| \leq M$, and $\|e_\beta f_\beta\| \leq \|e_\beta\| \|f_\beta\| \leq N$.

Step 2. $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a right approximate identity because

$$\begin{aligned}
&\left\| [a, e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \right\| \\
&\quad = \left\| [a, e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha] + [a, e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \right\| \\
&\quad \le \left\| [a, e_\beta f_\beta, e_\beta f_\beta] - a \right\| + \left\| [a, e_\alpha f_\alpha, e_\beta f_\beta] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta] \right\| \\
&\qquad + \left\| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\alpha f_\alpha] \right\| + \left\| [a, e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\beta f_\beta, e_\alpha f_\alpha] \right\| \\
&\qquad + \left\| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\beta f_\beta e_\alpha f_\alpha] \right\| \\
&\quad \le \left\| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \right\| + \left\| [a, e_\beta, f_\beta] - a \right\| \\
&\qquad + \left\| [a, e_\alpha, f_\alpha] - [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] \right\| \left\| e_\beta f_\beta \right\| + \left\| [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] - [a, e_\alpha, f_\alpha] \right\| \left\| e_\alpha f_\alpha \right\| \\
&\qquad + \left\| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \right\| \left\| e_\alpha f_\alpha \right\| \\
&\qquad + \left\| [[a, e_\alpha f_\alpha, e_\beta f_\beta], e_\beta f_\beta] - [a, e_\alpha f_\alpha, e_\beta f_\beta] \right\| \left\| e_\alpha f_\alpha \right\| \\
&\quad \le \frac{2\epsilon}{MN+1} + \frac{3M\epsilon}{MN+1} + \frac{N\epsilon}{MN+1} < \epsilon.
\end{aligned} \tag{2.10}$$

Step 3. By the similar method, we show that the net $(c_{\alpha,\beta}, d_{\alpha,\beta})$ is a middle approximate identity:

$$\begin{aligned}
&\left\| [e_\alpha f_\alpha \circ e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \right\| \\
&\quad = \left\| [e_\alpha f_\alpha, a, e_\beta f_\beta \circ e_\alpha f_\alpha] + [e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \right\| \\
&\quad \le \left\| [e_\alpha f_\alpha, a, e_\beta f_\beta] - a \right\| + \left\| [e_\alpha f_\alpha, a, e_\alpha f_\alpha] - [e_\alpha f_\alpha, a, e_\beta f_\beta e_\alpha f_\alpha] \right\| \\
&\qquad + \left\| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta] - [e_\beta f_\beta, a, e_\beta f_\beta] \right\| + \left\| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \right\| \\
&\qquad + \left\| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \right\| \\
&\quad \le \left\| [e_\alpha, f_\alpha, [a, e_\beta, f_\beta]] - [a, e_\beta, f_\beta] \right\| + \left\| [a, e_\beta, f_\beta] - a \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, a] - [[e_\alpha, f_\alpha, a], e_\beta, f_\beta] \right\| \left\| e_\alpha f_\alpha \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta]] - [e_\beta f_\beta, a, e_\beta f_\beta] \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \right\| \\
&\qquad + \left\| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \right\| \\
&\quad \le \frac{5\epsilon}{MN+1} + \frac{M\epsilon}{MN+1} < \epsilon.
\end{aligned} \tag{2.11}$$

This completes the proof of theorem. □

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