

Research Article

The Solution of a Class of Third-Order Boundary Value Problems by the Reproducing Kernel Method

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This paper expands the application of reproducing kernel method to a class of third-order boundary value problems with mixed nonlinear boundary conditions. The analytical solution is represented in the form of series in the reproducing kernel space. The n -term approximation is obtained and is proved to converge to the analytical solution. The numerical examples are given to demonstrate the computation efficiency of the presented method. Results obtained by the method indicate that the method is simple and effective.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, or gravity driven flows. Third-order boundary value problems were discussed in many papers in recent years, for instance, see [1–6] and references therein. In [1–3], the authors used the spline functions to solve boundary value problems. In [4], the authors developed a second-order method for solving third-order three-point boundary value problems based on Padé approximant in a recurrence relation. In [5], the authors introduced Adomian decomposition method for multipoint boundary value problems (BVPs). In this paper, we use reproducing kernel to solve singular third-order boundary value problems with mixed boundary conditions. Recently, the reproducing kernel methods [7–10] emerge one after another. Using the reproducing kernel methods, the authors discussed two-point boundary value problems and periodic boundary value problems. For third-order boundary value problems with mixed nonlinear boundary conditions, however, this method has not yet been applied. In previous work, the reproducing kernel method cannot be used directly to solve third-order boundary value problems with nonlinear

boundary conditions. Our work is to present a numerical algorithm for solving a class of singular third-order boundary value problems. By using this method, the analytical solution and approximate solution are given and uniformly converge to the exact solution and its corresponding derivatives. The algorithms are efficiently applied to solve some model problems.

Let us consider the following singular problems of third-order ordinary differential equations:

$$\begin{aligned} u'''(x) + p_1(x)u''(x) + p_2(x)u'(x) + p_3(x)u(x) &= F(x), \quad x \in (0, 1), \\ \lambda_i u &= r_i, \quad (i = 1, 2, 3), \end{aligned} \quad (1.1)$$

where $p_j(x), f(x) \in L^2[0, 1]$, ($j = 1, 2, 3$) are known functions. $\lambda_i u$, ($i = 1, 2, 3$) are linear independence conditions of determining the solution. We assume that (1.1) has a unique solution which belongs to $W_2^4[0, 1]$, where $W_2^4[0, 1]$ which is a reproducing kernel space is defined in the second section.

In order to solve (1.1), let $Lu = u'''(x) + p_1(x)u''(x) + p_2(x)u'(x) + p_3(x)u(x)$. It is easy to prove that $L : W_2^4[0, 1] \rightarrow L^2[0, 1]$ is bounded linear operator. On the other hand, we assume that the conditions of determining the solution can be homogenized; after homogenization of these conditions, we put the conditions into the reproducing kernel space $W_2^4[0, 1]$ constructed in the following section. Equation (1.1) can be transformed into the following form in $W_2^4[0, 1]$:

$$(Lu)(x) = F(x). \quad (1.2)$$

To solve problem (1.2), we give a space $W_2^4[0, 1] = \{u \mid u \in \overline{W}_2^4[0, 1] \text{ and } \lambda_i u = 0, i = 1, 2, 3\}$. The inner product in $W_2^4[0, 1]$ is given by $\langle u(x), v(x) \rangle = \sum_{i=0}^3 u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(t)v^{(4)}(t)dt$. Like in [8], we can get the following reproducing kernel space.

Theorem 1.1. *The space $W_2^4[0, 1]$ is a reproducing kernel space and its reproducing kernel is $K(x, y)$, and*

$$K(x, y) = R_y(x) - \frac{h_1(x)h_1(y)}{\|h_1(x)\|^2} - \frac{h_2(x)h_2(y)}{\|h_2(x)\|^2} - \frac{h_3(x)h_3(y)}{\|h_3(x)\|^2}, \quad (1.3)$$

where

$$R_y(x) = \begin{cases} \frac{5040 - y^7 + 35x^3y^3(4+y) - 21x^2y^2(-60+y^3) + 7xy(720+y^5)}{5040}, & x < y \\ \frac{5040 - x^7 + 5040xy + 7x^6y + 1260x^2y^2 - 21x^5y^2 + 140x^3y^3 + 35x^4y^3}{5040}, & y < x, \end{cases} \quad (1.4)$$

$h_1(x) = \lambda_{1y}R_y(x)$, $h_2(x) = \lambda_{2y}(R(x, y) - (h_1(x)h_1(y)/\|h_1(x)\|^2))$, $h_3(x) = \lambda_{3y}(R(x, y) - (h_1(x)h_1(y)/\|h_1(x)\|^2) - (h_2(x)h_2(y)/\|h_2(x)\|^2))$, the symbol λ_{iy} indicates that the operator λ_i applies to the function of y .

