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Research Article

Stability of Analytic and Numerical Solutions for Differential Equations with Piecewise Continuous Arguments

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In this paper, the asymptotic stability of the analytic and numerical solutions for differential equations with piecewise continuous arguments is investigated by using Lyapunov methods. In particular, the linear equations with variable coefficients are considered. The stability conditions of the analytic solutions of those equations and the numerical solutions of the θ-methods are obtained. Some examples are illustrated.

1. Introduction

This paper deals with the stability of both analytic and numerical solutions of the following differential equation:

$$\begin{gathered} x'(t) = f(t, x(t), x[t]), \qquad t > 0, \\ x(0) = x_0, \end{gathered} \tag{1.1}$$

where $x_0 \in \mathbb{R}^d$, $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous*, $f(t, 0, 0) = 0$, and $[\cdot]$ denotes the greatest integer function. This kind of equations has been initiated by Wiener [1, 2], Cooke and Wiener [3], and Shah and Wiener [4]. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [5].

It seems to us that the strong interest in differential equation with piecewise constant arguments is motivated by the fact that it describes hybrid dynamical system (a combination of continuous and discrete). These equations have the structure of continuous dynamical systems within intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals implies recurrent relations for the values of the solution at such points. Therefore, they combine the properties of differential equations and difference equations.

There are also some authors who have considered the stability of numerical solutions (see [6–8]). However, all of the above results are based on the linear autonomous equations. In this paper, we will use Lyapunov methods to investigate the analytic and numerical solution of the generalized equation (1.1).

Definition 1.1 (see [5]). A solution of (1.1) on $[0, \infty)$ is a function $x(t)$ that satisfies the following conditions:

(1) $x(t)$ is continuous on $[0, \infty)$,

(2) the derivative $x'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$, where one-sided derivatives exist,

(3) Equation (1.1) is satisfied on each interval $[k, k+1) \subset [0, \infty)$ with integral endpoints.

2. The Stability of the Analytic Solution

Definition 2.1 (see [9–11]). The trivial solution of (1.1) is said to be

(1) stable if for any given $\varepsilon > 0$, there exists a number $\eta = \eta(\varepsilon) > 0$ such that if $|x_0| \le \eta$, then $|x(t, x_0)| \le \varepsilon$ for all $t > 0$,

(2) asymptotically stable if it is stable, and there exists an $\eta > 0$ such that for any given $\gamma > 0$, there exists a number $T = T(\eta, \gamma) > 0$ such that if $|x_0| \le \eta$, then $|x(t, x_0)| \le \gamma$ for all $t \ge T$,

(3) globally asymptotically stable if it is asymptotically stable and $\eta = \infty$,

(4) unstable if stability fails to hold,

where $|\cdot|$ is a norm in $\mathbb{R}^d$.

Definition 2.2. Given a continuous function $V : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^+$, the derivative of V along the solution of (1.1) is defined by

$$V'(t, x(t)) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x(t+h)) - V(t, x(t))], \tag{2.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$.

It is easy to see that if $V(t, x)$ has continuous partial derivatives with respect to t and x, then (2.1) can be represented by

$$V'(t, x(t)) = \frac{\partial V(t, x(t))}{\partial t} + \frac{\partial V(t, x(t))}{\partial x} f(t, x(t), x([t])). \tag{2.2}$$

Theorem 2.3. *Suppose that* $f : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \times\mathbb{R}^d$ *is continuous,* $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous, strictly increasing functions satisfying* $a(0) = b(0) = 0$*. Let constants* $M > 0$ *and* $q \in [0, 1)$ *exist such that for* $n \in \mathbb{Z}$*,*

$$a(|x|) \le V(t, x) \le b(|x|), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d, \tag{2.3}$$

$$V(n+1, x(n+1)) \le qV(n, x(n)), \tag{2.4}$$

$$V(t, x(t)) \le MV(n, x(n)), \quad \textit{for } t \in (n, n+1). \tag{2.5}$$

Then the trivial solution of (1.1) *is asymptotically stable.*

Proof. Let $\varepsilon > 0$ be given, then there exist $\varepsilon_1 > 0$ and $\delta = \delta(\varepsilon)$ such that $\varepsilon_1 \le \varepsilon$, $a(\varepsilon_1) \le a(\varepsilon)/M$, and $b(\delta) < a(\varepsilon_1)$. Let $|x_0| < \delta$, and Let $x(t)$ be a solution of (1.1), then it follows from (2.4) that the function $V(n, x(n))$ is decreasing with respect to n. Making use of (2.3) and (2.4), we obtain successively the inequalities for any integer n,

$$a(|x(n)|) \le V(n, x(n)) \le V(0, x_0) \le b(\delta) < a(\varepsilon_1), \tag{2.6}$$

so $|x(n)| < \varepsilon_1 \le \varepsilon$.

