

## Research Article

# Symmetries of $n$ th-Order Approximate Stochastic Ordinary Differential Equations

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Symmetries of  $n$ th-order approximate stochastic ordinary differential equations (SODEs) are studied. The determining equations of these SODEs are derived in an Itô calculus context. These determining equations are not stochastic in nature. SODEs are normally used to model nature (e.g., earthquakes) or for testing the safety and reliability of models in construction engineering when looking at the impact of random perturbations.

## 1. Introduction

The modelling power of a SODE has been applied to many diverse fields of research, from the modelling of turbulent diffusion to neuronal activity in the brain. Models such as these are often influenced by more than one Wiener process. In these models, we assume that the Wiener processes are independent of one another. As a result of this increase in the number of Wiener processes affecting the model, the form of the Itô formula is slightly different to the one used in Fredericks and Mahomed [1, 2]. The Itô formula is able to relate an arbitrary sufficiently smooth function  $F(t, x)$  of time and space to a particular SODE, of which it is a solution. This formula, however, needs the SODE of the spatial random process  $X(t, \omega)$  which drives the arbitrary function  $F(X(t, \omega), \omega)$ . The application of an SODE to an approximate analysis algorithm has been done by Ibragimov et al. [3] for scalar SODEs of first order. We extend this work for higher dimensions and order. We derive a similar conditioning on the temporal infinitesimal  $\tau$  as had been performed by Ünal [4] and Fredericks and Mahomed [2]. We introduce operators to write the determining equations in a neater form.

Wafu Soh and Mahomed [5] gave an algorithm to obtain Lie point symmetries for both first- and  $n$ th-order SODEs. We briefly review their work and follow up with an extension from point symmetries to generalized symmetries.

The first section begins with the transformations of the spatial, temporal, and Wiener variables for an  $n$ th-order Itô process. These transformations have the same properties as stated in Fredericks and Mahomed [6].

Using the Itô formula in conjunction with the infinitesimal transformations which preserve form invariance, we derive conditions for  $n$ th-ordered SODEs that ensure the recovery of invariance preserving finite transformations from the infinitesimal ones. This has not been done in the past.

This is followed up with the development of recursive relations needed for finding the prolonged spatial infinitesimals in a SODE context by using the concept of form invariance. This differs from the methodology used by [5], where the recursive relation defined was predefined from an ODE context. As a result we also derive a conditioning on these prolonged spatial infinitesimal variables. We further derive a conditioning on the diffusion coefficient of the temporal generalised symmetry  $\tau$ , which is similar to that of Ünal [4]. We conclude the article with an introduction of operators which generalize the determining equations for any order SODE that is adaptable to both point and generalized symmetries.

The symmetries of high-ordered multidimensional stochastic ordinary differential equations (SODEs) are found using form invariance arguments on both the instantaneous drift and diffusion properties of the SODEs. We then apply this work to a generalized approximation analysis algorithm. The determining equations of SODEs are derived in an Itô calculus context. We also use the random time change formula used by Wafu Soh and Mahomed [5] to transform the Wiener processes.

## 2. Review of Wafu Soh and Mahomed [5] for $n$ th-Order SODEs

An  $n$ th-order Itô process has the following vector form:

$$d\mathbf{X}^{(n-1)}(t) = \mathbf{f}(t, \mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t))dt + \mathbf{G}(t, \mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t))d\mathbf{W}(t), \quad (2.1)$$

$$dX_j^{(k)}(t) = X_j^{(k+1)}(t)dt, \quad (2.2)$$

$$X_j^{(0)}(t) = X_j(t) \quad (2.3)$$

for  $k = 0, 1, \dots, n - 2$ . Since the instantaneous mean,  $\mathbf{f}$ , is an  $N$ -vector valued function, the index  $j$  runs from one to  $N$ , that is,  $j = 1, \dots, N$ . The diffusion coefficient  $\mathbf{G}$  is an  $N \times M$ -matrix valued function and  $\mathbf{W}(t)$  is an  $M$ -dimensional standard Wiener process. From here onwards we denote  $\{\mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t)\}$  by  $\mathcal{X}^{(n-1)}(t)$ . The context of this processes is that both the instantaneous drift and diffusion coefficients are Lipschitz continuous with respect to the right norm. A good example of the type of norm used for this is given by [7] in their seventh chapter.

The Lie point transformation methodology used by Wafu Soh and Mahomed [5] does all calculations to  $\mathcal{O}(\theta)$ . As a result, the recoverability of the finite transformations, which

keep invariance, from the infinitesimal ones is not verified. The symmetry operator  $H$  of point symmetry is

$$H = \tau(x, t) \frac{\partial}{\partial t} + \xi_j(x, t) \frac{\partial}{\partial x_j}, \quad (2.4)$$

where there is summation  $j = 1, N$ . However since we are dealing with an  $n$ th-ordered SODE, prolongation formulation is necessary. In the Banach space, the transformation for the  $(n - 1)$ th-ordered spatial derivative is

$$\begin{aligned} \bar{x}^{(n-1)} &= e^{\theta H^{[n-1]}} x^{(n-1)}, \\ \bar{x}^{(k)} &= e^{\theta H^{[n-1]}} x^{(k)} \\ &= e^{\theta H^{[k]}} x^{(k)}, \quad k < n - 1, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} H^{[n-1]} &= H^{(n-2)} + \xi_j^{[n-1]} \frac{\partial}{\partial x_j^{(n-1)}}, \quad n \geq 1, \\ H^{[0]} &= H. \end{aligned} \quad (2.6)$$