2. The Reproducing Kernel Method

Let $\varphi_i(x) = (L_y K_x(y))(x_i)$, $i = 1, 2, \dots$, Practise Gram-Schmidt orthonormalization for $\{\varphi_i(x)\}_{i=1}^\infty$, we get

$$\bar{\varphi}_i(x) = \sum_{k=1}^i \beta_{ik} \varphi_k(x), \quad (2.1)$$

where β_{ik} are coefficients of Gram-Schmidt orthonormalization.

Theorem 2.1. *If $\{x_i\}_{i=1}^\infty$ is distinct points dense in $[0, 1]$ and L^{-1} is existent, then*

$$u(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k) \bar{\varphi}_i(x) \quad (2.2)$$

is an analytical solution of the problem (1.2).

Proof. Since $\{\bar{\varphi}_i(x)\}_{i=1}^\infty$ is an orthonormal systems, $u(x)$ is expressed as

$$\begin{aligned} u(x) &= \sum_{i=1}^\infty \langle u(x), \bar{\varphi}_i(x) \rangle \bar{\varphi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle u(x), (L_s K_x(s))(x_k) \rangle \bar{\varphi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} (L_s \langle u(x), K_x(s) \rangle)(x_k) \bar{\varphi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} (L_s u(s))(x_k) \bar{\varphi}_i(x) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k) \bar{\varphi}_i(x). \end{aligned} \quad (2.3)$$

□

We denote the approximate solution of $u_n(x)$ by

$$u_n(x) = \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} F(x_k) \bar{\varphi}_i(x). \quad (2.4)$$

Theorem 2.2. *Let $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2$ where $u(x)$, $u_n(x)$ are given by (2.2) and (2.4), then the sequence of number $\varepsilon_n(x)$ is monotone decreasing and $\varepsilon_n(x) \rightarrow 0$.*

Proof. Because

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 = \sum_{i=n+1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) = \sum_{i=n+1}^{\infty} (\langle u(x), \bar{\psi}_i(x) \rangle)^2, \quad (2.5)$$

clearly $\varepsilon_{n-1} \geq \varepsilon_n$ and consequently $\{\varepsilon_n\}$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^4}$. By Theorem 2.1, we know $\sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^4}$, then we have

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 \longrightarrow 0. \quad (2.6)$$

Hence, $\varepsilon_n \rightarrow 0$. □

Theorem 2.3 (convergence analysis). *If $u(x)$, $u_n(x)$ are given by (2.2) and (2.4), then $u_n(x)$ and $u_n^{(k)}(x)$ uniformly convergent to $u(x)$ and $u^{(k)}(x)$, where $k = 0, 1, 2, 3$.*

Proof. For any $x \in [0, 1]$, $k = 0, 1, 2, 3$

$$\left| u_n^{(k)}(x) - u^{(k)}(x) \right| = \left| \left\langle u_n(t) - u(t), \frac{\partial^k K(x, t)}{\partial x^k} \right\rangle \right| \leq \|u_n(t) - u(t)\| \cdot \left\| \frac{\partial^k K(x, t)}{\partial x^k} \right\|, \quad (2.7)$$

then there exists $C_k > 0$ such that, $|u_n^{(k)}(x) - u^{(k)}(x)| \leq C_k \|u_n(t) - u(t)\| = C_k \varepsilon_n \rightarrow 0$. □

Theorem 2.4. *If $\{x_k\}_{k=1}^{\infty}$ is distinct points dense in $[0, 1]$ and $u(x)$, $u_n(x)$ are given by (2.2) and (2.4), then $Lu(x_k) = Lu_n(x_k)$.*

