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Research Article

Weak Solution to a Parabolic Nonlinear System Arising in Biological Dynamic in the Soil

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We study a nonlinear parabolic system governing the biological dynamic in the soil. We prove global existence (in time) and uniqueness of weak and positive solution for this reaction-diffusion semilinear system in a bounded domain, completed with homogeneous Neumann boundary conditions and positive initial conditions.

1. Introduction

Modelling biological dynamic in the soil is of great interest during these last years. Several attempts are made in $1D$, $2D$, and rarely in $3D$. For more details, readers are referred to [1–3]. We deal here with the mathematical study of the model described in [2].

Let $T > 0$ be a fixed time, $\Omega \subset \mathbb{R}^3$ an open smooth bounded domain, $Q_T =]0,T[\times \Omega$, and $\Gamma_T =]0,T[\times \partial\Omega$.

The set of equations describing the organic matter cycle of decomposition in the soil is given by the following system:

$$(\mathcal{S}) \begin{cases} \dfrac{\partial u_i}{\partial t} - D_i \Delta u_i + q_i(u) u_i = f_i(u) & \text{in } Q_T, \\ \dfrac{\partial u_i}{\partial n} = 0 & \text{over } \Gamma_T, \\ u_i(0,x) = u_{0i}(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

for $i = 1, \dots, 6$.

We have noticed $u = (u_1, u_2, \dots, u_6)^T$ with u_1 is the density of microorganisms (MB), u_2 is the density of DOM, u_3 is the density of SOM, u_4 is the density of FOM, u_5 is the density of enzymes, and u_6 is the density of CO_2,

$$\begin{gathered}
q_1(u) = -\frac{ku_2}{K_s + u_2} + \mu + r + \nu, \qquad q_2(u) = \frac{ku_1}{K_s + u_2}, \\
q_3(u) = \frac{c_1 u_5}{K_m + u_5}, \qquad q_4(u) = \frac{c_2 u_5}{K_m + u_5}, \qquad q_5(u) = \zeta, \qquad q_6(u) = 0, \\
f_1(u) = 0, \qquad f_2(u) = \frac{u_5}{K_m + u_5}(c_1 u_3 + c_2 u_4) + \frac{\zeta u_5 + \mu u_1}{2}, \\
f_3(u) = \frac{\zeta u_5 + \mu u_1}{2}, \qquad f_4(u) = 0, \qquad f_5(u) = \nu u_1, \qquad f_6(u) = r u_1,
\end{gathered} \tag{1.2}$$

with μ mortality rate, r is the breathing rate, ν is the enzymes production rate, ζ is the transformation rate of deteriorated enzymes, c_1 is the maximal transformation rate of SOM, c_2 is the maximal transformation rate of FOM, k maximal growth rate, K_m and K_s represent half-saturation constants, and D_i, $i = 1$ to 6, are strictly positive constants.

System $(\mathcal{S})$ is introduced in [2]. To our knowledge, it is the first time that diffusion is used to model biological dynamics and linking it to real soil structure described by a 3D computed tomography image.

Similar systems to $(\mathcal{S})$ operate in other situations. It comes in population dynamics as Lotka-Voltera equation which corresponds to the case $f = 0$, u_i denoting the densities of species present and q_i growth rate. This system is also involved in biochemical reactions. In this case, the u_i are the concentrations of various molecules, q_i is the rate of loss, and f_i represents the gains.

For models in biology, interested reader can consult with profit [4] where the author presents some models based on partial differential equations and originating from various questions in population biology, such as physiologically structured equations, adaptative dynamics, and bacterial movement. He describes original mathematical methods like the generalized relative entropy method, the description of Dirac concentration effects using a new type of Hamilton-Jacobi equations, and a general point of view on chemotaxis including various scales of description leading to kinetic, parabolic, or hyperbolic equations.

Theoretical study of semilinear equations is widely investigated. Some interesting mathematical difficulties arise with these equations because of blowup in finite time, nonexistence and uniqueness of solution, singularity of the solutions, and noncontinuity of the solution regarding data.

In [5], the authors prove the blowup in finite time for the system in $1D$,

$$\begin{gathered}
u_t = u_{xx} - a(x,t) f(u) \quad 0 < x < 1, \ t \in (0,T), \\
u_x(0,t) = 0 \quad t \in (0,T), \\
u_x(1,t) = b(t) g(u(1,t)) \quad t \in (0,T).
\end{gathered} \tag{1.3}$$

A sufficient condition for the blowup of the solution of parabolic semilinear second-order equation is obtained in [6] with nonlinear boundary conditions, and so the set in which the explosion takes place. He also gives a sufficient condition for the solution of this equation which tends to zero, and its asymptotic behavior.

Existence and uniqueness of weak solutions for the following system are considered in [7]:

$$(\partial_t + \mathcal{L})u + F(t,x,u,\nabla u) = 0 \quad \forall(t,x) \in [0,T[\, \times \mathbb{R}^d, \\ u(x,T) = g(x) \quad \forall x \in \mathbb{R}^d, \tag{1.4}$$

with

$$F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m, \quad b \in C_b^2\left(\mathbb{R}^d;\mathbb{R}^d\right), \quad \sigma \in C_b^3\left(\mathbb{R}^d;\mathbb{R}^{d\times d}\right), \quad a = \sigma\sigma^t, \\ \mathcal{L} = \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.5}$$

with obstacles, giving a probabilistic interpretation of solution. This problem is solved using a probabilistic method under monotony assumptions.