Applying Itô's formula on an arbitrarily ordered prolongation of a spatial infinitesimal,  $\xi_j^{[r]}(t, \mathcal{X}^{(r-1)}(t))$ , gives

$$d\xi_j^{[r]}(t, \mathcal{X}^{(r-1)}(t)) = f_{(\xi^{[r]})j}(t, \mathcal{X}^{(r-1)}(t))dt + G_{(\xi^{[r]})j}^l(t, \mathcal{X}^{(r-1)}(t))dW_i(t), \quad (2.7)$$

where

$$\begin{aligned} f_{(\xi^{[r]})j}(t, \mathcal{X}^{(r-1)}(t)) &= \frac{\partial \xi_j^{[r]}}{\partial t} + f_i \frac{\partial \xi_j^{[r]}}{\partial x_i^{(n-1)}} + \frac{1}{2} \sum_{k=1}^M G_i^k G_p^k \frac{\partial^2 \xi_j^{[r]}}{\partial x_i^{(n-1)} \partial x_p^{(n-1)}} \\ &\quad + \sum_{\alpha=0}^{n-2} x_p^{(\alpha+1)} \frac{\partial \xi_j^{[r]}}{\partial x_p^{(\alpha)}}, \quad \text{where } n \geq 2; \\ G_{(\xi^{[r]})j}^l(t, \mathcal{X}^{(r-1)}(t)) &= \frac{\partial \xi_j^{[r]}}{\partial x_i^{(n-1)}} G_i^l r, \quad \text{for each } j \text{ ranging from } 1 \text{ to } N. \end{aligned} \quad (2.8)$$

If the summation operator runs from a nonnegative value, for example, 0, to a negative one; that is,  $-1$ , the outcome of the entire summation is set to zero. With this convention, we are able to recover the Itô formula for first-order SODEs. Due to the repeated index summation convention, the spatial indices  $i$  and  $p$  both run from 1 to  $N$  in the summation; the Wiener

indices  $l$  and  $k$  run from 1 to  $M$ . Similarly, the Itô's formula for the temporal variable,  $\tau(x, t)$ , gives

$$d\tau = f_{(\tau)}(\mathbf{X}(t, \omega), t)dt + G_{(\tau)}^l(\mathbf{X}(t, \omega), t)dW_l(\mathbf{X}(t, \omega), t), \quad (2.9)$$

where

$$f_{(\tau)}(\mathbf{X}(t, \omega), t) = \frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(n-1)}} + \frac{1}{2} \sum_{k=1}^M G_i^k G_p^k \frac{\partial^2 \tau}{\partial x_i^{(n-1)} \partial x_p^{(n-1)}} + \sum_{\alpha=0}^{n-2} x_p^{(\alpha+1)} \frac{\partial \tau}{\partial x_p^{(\alpha)}}, \quad (2.10)$$

which reduces to the total derivative, since the temporal infinitesimal is a point transformation

$$f_{(\tau)}(\mathbf{X}(t, \omega), t) = D(\tau), \quad (2.11)$$

where the total derivative is defined as

$$D = \frac{\partial}{\partial t} + \sum_{\alpha=0}^{n-2} x_p^{(\alpha+1)} \frac{\partial}{\partial x_p^{(\alpha)}}. \quad (2.12)$$

The diffusion coefficient of the temporal infinitesimal is given by

$$G_{(\tau)}^l(\mathbf{X}(t, \omega), t) = G_i^l \frac{\partial \tau}{\partial x_i^{(n-1)}} \quad (2.13)$$

reduces to zero as well because of the fact that we are dealing with point transformations, that is,

$$G_{(\tau)}^l(\mathbf{X}(t, \omega), t) = 0. \quad (2.14)$$

The drift and diffusion coefficients of the  $(n-1)$ th-order spatial derivative are, respectively, transformed as

$$f_j \left( \overline{\mathcal{X}}^{(n-1)}(\bar{t}), \bar{t} \right) = f_j \left( \mathcal{X}^{(n-1)}(t), t \right) + \theta H^{[n-1]} \left( f_j \left( \mathcal{X}^{(n-1)}(t), t \right) \right) + \mathcal{O}(\theta^2), \quad (2.15)$$

$$G_j^l \left( \overline{\mathcal{X}}^{(n-1)}(\bar{t}), \bar{t} \right) = G_j^l \left( \mathcal{X}^{(n-1)}(t), t \right) + \theta H^{[n-1]} \left( G_j^l \left( \mathcal{X}^{(n-1)}(t), t \right) \right) + \mathcal{O}(\theta^2). \quad (2.16)$$

The Itô formula of the finite time index transformation is

$$d\bar{t} = \Gamma \left( e^{\theta H^{[n-1]}}(t) \right) dt + Y^l \left( e^{\theta H^{[n-1]}}(t) \right) dW_l, \quad (2.17)$$

which Wafo Soh and Mahomed [5] simply write as

$$d\bar{t} = dt(1 + \theta D(\tau)) + \mathcal{O}(\theta^2), \quad (2.18)$$

since the temporal infinitesimal is a point transformation. We also have that the transformed time index should keep invariance in the following probabilistic way:

$$\mathbb{E}_{\mathbb{Q}}[d\bar{t}(t, \omega)] = d\bar{t}(t, \omega). \quad (2.19)$$