Proof. We may set projective operator $P_n : W_2^4[0, 1] \rightarrow \{\sum_{m=1}^n c_m \psi_m(x), c_m \in R\}$. Hence,

$$\begin{aligned} Lu_n(x_k) &= \langle u_n(\xi), L_{x_k} K_{x_k}(\xi) \rangle = \langle u_n(\xi), \varphi_k(\xi) \rangle = \langle P_n u(\xi), \varphi_k(\xi) \rangle \\ &= \langle u(\xi), P_n \varphi_k(\xi) \rangle = \langle u(\xi), \varphi_k(\xi) \rangle = \langle u(\xi), L_{x_k} K_{x_k}(\xi) \rangle \\ &= L_{x_k} \langle u(\xi), K_{x_k}(\xi) \rangle = L_{x_k} u(x_k) = Lu(x_k). \end{aligned} \quad (2.8)$$

□

Theorem 2.5 (error estimate). *If $\{x_k\}_{k=1}^{\infty}$ is distinct points dense in $[0, 1]$ and $u(x)$, $u_n(x)$ are given by (2.2) and (2.4), then $|u(x) - u_n(x)| < (M/n)$, where $M = \|\sum_{i=n+1}^{\infty} \sum_{k=1}^i \beta_{ik} F(x_k) \bar{\psi}_i(x)\| \cdot \|\partial K_{\eta}(\xi) / \partial \eta\|$.*

Proof. For every given $x \in [0, 1]$, there is always $x_i \in \{x_k\}_{k=1}^{\infty}$ satisfying $x_i < x$ and $x - x_i = 1/n$. By Theorem 2.5 and $x_i \in \{x_k\}_{k=1}^{\infty}$ implying $Lu(x_i) = Lu_n(x_i)$, so we obtain

$$|Lu(x) - Lu_n(x)| = |Lu(x) - Lu(x_i) - [Lu_n(x) - Lu_n(x_i)]|. \quad (2.9)$$

For application reproducing kernel property, we have

$$u(x) = \langle u(\xi), K_x(\xi) \rangle, \quad Lu(x) = \langle u(\xi), LK_x(\xi) \rangle. \quad (2.10)$$

We also have

$$\begin{aligned} Lu(x) - Lu_n(x) &= Lu(x) - Lu(x_i) - [Lu_n(x) - Lu_n(x_i)] \\ &= \langle u(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle - \langle u_n(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle \\ &= \langle u(\xi) - u_n(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle. \end{aligned} \quad (2.11)$$

Moreover,

$$\begin{aligned} |u(x) - u_n(x)| &= \left| L^{-1} [Lu(x) - Lu_n(x)] \right| \\ &\leq \left| \left\langle u(\xi) - u_n(\xi), L^{-1} LK_x(\xi) - L^{-1} LK_{x_i}(\xi) \right\rangle \right| \\ &= |\langle u(\xi) - u_n(\xi), K_x(\xi) - K_{x_i}(\xi) \rangle| \\ &\leq \|u - u_n\| \|K_x(\xi) - K_{x_i}(\xi)\|. \end{aligned} \quad (2.12)$$

It is noted that we take norm of $\|K_x(\xi) - K_{x_i}(\xi)\|$ for variable ξ . The function $k_x(\xi)$ is derived on x in the interval of $[0, 1]$, so we have $K_x(\xi) - K_{x_i}(\xi) = (\partial K_\eta(\xi) / \partial \eta)(x - x_i)$. Hence,

$$\begin{aligned} |u(x) - u_n(x)| &\leq \|u - u_n\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta} (x - x_i) \right\| \\ &= \|u - u_n\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta} \right\| (x - x_i) = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^i \beta_{ik} F(x_k) \bar{\varphi}_i(x) \right\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta} \right\| (x - x_i) \leq \frac{M}{n}. \end{aligned} \quad (2.13)$$

□

3. Numerical Experiment

For showing the effectiveness of our method, we consider the following problems.

Example 3.1 (see [2, 3]). Considering the following third-order boundary values problem

$$\begin{aligned} u'''(x) - xu(x) &= (x^3 - 2x^2 - 5x - 3)e^x, \quad 0 \leq x \leq 1, \\ u(0) = u(1) &= 0, \quad u'(0) = 1, \end{aligned} \quad (3.1)$$

where the exact solution is $u_T(x) = x(1-x)e^x$. By the homogeneous process of the boundary condition, let $v(x) = u(x) - x(1-x)$, problem (3.8) can be transformed into the equivalent form

$$\begin{aligned} v'''(x) - xv(x) &= (1-x)x^2 + (x^3 - 2x^2 - 5x - 3)e^x, \quad 0 \leq x \leq 1, \\ v(0) &= v(1) = 0, \quad v'(0) = 0. \end{aligned} \quad (3.2)$$

The numerical results are presented in Tables 1, 2, and 3.