From (2.3), (2.5), and (2.6), we have for $t \in [n, n+1)$,

$$a(|x(t)|) \le V(t, x(t)) \le MV(n, x(n)) < Ma(\varepsilon_1) \le a(\varepsilon). \tag{2.7}$$

Hence, $|x(t)| < \varepsilon$ for all $t > 0$, which implies that the trivial solution is stable.

From (2.3), (2.4), and (2.5), we have for $t \in [n, n+1)$,

$$a(|x(t)|) \le V(t, x(t)) \le MV(n, x(n)) \le MqV(n-1, x(n-1)) \le \cdots \le Mq^nV(0, x(0)). \tag{2.8}$$

Hence, $\lim_{t\to\infty} x(t) = 0$. □

Example 2.4. The trivial solution of the following system:

$$\begin{aligned} \dot{x}_1(t) &= -\tan\frac{x_1(t)}{2} + x_2(t), \\ \dot{x}_2(t) &= -\sin x_1(t) - x_2^2([t])x_2(t) - \frac{x_2(t)}{2} \end{aligned} \qquad t > 0, \tag{2.9}$$

is asymptotically stable.

Proof. Let $h > 0$ be a constant such that $|x| \le h$, $V(t,x(t)) = 1 - \cos x_1(t) + (x_2^2(t)/2)$, $a(s) = (s/h)\min_{s\le|x|\le h} V(x)$, $b(s) = \max_{|x|\le s} V(x) + s$, $M = 1$, and $q = e^{-1}$, then

$$\begin{aligned}\dot{V}(t,x(t)) &= \sin x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t)\\ &= -\sin x_1(t)\tan\frac{x_1(t)}{2} + x_2(t)\sin x_1(t) - x_2(t)\sin x_1(t) - x_2^2(t)x_2^2([t]) - \frac{x_2^2(t)}{2}\\ &\le -2\sin^2\frac{x_1(t)}{2} - \frac{x_2^2(t)}{2} = \cos x_1(t) - 1 - \frac{x_2^2(t)}{2} = -V(t,x(t)).\end{aligned} \tag{2.10}$$

Hence, for $t \in [n, n+1)$, we have

$$\begin{gathered} V(t,x(t)) \le e^{-(t-n)}V(n,x(n)) \le MV(n,x(n)),\\ V(n+1,x(n+1)) \le qV(n,x(n)). \end{gathered} \tag{2.11}$$

Therefore, the trivial solution is asymptotically stable. □

In the following, we consider the following equation:

$$x'(t) = a(t)x(t) + b(t)x([t]), \quad x(0) = x_0, \tag{2.12}$$

where $a(t)$ and $b(t)$ are continuous.

Theorem 2.5. *The trivial solution of* (2.12) *is asymptotically stable if there exist constants $M > 0$ and $q \in (0,1)$ such that for $n = 1, 2, \ldots$,*

$$\begin{gathered} \max_{n\le t<n+1}\left| e^{\int_n^t a(s)ds} + e^{\int_n^t a(s)ds}\int_n^t e^{-\int_n^s a(u)du} b(s)ds \right| \le M,\\ \left| e^{\int_n^{n+1} a(s)ds} + e^{\int_n^{n+1} a(s)ds}\int_n^{n+1} e^{-\int_0^s a(u)du} b(s)ds \right| \le q. \end{gathered} \tag{2.13}$$

Proof. We define $V(t,x) = \alpha(x) = \beta(x) = x^2/2$, Then we have for $t \in [n, n+1)$,

$$V'(t,x(t)) = a(t)x^2(t) + b(t)x(t)x(n), \qquad V'(n,x(n)) = (a(t)+b(t))x^2(n) < 0. \tag{2.14}$$

