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*Research Article*

# The Dirichlet Problem on the Upper Half-Space

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A solution of the Dirichlet problem on the upper half-space is constructed by the generalized Dirichlet integral with a fast-growing continuous boundary function.

## 1. Introduction and Results

Let $\mathbf{R}^n$ ($n \geq 3$) denote the $n$-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set $D$ of $\mathbf{R}^n$ are denoted by $\partial D$ and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is $\partial H$. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x$, $y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'}. \tag{1.1}$$

Let $B(r)$ denote the open ball with center at the origin and radius $r$, and let $\sigma$ denote $(n-1)$-dimensional surface area measure. Let $[d]$ denote the integer part of the positive real number $d$. In the sense of Lebesgue measure, $dy' = dy_1 \cdots dy_{n-1}$ and $dy = dy' dy_n$.

Given a continuous function $f$ on $\partial H$, we say that $h$ is a solution of the (classical) Dirichlet problem on $H$ with $f$ if $\Delta h = 0$ in $H$ and $\lim_{x \in H, x \to z'} h(x) = f(z')$ for every $z' \in \partial H$.

The classical Poisson kernel for $H$ is defined by $P(x, y') = 2x_n \omega_n^{-1} |x - y'|^{-n}$, where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$.

To solve the Dirichlet problem on $H$, as in [1–6], we use the following modified Poisson kernel of order $m$ defined by

$$P_m(x,y') = \begin{cases} P(x,y') & \text{when } |y'| \le 1, \\ P(x,y') - \displaystyle\sum_{k=0}^{m-1} \frac{2x_n|x|^k}{\omega_n |y'|^{n+k}} C_k^{n/2}\left(\frac{x\cdot y'}{|x||y'|}\right) & \text{when } |y'| > 1, \end{cases} \tag{1.2}$$

where $m$ is a nonnegative integer, and $C_k^{n/2}(t)$ is the ultraspherical (Gegenbauer) polynomials [7]. The expression arises from the generating function for Gegenbauer polynomials

$$\left(1 - 2tr + r^2\right)^{-n/2} = \sum_{k=0}^{\infty} C_k^{n/2}(t) r^k, \tag{1.3}$$

where $|r| < 1$ and $|t| \le 1$. The coefficient $C_k^{n/2}(t)$ is called the ultraspherical (Gegenbauer) polynomial of degree $k$ associated with $n/2$, and the function $C_k^{n/2}(t)$ is a polynomial of degree $k$ in $t$.

Put

$$U_m(f)(x) = \int_{\partial H} P_m(x,y') f(y') dy', \tag{1.4}$$

where $f(y')$ is a continuous function on $\partial H$.

Using the modified Poisson kernel $P_m(x,y')$, Yoshida (cf. [6, Theorem 1]) and Siegel and Talvila (cf. [5, Corollary 2.1]) gave classical solutions of the Dirichlet problem on $H$, respectively. Motivated by their results, we consider the Dirichlet problem for harmonic functions of infinite order (e.g., see [8, Definition 4.1, page 2, Line 12] for the definition of harmonic functions).

To do this, we define a nondecreasing and continuously differentiable function $\rho(r) \ge 1$ on the interval $[0,+\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{r\to\infty} \frac{\rho'(r) r \log r}{\rho(r)} < 1. \tag{1.5}$$

Let $F(\rho,\beta)$ be the set of continuous functions $f$ on $\partial H$ such that

$$\int_{\partial H} \frac{|f(y')| dy'}{1 + |y'|^{\rho(|y'|)+n+\beta-1}} < \infty, \tag{1.6}$$

where $\beta$ is a positive real number.

Now, we have the following.

**Theorem 1.1.** *If $f \in F(\rho,\beta)$, then the integral $U_{[\rho(|y'|)+\beta]}(f)(x)$ is a solution of the Dirichlet problem on $H$ with $f$.*

*If one puts $[\rho(|y'|)+\beta] = m$ in Theorem 1.1, one immediately obtains the following (cf. [6, Theorem 1] and [5, Corollary 2.1]).*

**Corollary 1.2.** *If $f$ is a continuous function on $\partial H$ satisfying $\int_{\partial H}|f(y')|(1+|y'|)^{-n-m}dy'<\infty$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on $H$ with $f$.*

**Theorem 1.3.** *Let $u$ be harmonic in $H$ and continuous on $\overline{H}$. If $u\in F(\rho,\beta)$, then one has*

$$u(x)=U_{[\rho(|y'|)+\beta]}(u)(x)+h(x), \tag{1.7}$$

*for all $x\in\overline{H}$, where $h(x)$ is harmonic in $H$ and vanishes continuously on $\partial H$.*

## 2. Proof of Theorem 1.1

We need to use the following inequality (see [5, page 3]):

$$\left|P_m(x,y')\right|\le Mx_n|x|^m\left|y'\right|^{-n-m}, \tag{2.1}$$

for any $x\in H$ and $y'\in\partial H$ satisfying $|y'|\ge\max\{1,2|x|\}$, where $M$ is a positive constant.

For any $\epsilon$ $(0<\epsilon<1-\epsilon_0)$, there exists a sufficiently large positive number $R$ such that $r>R$, and by (1.5), we have

$$\rho(r)<\rho(e)(\ln r)^{(\epsilon_0+\epsilon)}, \tag{2.2}$$

which yields that there exists a positive constant $M(r)$ dependent only on $r$ such that

$$k^{-\beta/2}(2r)^{\rho(k+1)+\beta+1}\le M(r), \tag{2.3}$$

for any $k>k_r=[2r]+1$.