Example 3.2 (see [11, 14]). Considering the following third-order obstacle problems:

$$u'''(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{4}, \quad \frac{1}{4} \leq x \leq \frac{1}{4}, \\ u(x) - 1, & \frac{3\pi}{4} \leq x \leq 1, \end{cases} \quad (3.3)$$

where $u(0) = u'(0) = u'(1) = 0$, the exact solution is

$$u_T(x) = \begin{cases} \frac{1}{2}a_1x^2, & 0 \leq x \leq \frac{1}{4}, \\ 1 + a_2e^x + e^{-x/2} \left[a_3 \cos \frac{\sqrt{3}x}{2} + a_4 \sin \frac{\sqrt{3}x}{2} \right], & \frac{1}{4} \leq x \leq \frac{3}{4} \\ \frac{1}{2}a_5x(x-2) + a_6, & \frac{3}{4} \leq x \leq 1, \end{cases} \quad (3.4)$$

where

$$\begin{aligned} a_1 &= 0.24391096222648, & a_2 &= -0.17847234452746, & a_3 &= -0.81893573565615, \\ a_4 &= -0.30266818001866, & a_5 &= -0.24213890868443, & a_6 &= -0.06537630092111. \end{aligned} \quad (3.5)$$

The numerical results are presented in Table 4.

Example 3.3. Considering the following boundary value problems with nonclassical condition:

$$\begin{aligned} u'''(x) + \frac{1}{x\sqrt{1-x}}u''(x) + \frac{1}{x\sqrt{x(1-x)}}u'(x) + \frac{1}{x^2\sqrt{1-x}}u(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \quad u'(0) = \int_0^1 u(x)dx, \quad 2u(1) = u'(1). \end{aligned} \quad (3.6)$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = xe^x$. The numerical results are presented in Table 5.

Table 1: The absolute error of Example 3.1.

x	Absolute error						Ref. [3]	Ref. [2]	Present ($n = 500$)
	Present ($n = 20$)	Present ($n = 40$)	Present ($n = 80$)	Present ($n = 160$)	Present ($n = 320$)	Present ($n = 640$)			
0.0	0	0	0	0	0	0	0	0	0
0.1	1.52298×10^{-5}	2.85196×10^{-6}	5.80026×10^{-7}	1.27547×10^{-7}	1×10^{-7}	1×10^{-7}	0	1.18126×10^{-8}	
0.3	7.42332×10^{-5}	1.5986×10^{-5}	3.66544×10^{-6}	8.74382×10^{-7}	3×10^{-7}	3×10^{-7}	5×10^{-3}	8.65902×10^{-8}	
0.4	1.08810×10^{-4}	2.42167×10^{-5}	5.67330×10^{-6}	1.37023×10^{-6}	5×10^{-7}	5×10^{-7}	7×10^{-3}	1.36944×10^{-7}	
0.5	1.40256×10^{-4}	3.19274×10^{-5}	7.58382×10^{-6}	1.84582×10^{-6}	6×10^{-7}	6×10^{-7}	8×10^{-3}	1.85506×10^{-7}	
0.7	1.68693×10^{-4}	3.95170×10^{-5}	9.54427×10^{-6}	2.3440×10^{-6}	7×10^{-7}	7×10^{-7}	7×10^{-3}	2.37089×10^{-7}	
0.9	9.78854×10^{-5}	2.33293×10^{-5}	5.68883×10^{-6}	1.40422×10^{-6}	4×10^{-7}	4×10^{-7}	4×10^{-3}	1.42538×10^{-7}	
1.0	1.50887×10^{-5}	1.74768×10^{-5}	1.57219×10^{-5}	1.90655×10^{-5}	0	0	1.4×10^{-2}	1.49019×10^{-5}	
0 s < CPU runtime \leq 1 s CPU runtime \leq 10 s CPU runtime = 180 s									

Table 2: The numerical results of Example 3.1.