Let $y(t) = \sqrt{2V(t,x(t))}$, then

$$y'(t) = a(t)y(t) + b(t)\,\mathrm{sign}(x(t))x(n). \tag{2.15}$$

Hence,

$$y(t) = \left(e^{\int_n^t a(s)ds} + e^{\int_n^t a(s)ds}\int_n^t e^{-\int_n^s a(u)du} b(s)ds \right)\mathrm{sign}(x(t))x(n), \tag{2.16}$$

And we have from (2.13),

$$V(t,x(t)) = \left|e^{\int_n^t a(s)ds} + e^{\int_n^t a(s)ds}\int_n^t e^{-\int_n^s a(u)du}b(s)ds\right|^2 V(n,x(n)) \le MV(n,x(n)),$$

$$V(n+1,x(n+1)) = \left|e^{\int_n^{n+1} a(s)ds} + e^{\int_0^{n+1} a(s)ds}\int_n^{n+1} e^{-\int_0^s a(u)du}b(s)ds\right|^2 \times V(n,x(n)) \le qV(n,x(n)). \tag{2.17}$$

In view of (2.3), the theorem is proved. □

Assume $a(t) \equiv a$, $b(t) \equiv b$, then (2.12) and conditions (2.13) reduce to

$$x'(t) = ax(t) + bx([t]), \tag{2.18}$$

$$\max_{\substack{n\le t<n+1\\ n\in\mathbb{Z}}}\left|e^{a(t-n)}\left(1+\frac{b}{a}\right)-\frac{b}{a}\right| < M, \tag{2.19}$$

$$\left|e^{a}\left(1+\frac{b}{a}\right)-\frac{b}{a}\right| < 1. \tag{2.20}$$

If we choose $M = |b/a| + |1+(b/a)|e^{|a|}$, then (2.19) is automatically satisfied.

Remark 2.6. (1) The conditions (2.13) are necessary and sufficient for the trivial solution of (2.12) being asymptotically stable (see [1, Theorem 1.45]).

(2) The condition (2.20) is necessary and sufficient for the trivial solution of (2.18) being asymptotically stable (see [1, Corollary 1.2]).

Again we consider (2.12). Assume $|b(t)| \le -\alpha a(t)$ $(\alpha > 0)$, then for $t \in [n, n+1)$,

$$\begin{aligned}
&\left|e^{\int_n^t a(s)ds} + e^{\int_n^t a(s)ds}\int_n^t e^{-\int_n^s a(u)du}b(s)ds\right| \\
&\quad\le e^{\int_n^t a(s)ds} + e^{\int_n^t a(s)ds}\alpha\int_n^t e^{-\int_n^s a(u)du}(-a(s))ds \\
&\quad= e^{\int_n^t a(s)ds} + \alpha e^{\int_n^t a(s)ds} e^{-\int_n^s a(u)du}\Big|_n^t \\
&\quad= e^{\int_n^t a(s)ds} + \alpha e^{\int_n^t a(s)ds}\left[e^{-\int_n^t a(s)ds} - e^{-\int_n^n a(s)ds}\right] \\
&\quad= e^{\int_n^t a(s)ds} + \alpha - \alpha e^{\int_n^t a(s)ds} \\
&\quad= \alpha + (1-\alpha)e^{\int_n^t a(s)ds}.
\end{aligned} \tag{2.21}$$

Therefore, we have the following corollary.

Corollary 2.7. *Assume that $a(t) \leq -\beta$, $|b(t)| \leq -\alpha a(t)$, then the trivial solution of* (2.12) *is asymptotically stable if*

$$0 \leq \alpha < 1, \quad \beta > 0. \tag{2.22}$$

3. The Stability of the Discrete System

In this section, we will consider the discrete system with the form

$$x_{km+l+1} = \varphi(km + l, x_{km+l}, x_{km+l-1}, \dots, x_{km}), \tag{3.1}$$

where $k \in \mathbb{Z}$, $l = 0, 1, \dots, m-1$.

We assume that $\varphi(km + l, 0, 0, \dots, 0) = 0$ ($k \in \mathbb{Z}$, $l = 0, 1, \dots, m-1$) and (3.1) has a unique solution. The solution $x(n) \equiv 0$ is the trivial solution of (3.1). Like (2.1), we can define the stability and asymptotical stability.