By using bifurcation theory, in [8], authors determine the overall behavior of the dynamic system

$$\frac{\partial u}{\partial t} = \mathfrak{D}\Delta u + uf(x,u) \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0 \quad x \in \Omega. \tag{1.6}$$

A Cauchy problem for parabolic semilinear equations with initial data in $H_p^s(\mathbb{R}^n)$ is studied in [9]. Particularly the author solves local existence using distributions data.

Michel Pierre's paper, see [10], presents few results and open problems on reaction-diffusion systems similar to the following one:

$$\begin{aligned} &\forall i = 1,\dots,m, \\ &\partial_t u_i - d_i \Delta u_i = f_i(u_1,\dots,u_m) && \text{in}(0,T)\times\Omega, \\ &\alpha_i \frac{\partial u_i}{\partial n} + (1-\alpha_i)u_i = \beta_i && \text{on } (0,T)\times\partial\Omega, \\ &u_i(0) = u_{i0} && \text{in } \Omega, \end{aligned} \tag{1.7}$$

where the $f_i : \mathbb{R}^m \to \mathbb{R}$ are C^1 functions of $u = (u_1,\dots,u_m)$, and $d_i \in (0,\infty)$, $\alpha_i \in [0,1]$, $\beta_i \in C^2([0,T]\times\overline{\Omega})$, $\beta_i \geq 0$.

The systems usually satisfy the two main properties:

(i) the positivity of the solutions is preserved for all time,

(ii) the total mass of the components is uniformly controlled in time.

He recalls classical local existence result [11–13] under the above hypothesis.

It is assumed throughout the paper that

(i) all nonlinearities are quasipositives,

(ii) they satisfy a "mass-control structure"

$$\forall r = (r_1, \ldots, r_n), \quad \sum_{i=1}^{m} f(r) \leq C\left(1 + \sum_{i=1}^{m} r_i\right), \quad C > 0. \tag{1.8}$$

It follows that the total mass is bounded on any interval. Few examples of reactions-diffusion systems for which these properties hold are studied.

Systems where the nonlinearities are bounded in $L^1((0,T) \times \Omega)$ are also considered, for instance, for f_i in $L^1((0,T),\Omega)$ whose growth rate is less than $|u|^{(N+2)/N}$ when N tends to $+\infty$ [14].

Other situations are investigated, namely, when the growth of the nonlinearities is not small. But many questions are still unsolved, so several open problems are indicated.

A global existence result for the following system:

$$\begin{gathered}
\partial_t u - d_1 \Delta u = f(u,v), \\
\partial_t v - d_2 \Delta v = g(u,v), \\
u(0,\cdot) = u_0(\cdot) \geq 0, \qquad v(0,\cdot) = v_0(\cdot) \geq 0 \\
\text{with either: } \frac{\partial u}{\partial n} = \beta_1, \ \frac{\partial v}{\partial n} = \beta_2 \quad \text{on } (0,+\infty) \times \partial\Omega, \\
\text{or: } \quad u = \beta_1, \ v = \beta_2 \quad \text{on } (0,+\infty) \times \partial\Omega,
\end{gathered} \tag{1.9}$$

where $d_1, d_2 \in (0,+\infty)$, $\beta_1, \beta_2 \in [0,+\infty)$, and $f, g : [0,+\infty)^2 \to \mathbb{R}$ are C^1, holds for the additional following hypothesis:

$$\begin{gathered}
\forall u \geq U, \ \forall v \geq 0, \quad f(u,v) \leq C[1 + u + v], \quad U, C \geq 0, \\
\exists r \geq 1, \ \forall \ u, \ v \geq 0, \quad \left|g(u,v)\right| \leq C[1 + u^r + v^r].
\end{gathered} \tag{1.10}$$

This approach has been extended to $m \times m$ systems for which $f_1, f_1 + f_2, f_1 + f_2 + f_3, \ldots$ are all bounded by a linear of the u_i (see [15]).

However, $L^\infty(\Omega)$-blow up may occur in finite time for polynomial 2×2 systems as proved in [16, 17].

A very general result for systems which preserves positivity and for which the nonlinearities are bounded in L^1 may be found in [18]. It is assumed that, for all $i = 1, \ldots, m$,

$$\begin{gathered}
f_i : \ ]0,T[\times \Omega \times [0,+\infty)^m \longrightarrow \mathbb{R} \text{ is measurable}, \quad f_i(\cdot,0) \in L^1(]0,T[\times \Omega) \\
\exists K : \ ]0,T[\times \Omega \times [0,+\infty) \longrightarrow [0,+\infty) \quad \text{with } \forall M > 0, \ K(\cdot,M) \in L^1(]0,T[\times \Omega) \\
\text{and a.e.}(t,x) \in]0,T[\times \Omega, \quad \forall r, \tilde{r} \in [0,+\infty)^m \quad \text{with } |r|, |\tilde{r}| \ \leq M, \\
\left| f_i(t,x,r) - f_i(t,x,\tilde{r}) \right| \leq K(t,x,M)|r - \tilde{r}|, \quad \forall r \in [0,+\infty)^m, \\
f_i(t,x,r_1,\ldots,r_{i-1},0,r_{i+1},\ldots,r_m) \geq 0,
\end{gathered} \tag{1.11}$$

there is a sequence which converges in $L^1(]0,T[\times\Omega)$ to a supersolution of (1.7).

One consequence is that global existence of weak solutions for systems whose nonlinearities are at most quadratic with $u_0 \in L^2(\Omega)^m$ can be obtained.