x	$u'_T(x)$	$u'_{500}(x)$	$ u'_T(x) - u'_{500}(x) $	$u''_T(x)$	$u''_{500}(x)$	$ u''_T(x) - u''_{500}(x) $
0	1	1	0	0	0.0000123873	0.0000123873
0.1	0.983602	0.983602	2.217025×10^{-7}	-0.342603	-0.342601	1.904429×10^{-6}
0.3	0.823414	0.823414	4.842037×10^{-7}	-1.33636	-1.33636	6.462399×10^{-7}
0.5	0.41218	0.412181	4.459369×10^{-7}	-2.88526	-2.88526	1.127018×10^{-6}
0.7	-0.382613	-0.382613	1.172353×10^{-8}	-5.21562	-5.21562	3.578872×10^{-6}
0.9	-1.74632	-1.74632	1.0452710×10^{-6}	-8.63321	-8.63321	6.928269×10^{-6}
1.	-2.71828	-2.71828	1.8404898×10^{-6}	-10.8731	-10.8731	9.029439×10^{-6}

Table 3: The numerical results of Example 3.1.

x	$u^{(3)}_T(x)$	$u^{(3)}_{500}(x)$	$ u^{(3)}_T(x) - u^{(3)}_{500}(x) $	$u^{(4)}_T(x)$	$u^{(4)}_{500}(x)$	$ u^{(4)}_T(x) - u^{(4)}_{500}(x) $
0	-3	-3.01001	0.010006	-8	-3.01001	4.98999
0.2	-4.93447	-4.93447	8.518423×10^{-9}	-11.53	-11.5489	0.0188419
0.4	-7.69782	-7.69782	5.477735×10^{-8}	-16.3504	-16.3771	0.0266921
0.6	-11.5887	-11.5887	1.335948×10^{-8}	-22.8858	-22.9229	0.0370827
0.8	-17.0031	-17.0031	1.724353×10^{-8}	-31.6917	-31.7423	0.050624
1.	-24.4645	-24.4645	2.771117×10^{-13}	-43.4925	3.021444×10^{-16}	43.4925

Table 4: The observed maximum errors of Example 3.2.

Method	$n = 16$	$n = 32$	$n = 64$	$n = 128$
Our method	1.18×10^{-3}	5.47×10^{-4}	2.62×10^{-4}	1.29×10^{-4}
Nonpolynomial spline [11]	7.12×10^{-4}	4.05×10^{-4}	2.22×10^{-4}	1.15×10^{-4}
Quartic spline [12]	1.15×10^{-3}	5.32×10^{-4}	2.56×10^{-4}	1.26×10^{-4}
Finite difference [13]	1.96×10^{-4}	4.89×10^{-5}	1.22×10^{-5}	3.06×10^{-6}
Cubic spline [14]	1.23×10^{-3}	5.53×10^{-4}	2.61×10^{-4}	1.27×10^{-4}
Colloc. quantic spline [15]	1.26×10^{-3}	5.60×10^{-4}	3.10×10^{-4}	1.67×10^{-4}
Finite difference [16]	6.89×10^{-3}	7.11×10^{-3}	7.27×10^{-3}	7.36×10^{-3}
Quartic B spline [17]	1.13×10^{-3}	5.30×10^{-4}	5.52×10^{-4}	1.23×10^{-4}

Table 5: The numerical results of Example 3.3.

x	$u_T(x)$	$u_{100}(x)$	$ u_T(x) - u_{100}(x) $	$ u'_T(x) - u'_{100}(x) $	$ u''_T(x) - u''_{100}(x) $	$ u'''_T(x) - u'''_{100}(x) $
0	0	0	0	2.11892×10^{-5}	5.34582×10^{-5}	2.49157×10^{-2}
0.1	0.110517	0.110518	1.06299×10^{-6}	6.59012×10^{-6}	4.42396×10^{-6}	2.85088×10^{-4}
0.3	0.404958	0.404962	4.25342×10^{-6}	3.27006×10^{-5}	2.16465×10^{-4}	1.15677×10^{-3}
0.5	0.824361	0.824376	1.51582×10^{-5}	7.46045×10^{-5}	1.67531×10^{-4}	8.58014×10^{-4}
0.7	1.40963	1.40966	3.24736×10^{-5}	9.40379×10^{-5}	3.75442×10^{-5}	5.12073×10^{-4}
0.8	1.78043	1.78047	4.20348×10^{-5}	9.71987×10^{-5}	3.89027×10^{-5}	5.59346×10^{-4}
0.9	2.21364	2.21369	5.20699×10^{-5}	1.05201×10^{-4}	1.40742×10^{-4}	1.08743×10^{-3}
1.	2.71828	2.71835	6.36065×10^{-5}	1.27213×10^{-4}	1.9082×10^{-4}	1.07270×10^{-2}