Theorem 3.1. *Suppose $\varphi : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous, $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, strictly increasing functions satisfying $a(0) = b(0) = 0$. Let constants $M > 0$ and $q \in [0, 1)$ exist such that for $k \in \mathbb{Z}$*

$$a(|x|) \leq V(t, x) \leq b(|x|), \tag{3.2}$$

$$V\big((k+1)m, x_{(k+1)m}\big) \leq qV(km, x_{km}), \tag{3.3}$$

$$V(km + l, x_{km+l}) \leq MV(km, x_{km}), \quad l = 0, 1, \dots, m-1. \tag{3.4}$$

Then the trivial solution of (3.1) *is asymptotically stable.*

Proof. Firstly, we will prove the stability. Let $\varepsilon > 0$ be given, then there exists a $\varepsilon_1 > 0$ and $\delta = \delta(\varepsilon)$ such that $\varepsilon_1 \leq \varepsilon$, $a(\varepsilon_1) \leq a(\varepsilon)/M$, and $b(\delta) < a(\varepsilon_1)$. Let $|x_0| < \delta$, and Let x_{km+l} be a solution of (3.1), then it follows from (3.3) that the function $V(km, x_{km})$ is nonincreasing with respect to k. Making use of (3.2) and (3.3), we obtain successively the inequalities

$$a(|x_{km}|) \leq V(km, x_{km}) \leq V(0, x_0) \leq b(\delta) < a(\varepsilon_1), \tag{3.5}$$

so $|x_{km}| < \varepsilon_1 \leq \varepsilon$.

From (3.2), (3.4), and (3.5)

$$a(|x_{km+l}|) \leq V(km + l, x_{km+l}) \leq MV(km, x_{km}) < Ma(\varepsilon_1) \leq a(\varepsilon). \tag{3.6}$$

Therefore, for all $k \in \mathbb{Z}$, $l = 0, 1, \dots, m-1$, $|x_{km+l}| < \varepsilon$.

Nextly, we will prove the asymptotic stability. We have, from (3.2) and (3.4),

$$\begin{aligned} a(|x_{km+l}|) &\leq V(km+l, x_{km+l}) \leq MV(km, x_{km}) \\ &\leq MqV\left((k-1)m, x_{(k-1)m}\right) \leq \cdots \leq Mq^k V(0, x(0)), \end{aligned} \tag{3.7}$$

so $\lim_{k\to\infty} V(km, x_{km}) = 0$. The proof is complete. □

In the rest of the section, we consider the following scalar system:

$$\begin{aligned} x_{km+l+1} &= f(a_{km+l+1}, a_{km+l}, \ldots, a_{km}, b_{km+l+1}, b_{km+l}, \ldots, b_{km}) x_{km+l} \\ &\quad + g(a_{km+l+1}, a_{km+l}, \ldots, a_{km}, b_{km+l+1}, b_{km+l}, \ldots, b_{km}) x_{km}. \end{aligned} \tag{3.8}$$

Let $V(t, x) = |x(t)| = a(|x|) = b(|x|)$. The following corollary is easy to prove.

Corollary 3.2. *If there exists a* $\alpha \in [0, 1)$, *such that for* $k \in \mathbb{Z}$, $l = 0, 1, \ldots, m$,

$$S_{k,l} \triangleq \left|f(a_{km+l+1}, \ldots, a_{km}, b_{km+l+1}, \ldots, b_{km})\right| + \left|g(a_{km+l+1}, \ldots, a_{km}, b_{km+l+1}, \ldots, b_{km})\right| \leq \alpha, \tag{3.9}$$

then the trivial solution of (3.1) *is asymptotically stable.*

4. The Stability of the Numerical Solution

In this section, we will investigate the numerical asymptotic stability of θ-methods.