Results are also obtained in the weak sense for systems satisfying $W_t - \Delta Z \leq H$, where $W = \sum_i u_i$, $Z = \sum_i d_i u_i$, and $H \in L^2(]0,T[\times\Omega)$.

The aim of our paper is to study the global existence in time of solution for the system $(\mathcal{S})$. In our work, we use an approach based both on variational method and semigroups method to demonstrate existence and uniqueness of weak solution.

The difficulty is that u being in the denominator of some $q_i(u)$ and $f_i(u)$, it is necessary to guarantee that u is nonnegative to avoid explosion of these expressions, whereas the classical methods assume that these expressions are bounded.

For instance, to show that weak solution is positive with an initial positive datum, Stampachia's method uses majoration of $q_i(u)$ by a function of t.

In our work, we show existence and unicity of a global positive weak solution of System $(\mathcal{S})$ for an initial positive datum.

The work is organized as follows. In the first part, we recall some preliminary results concerning variational method and semigroups techniques. In the second part, we prove, using these methods, existence, uniqueness, and positivity of weak solution under assumptions of positive initial conditions.

2. Preliminary Results

2.1. Variational Method (See [19])

We consider two Hilbert spaces H and V such that V is embedded continuously and densely in H.

Then, we have duality $H' \hookrightarrow V'$. Using Riesz theorem, we identify H and H'. So we get $V \hookrightarrow H \hookrightarrow V'$.

Definition 2.1. We define the Hilbert space

$$W(0,T,V,V') = \left\{ u \in L^2(]0,T[,V) \text{ such that } \frac{\partial u}{\partial t} \in L^2(]0,T[,V') \right\}, \tag{2.1}$$

equipped with the norm

$$\|u\|_W^2 = \|u\|_{L^2(]0,T[,V)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(]0,T[,V')}^2. \tag{2.2}$$

We assume the two following lemmas, see [19].

Lemma 2.2. *There exists a continuous prolongation operator P from $W(0,T,V,V')$ to $W(-\infty, +\infty, V, V')$ such that*

$$Pu_{|]0,T[} = u \quad \forall u \in W(0,T,V,V'). \tag{2.3}$$

Lemma 2.3. *$\mathfrak{D}(\mathbb{R}, V)$ is dense in $W(-\infty,+\infty,V,V')$.*

Corollary 2.4. *$C^\infty([0,T],V)$ is dense in $W(0,T,V,V')$.*

Proof. If $u \in W(0,T,V,V')$, one takes a sequence u_n of $\mathfrak{D}(\mathbb{R},V)$ which converges in $W(-\infty,+\infty,V,V')$ toward Pu, and then $(u_n)_{|]0,T[}$ converges toward u and $(u_n)_{|]0,T[} \in C^\infty([0,T],V)$ for all $n \in \mathbb{N}$. □

Proposition 2.5. *Every element $u \in W(0,T,V,V')$ is almost everywhere equal to a function in $C^0([0,T],H)$.*

Furthermore, the injection of $W(0,T,V,V')$ into $C^0([0,T],H)$ is continuous when $C^0([0,T],H)$ is equipped with the supnorm.

Proof. See [19]. □

Application

For all $t \in [0,T]$, a bilinear form $(u,v) \to a(t;u,v)$ is given on $V \times V$ such that for u and v fixed, $t \to a(t;u,v)$ is measurable and

$$\exists M > 0, \text{ such that } |a(t;u,v)| \leq M\|u\|_V\|v\|_V \quad \forall u,v \in V. \tag{2.4}$$

For each fixed t, one defines a continuous linear application $A(\mathrm{t}) \in \mathcal{L}(V,V')$ by

$$\langle A(t)u,v\rangle = a(t;u,v). \tag{2.5}$$

Then, we have

$$\|A(t)\|_{\mathcal{L}(V,V')} \leq M. \tag{2.6}$$

Also we associate, for all fixed t, an unbounded operator in H whose domain is the set of $u \in V$ such that $v \to a(t;u,v)$ is continuous on V for the induced norm by H. It is exactly the set of $u \in V$ such that $A(t)u \in H$ and then

$$a(t;u,v) = (A(t)u,v)_H. \tag{2.7}$$

To simplify the writting the unbounded operator is noted $A(t)$.

Let $\varphi \in C^\infty([0,T],V)$, we have, for $v \in V$,

$$\frac{d}{dt}(\varphi(t),v)_H = \left(\frac{\partial\varphi}{\partial t},v\right)_H = \langle\varphi'(t),v\rangle, \tag{2.8}$$

where the bracket is the duality between V' and V because $V \hookrightarrow V'$. By density, if $u \in W(0,T,V,V')$, one has, for all $v \in V$,

$$\frac{d}{dt}(u(t),v)_H = \langle u'(t),v\rangle \quad \text{for a.e. } t. \tag{2.9}$$

The variational parabolic problem associated to the triple $(H, V, a(t; \cdot, \cdot))$ is the following.