This requires

$$Y^l(e^{\theta H^{[n-1]}}(t)) = 0, \quad l = 1, M, \quad (2.20)$$

which is automatically satisfied since  $\tau$  is point transformation. Condition (2.19) also forces

$$D(e^{\theta H^{[n-1]}}(t)) = \text{Constant}, \quad (2.21)$$

which is overlooked in [5]. Thus the finite transformation of the Wiener process is

$$d\bar{W}_1(\bar{t}, \omega) = \sqrt{D(e^{\theta H^{[n-1]}}(t))} dW_1(t, \omega), \quad (2.22)$$

which Wafo Soh and Mahomed [5] simplify as

$$d\bar{W}_1(\bar{t}, \omega) = dW_1(t, \omega) \left(1 + \frac{\epsilon}{2} D(\tau)\right) + \mathcal{O}(\theta^2), \quad (2.23)$$

where [5] used a generalized binomial expansion of the square-root of the derivative of the transformed time index with respect to time. The Itô SODE associated with Lie point  $n$ th-ordered spatial transformation is

$$d\bar{X}_j^{(n-1)}(\bar{t}) = dX_j^{(n-1)}(t) + \theta \left( f_{(\xi^{[n-1]})_j} dt + G_{(\xi^{[n-1]})_j}^l dW_1(t) \right) + \mathcal{O}(\theta^2). \quad (2.24)$$

Wafo Soh and Mahomed [5] make the assumption that only the system of  $n$ th order SODEs, (2.1), remains invariant under the symmetry operator (2.4), which implies that

$$d\bar{X}_j^{(n-1)}(\bar{t}) = f_j \left( \bar{t}, \bar{\mathcal{X}}^{(n-1)}(\bar{t}, \omega) \right) d\bar{t} + G_j^l \left( \bar{t}, \bar{\mathcal{X}}^{(n-1)}(\bar{t}, \omega) \right) d\bar{W}_l(\bar{t}), \quad (2.25)$$

where we denote  $\{\bar{X}(\bar{t}), \bar{X}(\bar{t}), \dots, \bar{X}^{(r-1)}(\bar{t})\}$  by  $\bar{\mathcal{X}}^{(r-1)}(\bar{t})$  for an arbitrary  $r \in \mathbb{N}$ .

Expanding the drift component  $f_j(\bar{t}, \bar{\mathcal{X}}^{(n-1)}(\bar{t}, \omega))d\bar{t}$  of (2.25) using (2.15) and (2.18) gives

$$\begin{aligned} \mathbf{f}(t, \mathcal{X}^{(n-1)}(t))d\bar{t} = & \left\{ \mathbf{f}(t, \mathcal{X}^{(n-1)}(t)) + \theta(D(\tau) + H^{[n-1]})\mathbf{f}(t, \mathcal{X}^{(n-1)}(t)) \right. \\ & + \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \left( (D(\tau) + H^{[n-1]})^k \mathbf{f}(t, \mathcal{X}^{(n-1)}(t)) \right. \\ & + \sum_{j=0}^{k-2} \binom{k}{k-j} H^j(\mathbf{f}(t, \mathcal{X}^{(n-1)}(t))) \\ & \left. \left. \times (D(H^{k-j}(t)) - [D(\tau)]^{k-j}) \right) \right\} dt. \end{aligned} \quad (2.26)$$

In order for the finite transformations to keep invariance, we require a higher ordered  $\theta$ -terms to be solely dependent on the  $\mathcal{O}(1)$  and  $\mathcal{O}(\theta)$  terms, this forces the condition

$$e^{\theta D(\tau)} = D(e^{\theta H^{[n-1]}}(t)), \quad (2.27)$$

which is satisfied as a result of condition (2.21). Whence the finite transformation of the Wiener process becomes

$$d\bar{W}_i(\bar{t}, \omega) = e^{\theta D(\tau)/2} dW_i(t, \omega). \quad (2.28)$$

The diffusion component of (2.25) can easily be expanded with the utility of (2.16) and (2.23)

$$\begin{aligned} G_j^l(t, \mathcal{X}^{(n-1)}(t))d\bar{W}_i = & \left\{ G_j^l(t, \mathcal{X}^{(n-1)}(t)) + \theta \left( \frac{D(\tau)}{2} + H^{[n-1]} \right) G_j^l(t, \mathcal{X}^{(n-1)}(t)) \right. \\ & \left. + \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \left( \frac{D(\tau)}{2} + H^{[n-1]} \right)^k G_j^l(t, \mathcal{X}^{(n-1)}(t)) \right\} dW_i. \end{aligned} \quad (2.29)$$

This allows us to make a comparison with the Itô SODE associated with the  $n$ th-ordered spatial transformation (2.24), which furnishes the determining equations used by Wafo Soh and Mahomed [5], that is,

$$\begin{aligned} f_{(\xi^{[n-1]})_j} &= (D(\tau) + H^{[n-1]})f_j(\mathcal{X}^{(n-1)}(t), t), \\ G_{(\xi^{[n-1]})_j}^l &= \left( \frac{D(\tau)}{2} + H^{[n-1]} \right) G_j^l(\bar{\mathcal{X}}^{(n-1)}(\bar{t}), \bar{t}). \end{aligned} \quad (2.30)$$