4.1. θ-Methods

Let $h = 1/m$ be a given stepsize with integer $m \geq 1$ and the gridpoints $t_n = nh (n = 0, 1, \ldots)$. The linear θ-method applied to (1.1) can be represented as follows:

$$x_{n+1} = x_n + h\left\{\theta f\left((n+1)h, x_{n+1}, x^h([(n+1)h])\right) + (1-\theta) f\left(nh, x_n, x^h([nh])\right)\right\}, \tag{4.1}$$

and the one-leg θ-method

$$x_{n+1} = x_n + h\left\{f\left((n+\theta)h, x^h((n+\theta)h), x^h([(n+\theta)h])\right)\right\}. \tag{4.2}$$

Table 1: Linear θ-methods for problem (5.1).

m	$\theta = 0$		$\theta = 1/2$		$\theta = 1$	
	AE	RE	AE	RE	AE	RE
3	$1.8930E-3$	$6.5172E-1$	$1.9237E-4$	$6.6228E-2$	$3.0508E-3$	$1.0503E-0$
5	$1.2740E-3$	$4.3861E-1$	$6.9633E-5$	$2.3973E-2$	$1.6936E-3$	$5.8308E-1$
10	$6.8950E-4$	$2.3738E-1$	$1.7448E-5$	$6.0071E-3$	$7.9471E-4$	$2.7360E-1$
20	$3.5792E-4$	$1.2322E-1$	$4.3646E-6$	$1.5026E-3$	$3.8424E-4$	$1.3229E-1$
50	$1.4633E-4$	$5.0378E-2$	$6.9845E-7$	$2.4046E-4$	$1.5054E-4$	$5.1828E-2$
100	$7.3691E-5$	$2.5370E-2$	$1.7462E-7$	$6.0117E-5$	$7.4744E-5$	$2.5733E-2$
Ratio	1.9857	1.9857	3.9998	3.9999	2.0141	2.0141

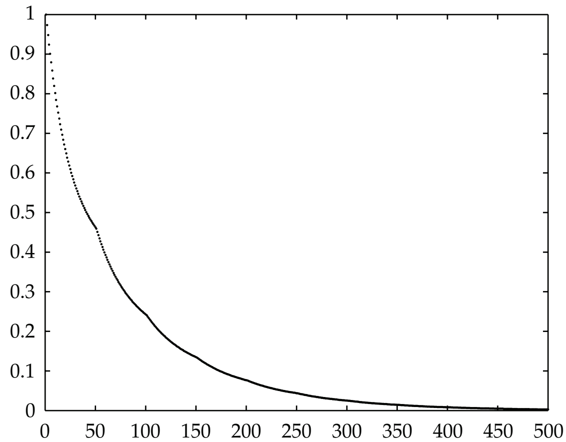

Figure 1: Linear θ-method with $\theta = 0$ for (5.1).

Here, $n = 0, 1, \ldots, \theta$ is a parameter with $0 \le \theta \le 1$, specifying the method, $x^h([t])$ denotes an approximation to $x([t])$, and $x^h(t)$ is an approximation to $x(t)$ defined by

$$x^h(t) = \frac{t - nh}{h} x_{n+1} + \frac{(n+1)h - t}{h} x_n, \quad \text{for } nh < t \le (n+1)h, \; n = 0, 1, \ldots. \tag{4.3}$$

4.2. Numerical Stability

Applying (4.1) and (4.2) to (2.12), we arrive at the following recurrence relations, respectively:

$$\begin{aligned} x_{n+1} &= x_n + h\Big\{\theta\Big(a(t_{n+1})x_{n+1} + b(t_{n+1})x^h([(n+1)h])\Big) \\ &\qquad + (1-\theta)\Big(a(t_n)x_n + b(t_n)x^h([nh])\Big)\Big\}, \\ x_{n+1} &= x_n + h\Big\{a(t_{n+\theta})(\theta x_{n+1} + (1-\theta)x_n) + b(t_{n+\theta})x^h([(n+\theta)h])\Big\}. \end{aligned} \tag{4.4}$$

Table 2: One-leg θ-methods for problem (5.1).

m	$\theta = 0$		$\theta = 1/2$		$\theta = 1$	
	AE	RE	AE	RE	AE	RE
3	$1.8930E-3$	$6.5172E-1$	$1.4260E-4$	$4.9094E-2$	$3.0508E-3$	$1.0503E+000$
5	$1.2740E-3$	$4.3861E-1$	$5.1570E-5$	$1.7754E-2$	$1.6936E-3$	$5.8308E-1$
10	$6.8950E-4$	$2.3738E-1$	$1.2917E-5$	$4.4470E-3$	$7.9471E-4$	$2.7360E-1$
20	$3.5792E-4$	$1.2322E-1$	$3.2307E-6$	$1.1123E-3$	$3.8424E-4$	$1.3229E-1$
50	$1.4633E-4$	$5.0378E-2$	$5.1699E-7$	$1.7799E-4$	$1.5054E-4$	$5.1828E-2$
100	$7.3691E-5$	$2.5370E-2$	$1.2925E-7$	$4.4498E-5$	$7.4744E-5$	$2.5733E-2$
Ratio	1.9857	1.9857	3.9999	4.0000	2.0141	2.0141