Given $f(t) \in L^2(]0, T[; V')$ and $u_0 \in H$, find $u \in W(0, T, V, V')$ such that

$$(\mathcal{P}) \begin{cases} \dfrac{d}{dt}(u(t), v)_H + a(t; u(t), v) = (f(t), v) \quad \forall v \in V, \\ u(0) = u_0. \end{cases} \tag{2.10}$$

This problem is equivalent to

$$\begin{gathered} \frac{du(t)}{dt} + A(t)u(t) = f(t), \\ u(0) = u_0. \end{gathered} \tag{2.11}$$

Definition 2.6. The form a is coercive or V coercive if $\alpha > 0$ exists such that

$$a(t; u, u) \geq \alpha \|u\|_V^2 \quad \forall (t, u) \in [0, T] \times V. \tag{2.12}$$

Theorem 2.7. *If the form is coercive, then the problem* $(\mathcal{P})$ *admits a unique solution.*

Proof. See Dautray-Lions [19]. □

Definition 2.8. The form is H coercive if there exist two constants λ and $\alpha > 0$ such that

$$a(t; u, u) + \lambda \|u\|_H^2 \geq \alpha \|u\|_V^2, \quad \forall t \in [0, T], \ \forall u \in V. \tag{2.13}$$

If we set $u(t) = e^{\lambda t} w(t)$, then u is solution of $(\mathcal{P})$ if and only if w is solution of

$$(\mathcal{P}') \begin{cases} \dfrac{d}{dt}(w(t), v)_H + a(t; w, v) + \lambda (w, v)_H = \langle e^{-\lambda t} f(t), v \rangle \quad \forall v \in V, \\ w(0) = u_0. \end{cases} \tag{2.14}$$

Writing

$$b(t; u, v) = a(t; u, v) + \lambda (u, v)_H, \tag{2.15}$$

b is a coercive form, and then $(\mathcal{P}')$ admits a unique solution, and therefore $(\mathcal{P})$ too. We apply Theorem 2.7 in the following case:

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \tag{2.16}$$

and defining

$$a(t;u,v) = \sum_{i,j=1}^{3} \int_{\Omega} a_{ij}(t,x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \tag{2.17}$$

we assume that

$$a_{ij}(t,x) \in L^{\infty}(]0,T[\times \Omega) \quad \forall i,j, \tag{2.18}$$

and there exists $\alpha > 0$ such that, for all $\zeta = (\zeta_i)_{i=1,2,3}$, we have

$$\sum_{i,j=1}^{3} a_{ij}(t,x)\zeta_i\zeta_j \geq \alpha\|\zeta\|_{\mathbb{R}^3}^2 \text{ a.e. in }]0,T[\times \Omega. \tag{2.19}$$

Then, we deduce that

$$a(t;u,u) \geq \alpha \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}^2. \tag{2.20}$$

The form is then H coercive, and it suffices to take $\lambda = \alpha$.

In addition, let us take $a_0 \in L^{\infty}(]0,T[\times\Omega)$ with $a_0(t,x) \geq 0$ for all t, x.

The form

$$b(t;u,v) = a(t;u,v) + (a_0(t)u,v)_H \tag{2.21}$$

is still H coercive. We have the following theorem.

Theorem 2.9. *Under the previous hypothesis, problem* $(\mathcal{P})$ *associated to the triple* (H,V,b) *admits a unique solution for all* $u_0 \in H$ *and* $f \in L^2(]0,T[;V')$.

Moreover, if $u_0 \geq 0$ *and*

$$\langle f(t),v\rangle \geq 0 \quad \forall v \in V \text{ such that } v \geq 0,\ t \in]0,T[\text{ a.e.}, \tag{2.22}$$

one has $u(t) \geq 0$ *for all* $t \in [0,T]$.

Proof. It remains to show that the solution is nonnegative.

Given $u \in L^2(\Omega)$, we set $u^+ = \max(0,u(x))$ and $u^- = \max(0,-u(x))$. If $u \in H^1(\Omega)$, then we have u^+ and $u^- \in H^1(\Omega)$.

By replacing v by $u^-(t)$ in $(\mathcal{P})$, we obtain

$$\left\langle \frac{du}{dt}, u^-(t) \right\rangle + a\left(t;u(t),u^-(t)\right) + \left(a_0(t)u,u^-\right)_H = \langle f(t),u^-\rangle. \tag{2.23}$$

One gets

$$u = u^+ - u^-, \qquad u^+u^- = 0, \qquad \frac{du^+}{dt}u^- = 0, \tag{2.24}$$

and by linearity, we obtain

$$-\left\langle \frac{du^-}{dt}, u^- \right\rangle - a\left(t; u^-(t), u^-(t)\right) - \left(a_0(t)u^-, u^-\right)_H = \left\langle f(t), u^- \right\rangle, \tag{2.25}$$

but

$$\left\langle \frac{du^-}{dt}, u^- \right\rangle = \frac{1}{2}\frac{d}{dt}\left(u^-(t), u^-(t)\right)_H. \tag{2.26}$$

Since

$$\left(a(t)u^-, u^-\right) \geq 0, \qquad \left(a_0(t)u^-, u^-\right)_H \geq 0, \qquad \left\langle f(t), u^-(t)\right\rangle \geq 0, \tag{2.27}$$

it comes that

$$\frac{d}{dt}\left\|u^-(t)\right\|_H^2 \leq 0. \tag{2.28}$$

By integration over $]0,t[$, we deduce

$$\left\|u^-(t)\right\|_H^2 \leq \left\|u^-(0)\right\|_H^2, \quad \forall t. \tag{2.29}$$

But $u_0 \geq 0$, then $u_0^- = 0$, so $u^-(0) = 0$.
Hence, we conclude that $u^-(t) = 0$ for all t.
Instead of $a_0(t,x) \geq 0$, assume that

$$\exists C > 0 \quad \text{such that } a_0(t,x) \geq -C \quad \forall t, x. \tag{2.30}$$