Constructing the prolonged variables was carried out in [5] by using preexisting recursive relations based on the Lie point theory for ODEs, that is,

$$\xi_j^{[k]} = D\xi_j^{[k-1]} - x_j^{(k)}D_\tau, \quad \xi_j^{[0]} = \xi_j, \quad (2.31)$$

for  $k \leq n$ . The sketch of the methodology used for Lie point symmetries for  $n$ th-ordered SODEs by Wafo Soh and Mahomed [5] ends here. However, it is possible to construct the recursive relations using form invariance arguments on the SODEs described in (2.2), that is,

$$d\bar{X}_j^{(k)}(\bar{t}) = \bar{X}^{(k+1)}(\bar{t})d\bar{t}, \quad (2.32)$$

which after expanding yields the following  $\theta$ -ordered relations:

$$d\bar{X}_j^{(k)}(\bar{t}) = X^{(k+1)}dt + \theta(\xi_j^{[k+1]} + x_j^{(k+1)}D(\tau))dt + \mathcal{O}(\theta^2), \quad (2.33)$$

$$\Gamma(\xi^{[k]}) = (\Gamma(\tau) + H^{[k+1]})X_i^{(k+1)}, \quad (2.34)$$

$$G_i^j \frac{\partial \xi^{[k]}}{\partial x_i^{(n-1)}} = 0, \quad \text{for } k < n - 1 \quad (2.35)$$

a new condition, which is not mentioned in [5], which is automatically satisfied since the terms  $\xi^{[k]}$ , where  $k < n - 1$ , are not functions of  $x^{(n-1)}$ . In conjunction with this, we have the Itô SODE associated with the transformation of the  $k$ th-ordered spatial transformation, that is,

$$d\bar{X}_j^{(k)}(\bar{t}) = dX_j^{(k)}(t) + \theta\left(f_{(\xi^{[k]})_j}dt + G_{(\xi^{[k]})_j}^l dW_l(t)\right) + \mathcal{O}(\epsilon^2), \quad (2.36)$$

which reduces to

$$d\bar{X}_j^{(k)}(\bar{t}) = dX_j^{(k)}(t) + \theta D(\xi_j^{[k]})dt + \mathcal{O}(\theta^2) \quad (2.37)$$

as a result of the fact that the lower ordered prolongation infinitesimals  $\xi_j^{[k]}$ , are not a function of  $x^{(n-1)}$  for  $k < (n - 1)$ . Thus the recursive relations, defined by Wafo Soh and Mahomed [5] from an ODE context, are easily derived using a form invariance philosophy, namely

$$D(\xi_j^{[k]}) = \xi_j^{[k+1]} + x_j^{(k+1)}D(\tau). \quad (2.38)$$

We now adapt the relations (2.30), (2.31), and (2.35) to an approximate SODE.

### 3. Symmetries of $n$ th Order Multidimensional Approximate Stochastic Ordinary Differential Equations

We now consider the following:

$$\begin{aligned} d\mathbf{X}^{(n-1)}(\vec{t}) &= \mathbf{f}\left(t, \mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t), R_\mu\right) dt + \dots + \mathbf{G}\left(t, \mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t), R_\nu\right) d\mathbf{W}(t), \\ dX_i^{(k)}(t) &= X_i^{(k+1)} dt, \\ X_i^{(0)}(t) &= X_i(t) \end{aligned} \quad (3.1)$$

for  $k = 0, 1, \dots, n-2$ . The function  $\mathbf{f}$  is an approximate drift, which is an  $N$  vector-valued function,  $i = 1, \dots, N$ .  $\mathbf{G}$  is an  $N \times M$  matrix-valued function approximating diffusion and  $\mathbf{W}(t)$  is an  $M$ -dimensional Wiener process. Here  $\mathbf{f}$  and  $\mathbf{G}$  are defined as follows:

$$\mathbf{f}\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t), R_\mu\right) = e^{r\mu} \mathbf{f}^r\left(t, \dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t)\right), \quad (3.2)$$

where the repeated index  $r$  runs from 0 to  $R_\mu$ , where  $R_\mu$  is the largest positive integer such that  $\mu R_\mu < 2\rho$  and

$$\mathbf{G}\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t), R_\nu\right) = e^{r\nu} \mathbf{G}^r\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t)\right), \quad (3.3)$$

where the repeated index  $r$  runs from 0 to  $R_\nu$ ;  $R_\nu$  is the largest positive integer such that  $\nu R_\nu < 2\rho$ . The order of accuracy to which we choose to work is  $\rho$ .

The spatial and temporal variables of our infinitesimal generator

$$H = \tau(t, \mathbf{x}, \rho) \frac{\partial}{\partial t} + \xi_j(t, \mathbf{x}, \rho) \frac{\partial}{\partial x_j}, \quad (3.4)$$

are defined as

$$\begin{aligned} \tau(t, \mathbf{x}, \rho) &= e^r \tau^r(t, \mathbf{x}), \\ \xi(t, \mathbf{x}, \rho) &= e^r \xi^r(t, \mathbf{x}). \end{aligned} \quad (3.5)$$

The repeated index runs from 0 to  $\rho$ , since throughout this paper we will be working on  $O(\epsilon^\rho)$ . Using Itô's formula on the  $(n-1)$ th-prolongation of the spatial, we get