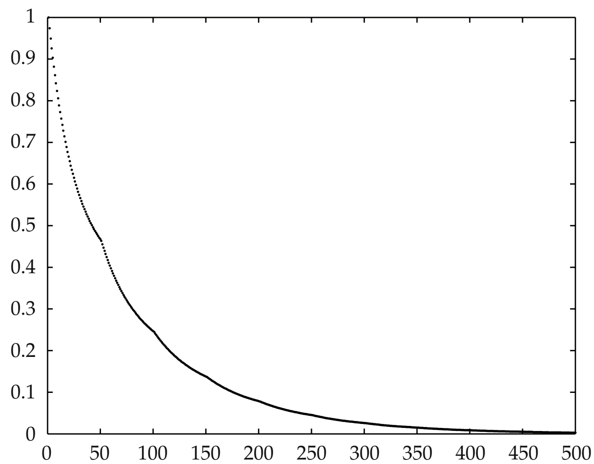

Figure 2: Linear θ-method with $\theta = 1/2$ for (5.1).

Let $n = km + l$ $(l = 0, 1, \dots, m-1)$, then we define $x^h([t_n + \delta h])$, $0 \le \delta \le 1$, as x_{km} according to Definition 1.1. As a result, (4.4) reduce to

$$x_{km+l+1} = \frac{1 + h(1-\theta)a(t_{km+l})}{1 - h\theta a(t_{km+l+1})} x_{km+l} + \frac{h\theta b(t_{km+l+1}) + h(1-\theta) b(t_{km+l})}{1 - h\theta a(t_{km+l+1})} x_{km},$$
$$x_{km+l+1} = \left[1 + \frac{h a(t_{km+l+\theta})}{1 - h\theta a(t_{km+l+\theta})}\right] x_{km+l} + \frac{h b(t_{km+l+\theta})}{1 - h\theta a(t_{km+l+\theta})} x_{km}. \tag{4.5}$$

In fact, in each interval $[n, n+1)$, (2.12) can be seen as ordinary differential equation. Hence, the θ-methods are convergent of order 1 if $\theta \ne 1/2$ and order 2 if $\theta = 1/2$.

Definition 4.1. (1) The numerical methods are called asymptotically stable if there exists an $h_0 > 0$, such that $x_n \to 0$ as $n \to \infty$ for any given x_0 and any stepsize $h < h_0$.

(2) The numerical methods are called general asymptotically stable if $x_n \to 0$ as $n \to \infty$ for any given x_0 and any stepsize.

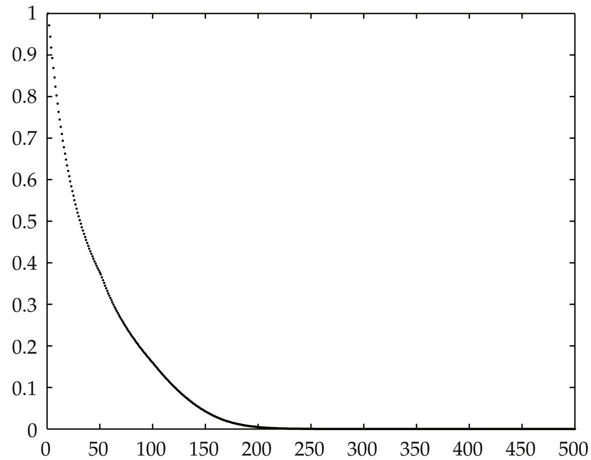

Figure 3: One-leg θ-method with $\theta = 1/2$ for (5.2).