As previously mentioned, if we set $u(t) = e^{\lambda t}w(t)$, $w(t)$ is solution of

$$\frac{d}{dt}(w(t), v)_H + a(t; w(t), v) + ((a_0(t) + \lambda)w, v)_H = \left\langle e^{-\lambda t} f(t), v \right\rangle, \tag{2.31}$$

with

$$a_0(t,x) + \lambda \geq \lambda - C. \tag{2.32}$$

It suffices to take $\lambda \geq C$ to reduce to the previous case, and $w(t) \geq 0$ implies $u(t) \geq 0$.
Then, we get. □

Corollary 2.10. *Consider the triple (H, V, a) satisfying assumptions of Theorem 2.9.*

If $a_0 \in L^\infty(]0,T[\times\Omega)$ and

$$\exists C > 0 \quad \text{such that } a_0(t,x) \geq -C \quad \text{for a.e. } (t,x) \in]0,T[\times\Omega, \tag{2.33}$$

then the variational problem

$$\frac{d}{dt}(u(t),v) + a(t;u,v) + (a_0(t)u,v)_H = \langle f(t),v\rangle \quad \forall v \in V, \qquad u(0) = u_0 \tag{2.34}$$

admits a unique solution in $W(0,T,V,V')$ for all $u_0 \in H$ and $f \in L^2(]0,T[,V')$.

Moreover, if $u_0 \geq 0$ and

$$\langle f(t),v\rangle \geq 0 \quad \forall v \in V,\ t \in]0,T[\text{a.e.}, \tag{2.35}$$

then one has $u(t) \geq 0$ for all $t \in [0,T]$.

Equivalence of the Variational Solution with the Initial Problem

We have

$$\partial Q_T = (\{0\}\times\Omega) \cup (\{T\}\times\Omega) \cup (]0,T[\times\partial\Omega). \tag{2.36}$$

For the sake of simplicity, we set $a_{ij} = \delta_{ij}$ which is the Kronecker symbol. Then, $A = -\Delta$ over $\mathfrak{D}(\Omega)$, and we have

$$\frac{\partial u}{\partial t} - \Delta u + a_0 u = f \quad \text{in }]0,T[\times\Omega. \tag{2.37}$$

We assume that $f \in L^2(]0,T[,H)$, then

$$\frac{\partial u}{\partial t} - \Delta u \in L^2(]0,T[,H) \tag{2.38}$$

if $u \in L^2(]0,T[,V)$. We set

$$v = (u, -\text{grad}(u))^T = \left(u, -\frac{\partial u}{\partial x_1}, -\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_3}\right)^T. \tag{2.39}$$

Consequently, we have $v \in (L^2(]0,T[\times\Omega))^4$ and

$$\text{div}(v) = \frac{\partial u}{\partial t} - \Delta u \in L^2(]0,T[\times\Omega). \tag{2.40}$$

Then $u(0,x)$, $u(T,x)$, and $\partial u/\partial n_{|]0,T[\times\partial\Omega}$ are well defined.

It remains to show that $\partial u/\partial n_{|]0,T[\times\partial\Omega} = 0$.

Let $\phi \in \mathfrak{D}(]0,T[)$, and we multiply $(2.34)_1$ by ϕ, and by integration over $]0,T[$, one gets

$$\int_0^T\int_\Omega \phi v\frac{\partial u}{\partial t}dt\,dx + \sum_{i=1}^{3}\int_0^T\int_\Omega \phi\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dt\,dx + \int_0^T\int_\Omega \phi a_0 uv\,dt\,dx = \int_0^T\int_\Omega f v\phi dt\,dx \quad \forall v \in H^1(\Omega). \tag{2.41}$$

Using Green formula with $(u, -\text{grad}(u))^T$, we have

$$\int_0^T\int_\Omega \phi v\left(\frac{\partial u}{\partial t} - \Delta u + a_0 u\right)dx\,dt = \int_0^T\int_\Omega f\phi v dx\,dt - \int_0^T\int_{\partial\Omega} \phi v\frac{\partial u}{\partial n}ds\,dt. \tag{2.42}$$

As

$$\frac{\partial u}{\partial t} - \Delta u + a_0 u = f, \tag{2.43}$$

we can conclude the following statement:

$$\int_0^T\int_{\partial\Omega} \phi v\frac{\partial u}{\partial n}dt\,ds = 0 \quad \forall(\phi, v) \in \mathfrak{D}(]0,T[) \times H^1(\Omega). \tag{2.44}$$

We deduce that

$$\left\langle v, \frac{\partial u}{\partial n}\right\rangle_{H^{1/2}H^{-1/2}} = 0 \quad \forall v \in H^1(\Omega) \text{ and } t \text{ a.e.} \tag{2.45}$$

Function $v \to v_{|\partial\Omega}$ from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$ being surjective, we deduce that

$$\frac{\partial u}{\partial n} = 0 \quad \text{in } H^{-1/2}(\Gamma) \quad \text{for } t \text{ a.e.} \tag{2.46}$$

2.2. Semigroup Method

Consider the variational triple (H, V, a) where a is independent of t. We associate operators $A \in \mathcal{L}(V, V')$ and A_H in H with

$$D(A_H) = \{u \in V \text{ such that } Au \in H\}. \tag{2.47}$$

Assume that a is H coercive, then A_H is the infinitesimal generator of semigroup $t \mapsto G(t)$ of class $\mathcal{C}^0$ over H, and $G(t)$ operates over V and V'. If we note $\widetilde{\mathcal{G}(t)}$ the extension of $G(t)$ by 0 for $t < 0$, then the Laplace transform of $\widetilde{\mathcal{G}(t)}$ is the resolvent of A_H.