Table 6: The numerical results of Example 3.4, $p = q = \beta = 1$, $\alpha = 2$, $k = 2$, $n = 20$.

x	u_T	u_{20}	$ u_T - u_{20} $	u'_T	u'_{20}	$ u'_T - u'_{20} $	$ u''_T - u''_{20} $
0.08	-0.0286650	-0.0286650	5.20529×10^{-9}	0.0273871	0.0273873	2.32501×10^{-7}	7.75366×10^{-6}
0.16	-0.0255710	-0.0255710	6.94285×10^{-9}	0.0490631	0.0490632	7.15527×10^{-8}	2.10654×10^{-6}
0.24	-0.0209523	-0.0209523	9.70246×10^{-9}	0.0655842	0.0655842	1.41703×10^{-8}	5.37958×10^{-7}
0.32	-0.0152036	-0.0152035	9.39437×10^{-9}	0.0773741	0.0773741	1.84526×10^{-8}	3.37592×10^{-7}
0.48	-0.0017588	-0.0017588	1.71495×10^{-9}	0.0878568	0.0878567	8.12846×10^{-8}	4.48111×10^{-7}
0.56	0.00525584	0.00525583	6.04974×10^{-9}	0.0868185	0.0868184	1.11842×10^{-7}	4.02874×10^{-7}
0.64	0.0120206	0.0120205	1.65310×10^{-8}	0.0815939	0.0815937	1.53109×10^{-7}	6.25311×10^{-7}
0.72	0.0181958	0.0181957	3.10494×10^{-8}	0.0720489	0.0720486	2.13478×10^{-7}	9.11482×10^{-7}
0.88	0.0273346	0.0273346	8.10037×10^{-8}	0.0389010	0.0389005	4.47861×10^{-7}	1.32613×10^{-6}
0.96	0.0295066	0.0295065	1.25247×10^{-7}	0.0144477	0.0144469	8.40832×10^{-7}	9.84025×10^{-6}

Table 7: The numerical results of Example 3.5.

x	u_T	u_{20}	$ u_T - u_{20} $	u'_T	u'_{20}	$ u'_T - u'_{20} $	$ u''_T - u''_{20} $
0.08	0.000256	0.00025618	1.87335×10^{-7}	0.0096	0.009597	3.13184×10^{-6}	1.14007×10^{-5}
0.16	0.002048	0.0020481	1.72967×10^{-7}	0.0384	0.038399	8.39162×10^{-7}	2.89347×10^{-5}
0.24	0.006912	0.0069121	1.38265×10^{-7}	0.0864	0.0864	1.89713×10^{-7}	1.24163×10^{-5}
0.32	0.016384	0.0163841	1.13116×10^{-7}	0.1536	0.1536	2.05885×10^{-7}	5.02804×10^{-7}
0.4	0.032000	0.0320001	9.39829×10^{-8}	0.2400	0.2400	2.21979×10^{-7}	1.40284×10^{-6}
0.48	0.055296	0.0552961	7.77508×10^{-8}	0.3456	0.3456	1.93052×10^{-7}	5.56607×10^{-7}
0.56	0.087808	0.0878081	6.38527×10^{-8}	0.4704	0.4704	1.60088×10^{-7}	3.15123×10^{-7}
0.64	0.131072	0.131072	5.21361×10^{-8}	0.6144	0.6144	1.33773×10^{-7}	3.10240×10^{-7}
0.72	0.186624	0.186624	4.23781×10^{-8}	0.7776	0.7776	1.13440×10^{-7}	1.75032×10^{-7}
0.8	0.256000	0.256000	3.44071×10^{-8}	0.9600	0.9600	9.20785×10^{-8}	3.49872×10^{-7}
0.88	0.340736	0.340736	2.78679×10^{-8}	1.1616	1.1616	4.41977×10^{-8}	3.00596×10^{-8}
0.96	0.442368	0.442368	2.08726×10^{-8}	1.3824	1.3824	2.08167×10^{-8}	9.08609×10^{-6}

Example 3.4. Considering the following singular third-order three points boundary value problems with nonlinear condition

$$u'''(x) + \frac{p}{x^\alpha} u''(x) - \frac{k^2}{x^\alpha (1-x)^\beta} u'(x) + \frac{q \sin(x)}{x^\alpha (1-x)^\beta} u(x) = f(x), \quad 0 < x < 1, \quad (3.7)$$

$$u'(1) = 0, \quad 2u'(0) = (u''(0))^2, \quad u\left(\frac{1}{2}\right) = 0.$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = r(k(2x-1) - 2 \sinh(kx) + 2(\cosh(kx)) \tanh(k/2))/2k^3$. The numerical results are presented in Table 6.