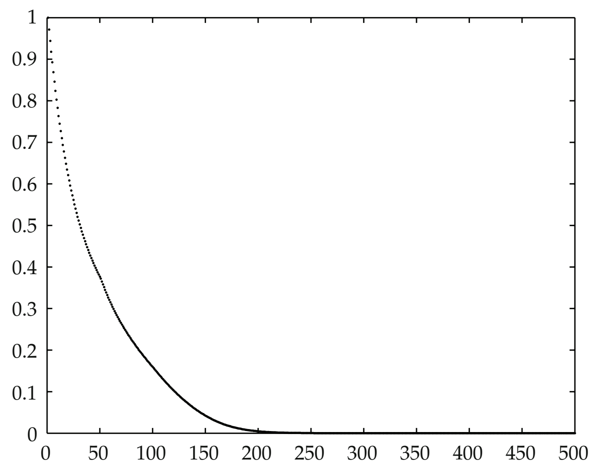

Figure 4: One-leg θ-method with $\theta = 1$ for (5.2).

Theorem 4.2. *Assume that* $a(t) \le -\beta$, $|b(t)| \le -\alpha a(t)$ $(\beta > 0,\ 0 \le \alpha < 1)$, *and there exists a* $r > 0$ *such that* $-r \le a(t)$, *then*

(1) *the linear* θ*-method and the one-leg* θ*-method are asymptotically stable if* $h < 1/(1-\theta)r$,

(2) *the one-leg* θ*-method is general asymptotically stable if* $(1 + a/2) \le \theta \le 1$, *and the linear* θ*-method is general asymptotically stable if* $(1 + a/2) \le \theta \le 1$ *and* $a(t)$ *is nonincreasing.*

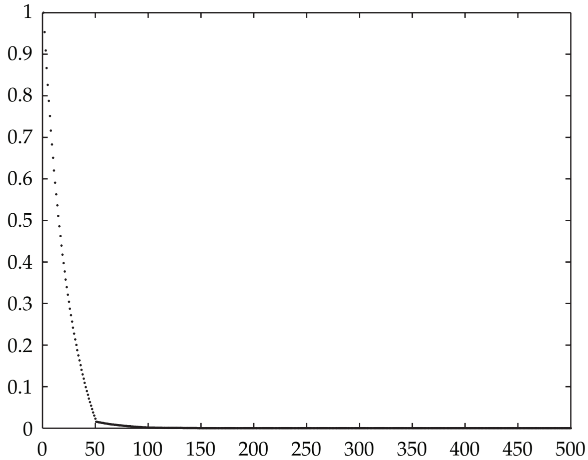

Figure 5: One-leg θ-method with $\theta = 0$ for (5.3).

Proof. Denote $a_{km+l} = a(t_{km+l})$, $a_{km+l+\theta} = a(t_{km+l+\theta})$, $b_{km+l} = b(t_{km+l})$, and $b_{km+l+\theta} = b(t_{km+l+\theta})$.

(1) For any integer k, and $l = 0, 1, \ldots, m-1$, we have, from $h < 1/(1-\theta)r$,

$$1 + h(1-\theta)a_{km+l} > 0, \qquad 1 + h(1-\theta)a_{km+l+\theta} > 0. \tag{4.6}$$

For the linear θ-method,

$$\begin{aligned} S_{k,l} &= \left| \frac{1 + h(1-\theta)a_{km+l}}{1 - \theta h a_{km+l+1}} + \frac{\theta h b_{km+l+1}}{1 - \theta h a_{km+l+1}} + \frac{(1-\theta)hb_{km+l}}{1 - \theta h a_{km+l+1}} \right| \\ &\le \frac{1 + h(1-\theta)a_{km+l} - \alpha\theta h a_{km+l+1} - \alpha(1-\theta)h a_{km+l}}{1 - \theta h a_{km+l+1}} \\ &= 1 + \frac{(1-\theta)(1-\alpha)h a_{km+l} + \theta(1-\alpha)h a_{km+l+1}}{1 - \theta h a_{km+l+1}} \\ &\le 1 + \frac{-(1-\theta)(1-\alpha)h\beta - \theta(1-\alpha)h\beta}{1 + \theta h r} \\ &= 1 + \frac{-(1-\alpha)h\beta}{1 + \theta h r} < 1. \end{aligned} \tag{4.7}$$

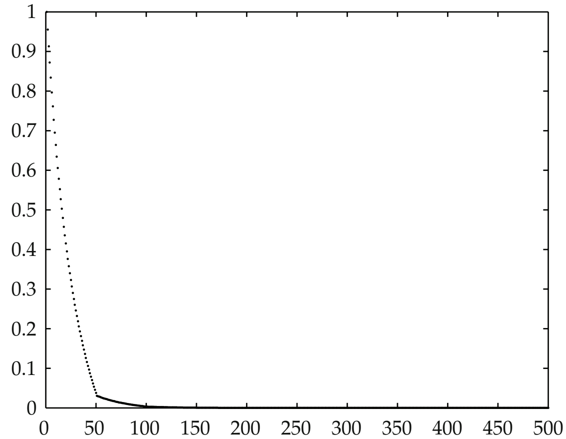

Figure 6: Linear θ-method with $\theta = 1$ for (5.3).