Proposition 2.11. *For $u_0 \in H$ and $f \in L^2(]0,T[,V')$, problem $(\mathcal{P})$ which consists in finding $u \in W(0,T,V,V')$ such that*

$$\frac{du}{dt} + Au = f \quad \text{with } u(0) = u_0 \tag{2.48}$$

admits a unique solution given by

$$u(t) = G(t)u_0 + \int_0^t G(t-s)f(s)ds. \tag{2.49}$$

Proof. Note $\tilde{u}$ and $\tilde{f}$ the extensions by 0 of u and f outside $]0,T[$, then we have

$$\frac{d\tilde{u}}{dt} = \frac{du}{dt} + u(0)\delta(t) - u(T)\delta(T-t), \tag{2.50}$$

with $\delta(t)$ the Dirac measure on $\mathbb{R}$.

Thus,

$$\begin{aligned} \frac{d\tilde{u}}{dt} + A\tilde{u} &= \frac{du}{dt} + \widetilde{Au} + u(0)\delta(t) - u(T)\delta(T-t) \\ &= \tilde{f} + u(0)\delta(t) - u(T)\delta(T-t). \end{aligned} \tag{2.51}$$

Hence, an equation of the form

$$\frac{dU}{dt} + AU = F \quad \text{in } \mathfrak{D}'_+(V'), \tag{2.52}$$

where $\mathfrak{D}'_+(V')$ is the space of distributions over $\mathbb{R}$ into V'whose support is in $[0,+\infty[$. By Laplace transform, one is reduced to

$$(A + PI)\mathcal{L}(U) = \mathcal{L}(F), \tag{2.53}$$

where

$$\mathcal{L}(U) = (A + PI)^{-1}\mathcal{L}(F), \tag{2.54}$$

and therefore, we have $U = \tilde{G} * F$.

But since

$$\operatorname{supp}(\delta(T-t)) = T, \qquad \operatorname{supp}\left(\tilde{G} * (U(T)\delta(T-t))\right) \subset [T,+\infty[, \tag{2.55}$$

we have

$$U_{|]0,T[} = u = \left(\widetilde{G} * F\right)_{|]0,T[}. \tag{2.56}$$

Hence, we get the result. □

3. System $(\mathcal{S})$ Resolution

In this part, we go back to system $(\mathcal{S})$ with assumptions and will analyze this problem by using the framework described in the previous section.

We define $H = L^2(\Omega)$ and $V = H^1(\Omega)$ and the following hypothesis for initial conditions:

$$u_{01} \in L^\infty(\Omega), \qquad u_{0i} \in H \quad \text{for } i \in \{2,\dots,6\}, \qquad u_{0i} \geq 0 \quad \text{for } i \in \{1,\dots,6\}. \tag{3.1}$$

We will make a resolution component by applying Theorem 2.7 with, for each i, the form

$$\mathrm{a}(t; u, v) = \sum_{j=1}^{3} D_i \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx. \tag{3.2}$$

One approaches the solution by a sequence of solutions of linear equations.

3.1. Recursive Sequence of Solutions

For $n = 0$, we note that u_i^0 is the solution of

$$\begin{aligned} &\frac{\partial u_i^0}{\partial t} - D_i \Delta u_i^0 = 0 \quad \text{in } Q_T, \\ &u_i^0(0) = u_{0i} \qquad\quad \text{in } \Omega, \\ &\frac{\partial u_i^0}{\partial n}_{|\partial\Omega} = 0. \end{aligned} \tag{3.3}$$

This equation admits strong solution and $u_i^0 \geq 0$.

By induction, we note that u_i^n is solution of equation

$$\begin{aligned} \frac{\partial u_i^n}{\partial t} - D_i \Delta u_i^n + q_i\left(u^{n-1}\right) u_i^n &= f_i\left(u^{n-1}\right) \quad \text{in } Q_T, \\ u_i^n(0) = u_{0i} &\qquad\qquad\qquad \text{in } \Omega, \\ \frac{\partial u_i^n}{\partial n}_{|\partial\Omega} = 0. & \end{aligned} \tag{3.4}$$

It is a linear equation within the framework of Corollary 2.10 with $a_0 = q_i(u^{n-1})$ and $f(t) = f_i(u^{n-1}(t))$. Let us suppose that there exists a unique nonnegative solution u^{n-1}. Assuming by induction that $u_i^j \geq 0$ for $0 \leq j \leq n-1$, we have

$$0 \leq \frac{k u_2^{n-1}}{K_s + u_2^{n-1}} \leq k, \tag{3.5}$$

which implies that

$$\mu + r + \nu - k \leq q_1\left(u^{n-1}\right) \leq \mu + r + \nu. \tag{3.6}$$

u^{n-1} is nonnegative also that implies that there are two positive constants C_1, C_2 such that

$$0 \leq q_3\left(u^{n-1}\right) \leq C_1, \qquad 0 \leq q_4\left(u^{n-1}\right) \leq C_2. \tag{3.7}$$

For the rest, we notice that q_5 and q_6 are constant.

We have shown that $q_i(u^{n-1}) \in L^\infty(]0,T[\times\Omega)$ for $i \neq 2$. It remains to prove that the same property is satisfied by $q_2(u^{n-1})$.

To prove that $q_2(u^{n-1})$ is bounded, we need to show that $u_1^n \in L^\infty(0,T;L^\infty(\Omega))$.