$$\begin{aligned}
d\xi_j^{[n-1]} &= \left( \frac{\partial \xi_j^{[n-1]}}{\partial t} + f_i \frac{\partial \xi_j^{[n-1]}}{\partial x_i^{(n-1)}} + \frac{1}{2} \sum_{s=1}^M G_i^s G_k^s \frac{\partial^2 \xi_j^{[n-1]}}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}} + \sum_{\alpha=0}^{n-2} x_k^{(\alpha+1)} \frac{\partial \xi_j^{[n-1]}}{\partial x_k^{(\alpha)}} \right) dt \\
&\quad + \cdots + \frac{\partial \xi_j^{[n-1]}}{\partial x_i^{(n-1)}} G_i^k dW_k(t) \\
&= \left( \left( \frac{1}{2} \sum_{s=1}^M G_i^{0s} G_k^{0s} \frac{\partial^2 \xi_j^{[n-1]}}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}} + f_i \frac{\partial \xi_j^{[n-1]}}{\partial x_i} + \frac{\partial \xi_j^{[n-1]}}{\partial t} + \sum_{\alpha=0}^{n-2} x_k^{(\alpha+1)} \frac{\partial \xi_j^{[n-1]}}{\partial x_k^{(\alpha)}} \right) \epsilon^l \right. \\
&\quad + \cdots + \frac{1}{2} \sum_{s=1}^M G_i^{ps} G_k^{ps} \frac{\partial^2 \xi_j^{[n-1]}}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}} \epsilon^{l+2pv} + \sum_{s=1}^M G_i^{rs} G_k^{ps} \frac{\partial^2 \xi_j^{[n-1]}}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}} \epsilon^{l+v(r+p)} \\
&\quad \left. + \cdots + f_i \frac{\partial \xi_j^{[n-1]}}{\partial x_i^{(n-1)}} \epsilon^{l+\mu q} \right) dt + \cdots + G_i^{ps} \frac{\partial \xi_j^{[n-1]}}{\partial x_s^{(n-1)}} \epsilon^{l+vp} dW^{(i)}
\end{aligned} \tag{3.6}$$

and on the temporal infinitesimal

$$\begin{aligned}
d\tau &= D(\tau)dt \\
&= \epsilon^l \left( \frac{\partial \tau^l}{\partial t} + \sum_{\alpha=0}^{n-2} x_k^{(\alpha+1)} \frac{\partial \tau^l}{\partial x_k^{(n-1)}} \right) dt
\end{aligned} \tag{3.7}$$

with

$$G_i^s \frac{\partial \tau^l}{\partial x_s^{(n-1)}} \epsilon^{l+vp} = 0, \tag{3.8}$$

which is automatically satisfied since  $\tau$  is point symmetry. The repeated indices  $r, p, q$ , and  $l$  run from 0 to  $R_\nu - 1, R_\nu, R_\mu$ , and  $\rho$ , respectively in our repeated index summation convention;  $r < p$ . Thus, by substitution, we get

$$\begin{aligned}
d\bar{X}^{(n-1)} &= dX^{(n-1)} + \cdots + \theta \left( \frac{\partial \xi^{[n-1]}}{\partial t} + f_i \frac{\partial \xi^{[n-1]}}{\partial x_i^{(n-1)}} + \frac{1}{2} \sum_{k=1}^M G_i^k G_j^k \frac{\partial^2 \xi^{[n-1]}}{\partial x_i^{(n-1)} \partial x_j^{(n-1)}} \right. \\
&\quad \left. + \sum_{\alpha=0}^{n-2} x_j^{(\alpha+1)} \frac{\partial \xi^{[n-1]}}{\partial x_j^{(\alpha)}} \right) dt + \theta G_j^i \frac{\partial \xi^{[n-1]}}{\partial x_i^{(n-1)}} dW_t^{(j)} + \mathcal{O}(\theta^2)
\end{aligned}$$

$$\begin{aligned}
d\bar{t} &= dt + \theta D(\tau)dt + \mathcal{O}(\theta^2), \\
d\bar{W}_{\bar{t}} &= dW_t^{(l)} \left( 1 + \frac{\theta}{2} D(\tau) \right) + \mathcal{O}(\theta^2).
\end{aligned}
\tag{3.9}$$

The transformation of  $\mathbf{f}$  and  $\mathbf{G}$  under our prolonged infinitesimal generator  $H^{[\beta]}$  is

$$\begin{aligned}
f_i(\bar{t}, \widehat{\mathcal{X}}^{(n-1)}) &= f_i(t, \mathcal{X}^{(n-1)}) + \theta H^{[\beta]} f_i(t, \mathcal{X}^{(n-1)}) + \mathcal{O}(\theta^2) \\
&= f_i + \theta \left( \tau(t, \dot{\mathbf{X}}(t)) \frac{\partial f_i}{\partial t} + \xi_j^{[\beta]}(t, \mathcal{X}^{(n-1)}) \frac{\partial f_i}{\partial x_j^{(\beta)}} \right) + \mathcal{O}(\theta^2) \\
&= \epsilon^q f_i^q + \theta \xi_j^{l[\beta]} \frac{\partial f_i^q}{\partial x_j^{(\beta)}} \epsilon^{l+\mu q} + \theta \tau^l \frac{\partial f_i^q}{\partial t} \epsilon^{l+\mu q} + \mathcal{O}(\theta^2) \\
&= \epsilon^q f_i^q + \theta \epsilon^{l+\mu q} \left( \xi_j^{l[\beta]} \frac{\partial f_i^q}{\partial x_j^{(\beta)}} + \tau^l \frac{\partial f_i^q}{\partial t} \right) + \mathcal{O}(\theta^2) \\
G_k^i(\bar{t}, \widehat{\mathcal{X}}^{(n-1)}) &= G_k^i(t, \mathcal{X}^{(n-1)}) + \theta H^{[\beta]} G_k^i(t, \mathcal{X}^{(n-1)}) + \mathcal{O}(\theta^2) \\
&= G_k^i + \theta \left( \tau(t, \dot{\mathbf{X}}(t)) \frac{\partial G_k^i}{\partial t} + \xi_j^{l[\beta]}(t, \mathcal{X}^{(n-1)}) \frac{\partial G_k^i}{\partial x_j^{(\beta)}} \right) + \mathcal{O}(\theta^2) \\
&= \epsilon^p G_k^{pi} + \theta \epsilon^{\nu p+l} \left( \xi_j^{l[\beta]} \frac{\partial G_k^{pi}}{\partial x_j^{(\beta)}} + \tau^l \frac{\partial G_k^{pi}}{\partial t} \right) + \mathcal{O}(\theta^2),
\end{aligned}
\tag{3.10}$$

where  $\{X, \dot{X}, \dots, X^{(n-1)}\}$  is represented by  $\mathcal{X}^{(n-1)}$  and the transformed set  $\{\bar{X}, \dot{\bar{X}}, \dots, \bar{X}^{(n-1)}\}$  is represented by  $\widehat{\mathcal{X}}^{(n-1)}$ . The repeated indices  $q, p, l$ , and  $n$  run from 0 to  $R_\mu, R_\nu, \rho$ , and  $n-1$ , respectively.