For the one-leg θ-method,

$$\begin{aligned}\overline{S}_{k,l} &= \left|1 + \frac{ha_{km+l+\theta}}{1-\theta ha_{km+l+\theta}} + \frac{hb_{km+l+\theta}}{1-\theta ha_{km+l+\theta}}\right| \\ &\leq 1 + \frac{(1-\alpha)ha_{km+l+\theta}}{1-\theta ha_{km+l+\theta}} \leq 1 + \frac{-(1-\alpha)h\beta}{1+\theta h\beta} < 1.\end{aligned} \tag{4.8}$$

(2) For the linear θ-method, if $1 + (1-\theta)ha_{km+l} \geq 0$, then we have, from (1),

$$\begin{aligned}S_{k,l} &\leq 1 + \frac{(1-\theta)(1-\alpha)ha_{km+l} + \theta(1-\alpha)ha_{km+l+1}}{1-\theta ha_{km+l+1}} \\ &\leq 1 + \frac{-(1-\alpha)h\beta}{1+\theta hr} < 1.\end{aligned} \tag{4.9}$$

If $1 + (1-\theta)ha_{km+l} < 0$, then since $a(t)$ is nonincreasing, we have

$$\begin{aligned}S_{k,l} &\leq -1 + \frac{-(1-\theta)(1+\alpha)ha_{km+l} - \theta(1+\alpha)ha_{km+l+1}}{1-\theta ha_{km+l+1}} \\ &\leq -1 + \frac{-(1+\alpha)ha_{km+l+1}}{1-\theta a_{km+l+1}} \leq -1 + \frac{(1+\alpha)hr}{1+\theta hr} < 1.\end{aligned} \tag{4.10}$$

For one-leg θ-method, we have the following two cases.

If $1+(1-\theta)ha_{km+l+\theta} \geq 0$, then $\overline{S}_{k,l} \leq 1+((1+\alpha)ha_{km+l+\theta})/(1-\theta ha_{km+l+\theta}) \leq 1+(-(1-\alpha)h\beta)/(1+\theta h\beta) < 1$.

If $1+(1-\theta)ha_{km+l+\theta} < 0$, then $\overline{S}_{k,l} \leq -1+(-(1+\alpha)ha_{km+l+\theta})/(1-\theta ha_{km+l+\theta}) \leq -1+(1+\alpha)hr/(1+\theta hr) < 1$. □

5. Numerical Experiments

In this section, we will give some examples to illustrate the conclusions in the paper. We consider the following three problems:

$$\begin{gathered}\dot{x}(t) = \left(-e^{-t}-1\right)x(t) + \frac{e^{-t}+1}{3}x([t]), \quad t > 0, \\ x(0) = 1,\end{gathered} \tag{5.1}$$

$$\begin{gathered}\dot{x}(t) = (\sin t - 2)x(t) + \frac{e^{-t}}{2}x([t]), \quad t > 0, \\ x(0) = 1,\end{gathered} \tag{5.2}$$

$$\begin{gathered}\dot{x}(t) = \left(-e^{-t}-1\right)x(t) - \frac{1}{3}x([t]), \quad t > 0, \\ x(0) = 1.\end{gathered} \tag{5.3}$$

It is easy to verify that the above examples satisfy the conditions of Theorem 4.2. Hence, the solutions of three equations are asymptotically stable according to Corollary 2.7.

In Tables 1 and 2, we list the absolute errors (AEs) and the relative errors (REs) at $t = 10$ of the θ-methods for the first problem. We can see from these tables that the methods preserve their orders of convergence.

In Figures 1, 2, 3, 4, 5, and 6, we draw the numerical solutions of the θ-methods with $m = 50$. It is easy to see that the numerical solutions are asymptotically stable.

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