Case of u_1^0

Let $k \in \mathbb{N}^*$, we multiply $(3.3)_1$ by $(u_1^0)^{2k-1}$ and integrate it over Ω, and it comes that

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega \left(u_1^0\right)^{2k} dx + (2k-1)D_1 \int_\Omega \left(u_1^0\right)^{2(k-1)} \left|\nabla u_1^0\right|^2 dx = 0. \tag{3.8}$$

The second term is nonnegative, then we have

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega \left(u_1^0\right)^{2k} dx \leq 0. \tag{3.9}$$

By integrating over $]0,t[$, we obtain

$$\left\|u_1^0(t)\right\|_{L^{2k}(\Omega)} \leq \|u_{01}\|_{L^{2k}(\Omega)}. \tag{3.10}$$

When k tends to $+\infty$, it comes that,

$$\left\|u_1^0(t)\right\|_{L^\infty(\Omega)} \leq \|u_{01}\|_{L^\infty(\Omega)}, \tag{3.11}$$

which implies that $u_1^0 \in L^\infty(0,T;L^\infty(\Omega))$.

Case of u_1^n with $n \in \mathbb{N}^*$

By induction, we suppose that $u_1^0, u_1^1, \dots, u_1^{n-1}$ are in $L^\infty(0,T;L^\infty(\Omega))$.

Remark 3.1. We make the following change: $w_i^n = e^{-\lambda t}u_i^n,\ i = 1,\dots,6$. We obtain

$$\frac{\partial w_1^n}{\partial t} - D_1 \Delta w_1^n + \left(\lambda + q_1\left(\left(e^{\lambda t} w_i^{n-1}\right)_i\right)\right) w_1^n = 0. \tag{3.12}$$

The function q_1 being undervalued, we can choose $\lambda \geq 0$ such that

$$\lambda + q_1\left(\left(e^{\lambda t} w_i^{n-1}\right)_i\right) \geq 0. \tag{3.13}$$

We multiply (3.12) by $(w_1^n)^{2k-1}$ and integrate it over Ω. We obtain

$$\begin{aligned} &\frac{1}{2k}\frac{d}{dt}\int_\Omega (w_1^n)^{2k} dx + (2k-1)D_1 \int_\Omega (w_1^n)^{2(k-1)} \left|\nabla w_1^n\right|^2 dx \\ &\quad + \int_\Omega \left(\lambda + q_1\left(\left(e^{\lambda t} w_i^{n-1}\right)_i\right)\right)(w_1^n)^{2k} dx = 0. \end{aligned} \tag{3.14}$$

The second and third term being nonnegative, we can conclude as in the previous case that

$$\left\|w_1^n(t)\right\|_{L^{2k}(\Omega)} \leq \|u_{01}\|_{L^{2k}(\Omega)}. \tag{3.15}$$

Since k tends to ∞, it follows that

$$\left\|w_1^n(t)\right\|_{L^\infty(\Omega)} \leq \|u_{01}\|_{L^\infty(\Omega)}. \tag{3.16}$$

As a result, we have proved that $w_1^n \in L^\infty(0,\infty;L^\infty(\Omega))$, and since $u_1^n = e^{\lambda t} w_1^n$, we have $u_1^n \in L^\infty(0,T;L^\infty(\Omega))$.

Conclusion 1. With the previous demonstration, we obtain by induction that if $u_{01} \in L^\infty(\Omega)$ with $u_{01} \geq 0$, then $q_i(u^n) \in L^\infty(]0,T[\times\Omega)$ for all n and $i = 1,\dots,6$.

We also have $f_i(u^{n-1}) \geq 0$ and $f_i(u^{n-1}) \in L^2(]0,T[;V')$. Then by means of Corollary 2.10, there exists a unique solution $u_i^n \in W(0,T,V,V')$ with $u_i^n \geq 0$.

3.2. Boundedness of the Solution

Let us show that the sequence is bounded. u_i^n satisfies (2.34), so

$$\frac{\partial}{\partial t}(u_i^n, v)_H + D_i a(u_i^n, v) + \left(q_i\left(u^{n-1}\right)u_i^n, v\right)_H = \left\langle f_i\left(u^{n-1}\right), v\right\rangle \quad \forall v \in V. \tag{3.17}$$

We remark that

$$\frac{\partial}{\partial t}(u_i^n, v)_H = \left\langle \frac{\partial u_i^n}{\partial t}, v \right\rangle. \tag{3.18}$$

For $\varphi \in C^\infty([0,T],V)$,

$$\left\langle \frac{\partial \varphi(t)}{\partial t}, \varphi(t) \right\rangle = \frac{1}{2}\frac{d}{dt}(\varphi(t), \varphi(t))_H. \tag{3.19}$$

By density and choosing $v = u_i^n(t)$, we have

$$\left\langle \frac{\partial u_i^n(t)}{\partial t}, u_i^n(t) \right\rangle = \frac{1}{2}\frac{d}{dt}(u_i^n(t), u_i^n(t))_H. \tag{3.20}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_H^2 + D_i a(u_i^n, u_i^n) + \left(q_i\left(u^{n-1}\right)u_i^n, u_i^n\right)_H = \left\langle f_i\left(u^{n-1}\right), u_i^n \right\rangle. \tag{3.21}$$

We have seen that we can obtain problem $(\mathcal{P}')$ replacing $u(t)$ by $w(t) = e^{-\lambda t}u(t)$, and since $0 \le t \le T$, if w is bounded, u is also bounded.