### 3.1. Operators

Thus the determining equations (2.30), (2.31), and (2.35) become

$$\epsilon^{\mu q+l} \left( f_m^q \left( \Gamma^0(\tau^l) + \epsilon^{\nu(r+p)} \Upsilon_{rp}(\tau^l) + \epsilon^{2\nu p} \Upsilon_p(\tau^l) + \epsilon^{\mu j} \Psi_j(\tau^l) \right) + H_\beta^l(f_m^q) \right)$$

$$\begin{aligned}
 & + \dots - \epsilon^l \left( \Gamma^0(\xi_m^{l[n-1]}) + \epsilon^{\nu(r+p)} \mathcal{U}_{rp}(\xi_m^{l[n-1]}) \right. \\
 & \quad \left. + \epsilon^{2\nu p} \Upsilon_p(\xi_m^{l[n-1]}) + \epsilon^{\mu q} \Psi_q(\xi_m^{l[n-1]}) \right) = 0,
 \end{aligned} \tag{3.11}$$

$$\epsilon^{l+\nu p} \left( H_\beta^l(G_k^{pm}) + \frac{1}{2} G_k^{pm} \right) \tag{3.12}$$

$$\begin{aligned}
 & \times \left( \Gamma^0(\tau^l) + \epsilon^{\nu(r+p)} \mathcal{U}_{rp}(\tau^l) + \epsilon^{2\nu p} \Upsilon_p(\tau^l) + \epsilon^{\mu q} \Psi_q(\tau^l) \right) - \Upsilon_{n-1}^{pk}(\xi_m^{l[n-1]}) = 0, \\
 & \epsilon^{l+\nu p} \Upsilon_{n-1}^{pi}(\tau^l) = 0,
 \end{aligned} \tag{3.13}$$

$$\epsilon^{l+\nu p} \Upsilon_{n-1}^{pi}(\xi_i^{l[k]}) = 0, \quad k < n - 1, \tag{3.14}$$

$$\begin{aligned}
 & \epsilon^l \left( \Gamma^0(\xi_m^{l[k]}) + \epsilon^{\nu(r+p)} \mathcal{U}_{rp}(\xi_m^{l[k]}) + \epsilon^{2\nu p} \Upsilon_p(\xi_m^{l[k]}) + \epsilon^{\mu q} \Psi_q(\xi_m^{l[k]}) \right) \\
 & = \epsilon^l \left( x_i^{(k+1)} \left( \Gamma^0(\tau^l) + \epsilon^{\nu(r+p)} \mathcal{U}_{rp}(\tau^l) + \epsilon^{2\nu p} \Upsilon_p(\tau^l) + \epsilon^{\mu j} \Psi_j(\tau^l) \right) + \xi_i^{l[k+1]} \right),
 \end{aligned} \tag{3.15}$$

respectively, where

$$\Gamma^0 = \frac{1}{2} \sum_{s=1}^M G_i^{0s} G_k^{0s} \frac{\partial^2}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}} + f_i^0 \frac{\partial}{\partial x_i^{(n-1)}} + \frac{\partial}{\partial t} + \dots + \sum_{\alpha=0}^{n-1-1} x_k^{(\alpha+1)} \frac{\partial}{\partial x_k^{(n-1)}}, \tag{3.16}$$

$$\mathcal{U}_{rp} = \sum_{s=1}^M G_i^{rs} G_k^{ps} \frac{\partial^2}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}}, \quad (0 \leq r < p \leq R_\nu), \tag{3.17}$$

$$\Upsilon_p = \frac{1}{2} \sum_{s=1}^M G_i^{ps} G_k^{ps} \frac{\partial^2}{\partial x_i^{(n-1)} \partial x_k^{(n-1)}}, \quad (0 < p \leq R_\nu), \tag{3.18}$$

$$\Psi_q = f_i^q \frac{\partial}{\partial x_i^{(n-1)}}, \quad (0 < q, j \leq R_\mu), \tag{3.19}$$

$$\Upsilon_{n-1}^{pi} = G_i^{pk} \frac{\partial}{\partial x_k^{(n-1)}}, \quad (0 \leq p \leq R_\nu), \tag{3.20}$$

$$H_\beta^l = \tau^l \frac{\partial}{\partial t} + \xi_j^{l[\beta]} \frac{\partial}{\partial x_j^{(\beta)}}, \quad (0 \leq \beta \leq n - 1). \tag{3.21}$$

Note that we cannot cancel out the terms  $\epsilon^l$  and  $\epsilon^{l+\nu p}$  in (3.11) and (3.12), respectively, in order to simplify them. These terms are a part of the summation convention implied by the repeated indices. These terms contribute to the order of error as a result of this implication.

We now apply our generalized methodology for finding approximate symmetries to the Itô system considered in [3]. Our application should be consistent with the determining equations found in Ibragimov et al. [3].