For any $x\in H$ and $|x|\le r$, we have by (1.6), (2.1), (2.3), $1/p+1/q=1$, and Hölder's inequality

$$\begin{aligned}
&M\sum_{k=k_r}^{\infty}\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\frac{(2|x|)^{[\rho(|y'|)+\beta]+1}}{\left|y'\right|^{[\rho(|y'|)+\beta]+n}}\left|f(y')\right|dy'\\
&\quad\le M\sum_{k=k_r}^{\infty}(2r)^{\rho(k+1)+\beta+1}\left(\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\frac{\left|f(y')\right|^p}{\left|y'\right|^{\rho(|y'|)+n+p\beta/2-1}}dy'\right)^{1/p}\\
&\qquad\times\left(\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\left|y'\right|^{-q\{[\rho(|y'|)+\beta]+n-(\rho(|y'|)+n-1)/p-\beta/2\}}dy'\right)^{1/q}\\
&\quad\le M\sum_{k=k_r}^{\infty}\frac{(2r)^{\rho(k+1)+\beta+1}}{k^{\beta/2}}\int_{\{y'\in\partial H:k\le|y'|<k+1\}}\frac{\left|f(y')\right|}{\left|y'\right|^{\rho(|y'|)+n+\beta/2-1}}dy'\\
&\quad\le 2MM(r)\int_{\{y'\in\partial H:|y'|\ge k_r\}}\frac{\left|f(y')\right|}{1+\left|y'\right|^{\rho(|y'|)+n+\beta/2-1}}dy'<\infty.
\end{aligned} \tag{2.4}$$

Thus, $U_{[\rho(|y'|)+\beta]}(f)(x)$ is finite for any $x \in H$. Since $P_{[\rho(|y'|)+\beta]}(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$, $U_{[\rho(|y'|)+\beta]}(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_{[\rho(|y'|)+\beta]}(f)(x)$, we fix a boundary point $z' \in \partial H$, choose a large $t > |z'| + 1$, and write

$$U_{[\rho(|y'|)+\beta]}(f)(x) = X(x) - Y(x) + Z(x), \tag{2.5}$$

where

$$\begin{aligned} X(x) &= \int_{\{y'\in\partial H:|y'|\le t\}} P(x,y')f(y')dy', \\ Y(x) &= \sum_{k=0}^{[\rho(|y'|+\beta)]-1} \frac{2x_n|x|^k}{\omega_n} \int_{\{y'\in\partial H:1<|y'|\le t\}} \frac{1}{|y'|^{n+k}} C_k^{n/2}\left(\frac{x'\cdot y'}{|x||y'|}\right) f(y')dy', \\ Z(x) &= \int_{\{y'\in\partial H:|y'|>t\}} P_{[\rho(|y'|+\beta)]}(x,y')f(y')dy'. \end{aligned} \tag{2.6}$$

Notice that $X(x)$ is the Poisson integral of $f(y')\chi_{B(t)}(y')$, where $\chi_{B(t)}$ is the characteristic function of the ball $B(t)$. So it tends to $f(z')$ as $x \to z'$. Since $Y(x)$ are polynomial times $x_n$ and $Z(x) = O(x_n)$, both of them tend to zero as $x \to z'$. Thus, the function $U_{[\rho(|y'|)+\beta]}(f)(x)$ can be continuously extended to $\overline{H}$ such that $U_{[\rho(|y'|)+\beta]}(f)(z') = f(z')$, for any $z' \in \partial H$. Theorem 1.1 is proved.

## 3. Proof of Theorem 1.3

Consider that the function $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$, which is harmonic in $H$, can be continuously extended to $\overline{H}$ and vanishes on $\partial H$.

The Schwarz reflection principle [9, page 68] applied to $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$ shows that there exists a harmonic function $h(x)$ in $H$ such that $h(x^*) = -h(x) = -(u(x) - U_{[\rho(|y'|)+\beta]}(u)(x))$ for $x \in \overline{H}$, where $*$ denotes reflection in $\partial H$ just as $x^* = (x', -x_n)$.

Thus, $u(x) = h(x) + U_{[\rho(|y'|)+\beta]}(u)(x)$ for all $x \in \overline{H}$, where $h(x)$ is a harmonic function on $H$ vanishing continuously on $\partial H$. We complete the proof of Theorem 1.3.

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## References

[1] D. H. Armitage, "Representations of harmonic functions in half-spaces," *Proceedings of the London Mathematical Society*, vol. 38, no. 3, pp. 53–71, 1979.
[2] M. Finkelstein and S. Scheinberg, "Kernels for solving problems of Dirichlet type in a half-plane," *Advances in Mathematics*, vol. 18, no. 1, pp. 108–113, 1975.
[3] S. J. Gardiner, "The Dirichlet and Neumann problems for harmonic functions in half-spaces," *The Journal of the London Mathematical Society*, vol. 24, no. 3, pp. 502–512, 1981.
[4] D. Siegel and E. O. Talvila, "Uniqueness for the $n$-dimensional half space Dirichlet problem," *Pacific Journal of Mathematics*, vol. 175, no. 2, pp. 571–587, 1996.
[5] D. Siegel and E. Talvila, "Sharp growth estimates for modified Poisson integrals in a half space," *Potential Analysis*, vol. 15, no. 4, pp. 333–360, 2001.
[6] H. Yoshida, "A type of uniqueness for the Dirichlet problem on a half-space with continuous data," *Pacific Journal of Mathematics*, vol. 172, no. 2, pp. 591–609, 1996.
[7] G. Szegő, *Orthogonal Polynomials*, vol. 23 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 1975.
[8] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, Academic Press, London, UK, 1 edition, 1976.
[9] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, Springer, New York, NY, USA, 2nd edition, 1992.