Example 3.5. Considering the following boundary value problems with nonlinear condition:

$$\begin{aligned} u'''(x) - \frac{1}{\sqrt{1+x}} u'(x) + 2u(x) &= f(x), \\ u(0) = u'(0) &= 0, \quad (u'(1))^2 = 2 \int_0^1 u(x) dx. \end{aligned} \tag{3.8}$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = (1/2)x^3$. The numerical results are presented in Table 7.

4. Conclusions and Remarks

In this work, we present an algorithm for solving third-order mixed boundary value problems (BVPs) based on the reproducing kernel method. The method can be generalized to get reproducing kernel of problem with linear conditions. All computations are performed by the Mathematica 7.0 software package.

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References

- [1] J. Rashidinia and M. Ghasemi, "B-spline collocation for solution of two-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2325–2342, 2011.
- [2] H. N. Caglar, S. H. Caglar, and E. H. Twizell, "The numerical solution of third-order boundary-value problems with fourth-degree B-spline functions," *International Journal of Computer Mathematics*, vol. 71, no. 3, pp. 373–381, 1999.
- [3] A. Khan and T. Aziz, "The numerical solution of third-order boundary-value problems using quintic splines," *Applied Mathematics and Computation*, vol. 137, no. 2-3, pp. 253–260, 2003.
- [4] I. A. Tirmizi, E. H. Twizell, and Siraj-Ul-Islam, "A numerical method for third-order non-linear boundary-value problems in engineering," *International Journal of Computer Mathematics*, vol. 82, no. 1, pp. 103–109, 2005.
- [5] M. Tatari and M. Dehghan, "The use of the Adomian decomposition method for solving multipoint boundary value problems," *Physica Scripta*, vol. 73, no. 6, pp. 672–676, 2006.
- [6] M. Haque, M. H. Baluch, and M. F. N. Mohsen, "Solution of multiple point nonlinear boundary value problems by method of weighted residuals," *International Journal of Computer Mathematics*, vol. 18, no. 3, pp. 341–354, 1986.
- [7] Y.-I. Wang and L. Chao, "Using reproducing kernel for solving a class of partial differential equation with variable-coefficients," *Applied Mathematics and Mechanics*, vol. 29, no. 1, pp. 129–137, 2008.
- [8] Y. L. Wang, Z. Y. Li, Y. Cao, and X. H. Wan, "A new method for solving a class of mixed boundary value problems with singular coefficient," *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2768–2772, 2010.
- [9] Z. Chen and Z.-J. Chen, "The exact solution of system of linear operator equations in reproducing kernel spaces," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 56–61, 2008.
- [10] Y. L. Wang, C. L. Temuer, and J. Pang, "New algorithm for second-order boundary value problems of integro-differential equation," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 1–6, 2009.

- [11] S. Islam, S. I. A. Tirmizi, and M. A. Khan, "Non-polynomial splines approach to the solution of a system of third-order boundary-value problems," *Applied Mathematics Computation*, vol. 168, no. 1, pp. 125–163, 2005.
- [12] M. A. Noor and E. E. Al-Said, "Quartic splines solutions of third-order obstacle problems," *Applied Mathematics and Computation*, vol. 153, no. 2, pp. 307–316, 2004.
- [13] E. A. Al-Said, "Numerical solutions for system of third-order boundary value problems," *International Journal of Computer Mathematics*, vol. 78, no. 1, pp. 111–121, 2001.
- [14] E. A. Al-Said and M. A. Noor, "Cubic splines method for a system of third-order boundary value problems," *Applied Mathematics and Computation*, vol. 142, no. 2-3, pp. 195–204, 2003.
- [15] M. A. Noor and A. K. Khalifa, "A numerical approach for odd-order obstacle problems," *International Journal of Computer Mathematics*, vol. 54, no. 1, pp. 109–116, 1994.
- [16] E. A. Al-Said, M. A. Noor, and A. K. Khalifa, "Finite difference scheme for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 89, no. 2, pp. 453–459, 1996.
- [17] F. Gao and C.-M. Chi, "Solving third-order obstacle problems with quartic B-splines," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 270–274, 2006.