### 3.1.1. Example 1

For their approximate stochastic ordinary differential equations,  $n - 1 = 0$ ,  $\mu = 1$ ,  $\nu = 1/2$ ,  $R_\mu = 1$ ,  $R_\nu = 1$ , and  $\rho = 1$ . Thus the diffusion coefficient  $\mathbf{G}$ , which was taken to be constant, and the drift  $\mathbf{f}$  appeared as follows in the Itô system:

$$dx = (\mathbf{f}^0 + \epsilon \mathbf{f}^1) dt + \sqrt{\epsilon} \mathbf{G} d\mathbf{W}_t, \quad (3.22)$$

where the drift is a  $N \times 1$  vector and the constant diffusion coefficient is a matrix with dimension  $N \times M$ . The determining equations are

$$\begin{aligned} & -\frac{1}{2} \sum_{s=1}^M G_k^{1s} G_i^{1s} \epsilon \left( \frac{\partial^2 \xi_j^0}{\partial x_i \partial x_k} + \epsilon \frac{\partial^2 \xi_j^1}{\partial x_i \partial x_k} \right) + (\xi_i^0 + \epsilon \xi_i^1) \left( \frac{\partial f_j^0}{\partial x_i} + \epsilon \frac{\partial f_j^1}{\partial x_i} \right) - f_i^0 \left( \frac{\partial \xi_j^0}{\partial x_i} + \epsilon \frac{\partial \xi_j^1}{\partial x_i} \right) \\ & - \epsilon f_i^1 \left( \frac{\partial \xi_j^0}{\partial x_i} + \epsilon \frac{\partial \xi_j^1}{\partial x_i} \right) + (\tau^0 + \epsilon \tau^1) \left( \frac{\partial f_j^0}{\partial t} + \epsilon \frac{\partial f_j^1}{\partial t} \right) - \frac{\partial \xi_j^0}{\partial t} - \epsilon \frac{\partial \xi_j^1}{\partial t} + (f_j^0 + \epsilon f_j^1) \\ & \times \left( \frac{1}{2} \epsilon \sum_{s=1}^M G_k^{1s} G_i^{1s} \left( \frac{\partial^2 \tau^0}{\partial x_i \partial x_k} + \epsilon \frac{\partial^2 \tau^1}{\partial x_i \partial x_k} \right) + f_i^0 \left( \frac{\partial \tau^0}{\partial x_i} + \epsilon \frac{\partial \tau^1}{\partial x_i} \right) \right. \\ & \quad \left. + \epsilon f_i^1 \left( \frac{\partial \tau^0}{\partial x_i} + \epsilon \frac{\partial \tau^1}{\partial x_i} \right) + \frac{\partial \tau^0}{\partial t} + \epsilon \frac{\partial \tau^1}{\partial t} \right) = 0, \\ & \sqrt{\epsilon} (\xi_i^0 + \epsilon \xi_i^1) \frac{\partial G_k^{1j}}{\partial x_i} - \sqrt{\epsilon} G_k^{1i} \left( \frac{\partial \xi_j^0}{\partial x_i} + \epsilon \frac{\partial \xi_j^1}{\partial x_i} \right) + \sqrt{\epsilon} (\tau^0 + \epsilon \tau^1) \frac{\partial G_k^{1j}}{\partial t} + \frac{1}{2} \sqrt{\epsilon} G_k^{1j} \\ & \times \left( \frac{1}{2} \epsilon \left( \frac{\partial^2 \tau^0}{\partial x_i \partial x_l} + \epsilon \frac{\partial^2 \tau^1}{\partial x_i \partial x_l} \right) \sum_{s=1}^M G_i^{1s} G_l^{1s} + f_i^0 \left( \frac{\partial \tau^0}{\partial x_i} + \epsilon \frac{\partial \tau^1}{\partial x_i} \right) \right. \\ & \quad \left. + \epsilon f_i^1 \left( \frac{\partial \tau^0}{\partial x_i} + \epsilon \frac{\partial \tau^1}{\partial x_i} \right) + \frac{\partial \tau^0}{\partial t} + \epsilon \frac{\partial \tau^1}{\partial t} \right) = 0. \end{aligned} \quad (3.23)$$

Now since we are working to order  $\rho$ , we get the following groups of determining equations which are exactly what Ibragimov et al. [3] get

$$-\frac{\partial \xi_j^0}{\partial t} - f_i^0 \frac{\partial \xi_j^0}{\partial x_i} + \xi_i^0 \frac{\partial f_j^0}{\partial x_i} + \tau^0 \frac{\partial f_j^0}{\partial t} + f_j^0 f_i^0 \frac{\partial \tau^0}{\partial x_i} + f_j^0 \frac{\partial \tau^0}{\partial t} = 0, \quad (3.24)$$

which we get by comparing coefficients with no  $\epsilon$ 's

$$\begin{aligned}
 f_j^0 f_i^0 \frac{\partial \tau^1}{\partial x_i} - \frac{\partial \xi_j^1}{\partial x_i} f_i^0 + f_j^0 f_i^1 \frac{\partial \tau^0}{\partial x_i} + f_j^1 f_i^0 \frac{\partial \tau^0}{\partial x_i} + \frac{1}{2} \frac{\partial^2 \tau^0}{\partial x_i \partial x_l} \sum_{s=1}^M G_i^{1s} G_l^{1s} f_j^0 + \frac{\partial \tau^1}{\partial t} f_j^0 + \xi_i^1 \frac{\partial f_j^0}{\partial x_i} + \xi_i^0 \frac{\partial f_j^1}{\partial x_i} \\
 - f_i^1 \frac{\partial \xi_j^0}{\partial x_i} - \frac{1}{2} \frac{\partial^2 \xi_j^0}{\partial x_i \partial x_l} \sum_{s=1}^M G_i^{1s} G_l^{1s} + \frac{\partial f_j^0}{\partial t} \tau^1 + \frac{\partial f_j^1}{\partial t} \tau^0 - \frac{\partial \xi_j^1}{\partial t} + \frac{\partial \tau^0}{\partial t} f_j^1 = 0,
 \end{aligned}
 \tag{3.25}$$

which all share the same coefficient  $\epsilon$ . In a similar fashion, we get the following for  $\sqrt{\epsilon}$  and  $\epsilon$ , respectively

$$\frac{1}{2} G_k^{1j} \left( \frac{\partial \tau^0}{\partial t} + f_i^0 \frac{\partial \tau^0}{\partial x_i} \right) - G_k^{li} \frac{\partial \xi_j^0}{\partial x_i} = 0,
 \tag{3.26}$$

$$-G_k^{li} \frac{\partial \xi_j^1}{\partial x_i} + \frac{1}{2} G_k^{1j} \left( \frac{\partial \tau^1}{\partial t} + f_i^0 \frac{\partial \tau^1}{\partial x_i} \right) + \frac{1}{2} G_k^{1j} f_i^1 \frac{\partial \tau^0}{\partial x_i} + \frac{G_k^{1j}}{4} \sum_{s=1}^M G_i^{1s} G_l^{1s} \frac{\partial^2 \tau^0}{\partial x_i \partial x_l} = 0.
 \tag{3.27}$$

Notice that we used (2.21) and the fact that  $G$  was constant to simplify the above.

*Remark 3.1.* Our application is consistent with the findings of [3] for this example.

### 3.1.2. Example 2

We consider

$$d\dot{X} = -\omega^2 X dt + \sigma dW + \sqrt{\epsilon} X dW.
 \tag{3.28}$$

By applying the condition (2.35), we have that

$$\xi = \xi(t, x)
 \tag{3.29}$$

and thus the prolongation formula (2.34) becomes

$$\xi^{[1]} = D(\xi) - \dot{x}D(\tau),
 \tag{3.30}$$

where  $D$  is the total time derivative operator. Our determining equations at  $\epsilon^0$  are

$$\begin{aligned}
 -\omega^2 x \Gamma^0(\tau^0) + H^0(-\omega^2 x) &= \Gamma^0(\xi^{0[1]}), \\
 H^0(G^0) + \frac{1}{2} G^0 \Gamma^0(\tau^0) &= Y^0(\xi^{0[1]}).
 \end{aligned}
 \tag{3.31}$$

The determining equations at  $\epsilon$  are

$$-\omega^2 x \Gamma^0(\tau^1) + H^1(-\omega^2 x) - \omega^2 x \Upsilon_1(\tau^0) = \Gamma^0(\xi^{1[1]}) + \Upsilon_1(\xi^0), \quad (3.32)$$

$$H^1(G^0) + \frac{1}{2} G^0 \Gamma^0(\tau^1) = \Upsilon^0(\xi^{1[1]}). \quad (3.33)$$

At  $\epsilon^{1/2}$  the determining equations are

$$H^0(G^1) + \frac{1}{2} G^1 \Gamma^0(\tau^0) + \frac{1}{2} G^0 \mathcal{U}_{10}(\tau^0) = \Upsilon^1(\xi^{0[1]}) \quad (3.34)$$

and the final determining equation at  $\epsilon^{3/2}$  is

$$\frac{1}{2} G^0 \mathcal{U}_{12}(\tau^0) + \frac{1}{2} G^1 \Gamma^0(\tau^1) = \Upsilon^1(\xi^{1[1]}). \quad (3.35)$$

Equation (3.31) for the infinitesimals at the zeroth echelon, that is,  $\tau^0$  and  $\xi^0$ , have been solved earlier in the oscillating-spring mass example

$$\begin{aligned} \tau^0 &= C_0, \\ \xi^0 &= C_1 \cos(\omega t) + C_2 \sin(\omega t). \end{aligned} \quad (3.36)$$

Whence, (3.34) and (3.35) force

$$\begin{aligned} \xi^0 &= 0, \\ \tau^1 &= C_3. \end{aligned} \quad (3.37)$$

From (3.32), we get

$$-\omega^2 \xi^1 = D^2(\xi^1), \quad (3.38)$$

which solves as

$$\xi^1 = C_4 \cos(\omega t) + C_5 \sin(\omega t). \quad (3.39)$$

Therefore we have

$$\begin{aligned} \xi &= \epsilon(C_4 \cos(\omega t) + C_5 \sin(\omega t)), \\ \tau &= C_0 + \epsilon C_3. \end{aligned} \quad (3.40)$$

#### 4. Concluding Comments

Lie group analysis for  $n$ th-ordered Itô SODEs was first pursued in Wafo Soh and Mahomed [5]. Though it had only been done for point symmetries, it has led to many interesting findings in this paper. We have shown that it is possible to derive the prolongation formulas by using the philosophy of form invariance.

With the use of the philosophy that the properties of the Wiener processes should remain invariant under the Lie group transformations, we derive conditions on the temporal and lower level derivative spatial infinitesimals that are a generalization of the condition derived by Ünal [4] for one-dimensional SODEs.

In this more general approximate approach to higher order SODE, we derive the same conditioning as Ünal [4] did without recourse to the Itô's multiplication table for the transformed variables. Our results are consistent with that of [3] in the first-order case. However, we have a generalization to  $n$ th-order SODEs. We also applied our method to an example taken from [3] as well as another.

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