We take then $\lambda = \beta + \delta$, and one is reduced to

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_H^2 + D_i a(u_i^n, u_i^n) + \beta\|u_i^n\|_H^2 + \left(\left(\delta + q_i\left(u^{n-1}\right)\right)u_i^n, u_i^n\right)_H = \left\langle f_i\left(u^{n-1}\right), u_i^n \right\rangle. \tag{3.22}$$

The form $D_i a$ is H coercive, so we take β such that

$$D_i a(u,u) + \beta\|u\|_H^2 \ge \alpha\|u\|_V^2 \quad \forall u \in V. \tag{3.23}$$

The q_i are bounded, so

$$\exists \delta > 0 \quad \text{such that } \delta + q_i(u) \ge l \quad \forall i,\ u \ge 0. \tag{3.24}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_i^n\|_H^2 + \alpha\|u_i^n\|_V^2 + l\|u_i^n\|_H^2 &\le \|f_i(u^{n-1})\|_{V'}\|u_i^n\|_V \\ &\le \frac{\epsilon}{2}\|u_i^n\|_V^2 + \frac{1}{2\epsilon}\|f_i(u^{n-1})\|_{V'}^2 \quad \epsilon > 0. \end{aligned} \tag{3.25}$$

We take ϵ small enough such that $\alpha - \epsilon/2 = \gamma > 0$. Hence,

$$\frac{1}{2}\|u_i^n\|_H^2 + \gamma\|u_i^n\|_V^2 + l\|u_i^n\|_H^2 \le \frac{1}{2\epsilon}\left\|f_i\left(u^{n-1}\right)\right\|_{V'}^2. \tag{3.26}$$

For $i = 1$ and $i = 4$, $f_i = 0$. Therefore, by integration,

$$\frac{1}{2}\|u_i^n(t)\|_H^2 + \gamma \int_0^t \|u_i^n(s)\|_V^2 ds + l \int_0^t \|u_i^n(s)\|_H^2 \le \frac{1}{2}\|u_i^n(0)\|_H^2. \tag{3.27}$$

We deduce that (u_1^n) and (u_4^n) remain bounded in $C^0([0,T],H)$ and $L^2(]0,T[,V)$.

$f_5(u) = \nu u_1$, thus, $f_5(u^{n-1}) = \nu u_1^{n-1}$ remains bounded in $L^2(]0,T[,V)$; therefore, u_5^n has the same property as u_1^n, u_4^n. It is the same for u_6^n because $f_6(u) = r u_1$.

We have $f_3(u) = (\zeta u_5 + \mu u_1)/2$; therefore, we have the same conclusion for u_3^n and finally for u_2^n because $f_2(u)$ depends on u_1, u_2, u_4, and u_5.

3.3. Convergence of the Sequence

We deduce at this stage that the sequence $(u_i^n)_{n\ge 0}$ (one can extract subsequence $(u_i^m)_{m\ge 0}$) converges weakly in $L^2(]0,T[,V)$ to u_i and weakly star in $L^\infty(]0,T[,H)$ to u_i.

But it is not enough to pass to the limit in the equation, we need the pouctual convergence for almost all t to deduce that $u_i \ge 0$ for all i and to pass to the limit in $q_i(u^{n-1})$ and $f_i(u^{n-1})$. To pass to the limit, we need strong compactness. Using Proposition 2.11, for all n, we have

$$u_i^n(t) = \int_0^t G_i(t-s) g_i^n(s) ds + G_i(t) u_{0i}, \tag{3.28}$$

where $G_i(t)$ is the semigroup generated by the unbounded operator $-D_i A_H$. Let us denote

$$g_i^n(s) = -q_i\left(u^{n-1}(s)\right) u_i^n(s) + f_i\left(u^{n-1}(s)\right), \tag{3.29}$$

and we deduce $g_i^n \in L^2(]0,T[,V)$.

Moreover, the sequence $(u_i^n)_{n\ge 0}$ is bounded in $C^0([0,T],H)$ which implies that the sequence $(g_i^n)_{n\ge 0}$ is bounded in $C^0([0,T],H)$ for all i.

Then, we can conclude showing that operator $\mathcal{G}_i$ from $C^0([0,T],H)$ into $C^0([0,T],H)$ defined by

$$\mathcal{G}_i(f)(t) = \int_0^t G_i(t-s) f(s) ds \tag{3.30}$$

is compact.

One takes the triple $(L^2(\Omega), H^1(\Omega), a)$ with

$$a(u,v) = \sum_{j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx, \tag{3.31}$$

where Ω is regular and bounded. The unbounded variational operator A_H associated to a is a positive symmetric operator with compact resolvent. It admits a sequence $(\lambda_k)_k$ of positive eigenvalues with $\lim_{k\to+\infty}\lambda_k = +\infty$ and a Hilbert basis $(e_k)_k$ of H consisting of eigenvectors of A_H. If $(G(t))_{t>0}$ is the semigroup generated by $-A_H$, then for all $u_0 \in H$,

$$G(t)u_0 = \sum_{k=0}^{+\infty} e^{-t\lambda_k}(u_0, e_k)e_k, \tag{3.32}$$

which proves that the operator is compact for all $t > 0$ because

$$\lim_{k\to+\infty} e^{-t\lambda_k} = 0. \tag{3.33}$$

We have the same formula for $G_i(t)$, and it suffices to replace λ_k by $D_i\lambda_k$.

If we set

$$G_N(t)u = \sum_{k=0}^{N} e^{-t\lambda_k}(u, e_k)e_k, \tag{3.34}$$

then $G_N(t)$ is an operator with finite rank which converges to $G(t)$.

Theorem 3.2. *Let $t \to G(t)$ be an application from $[0,\infty[$ into $\mathcal{L}(H)$. One assumes that there exists a sequence of operators $(G_N(t))_{N\geq 0}$ of H with the following properties:*

(1) *for all N and all $t > 0$, $G_N(t)$ is finite rank independent of t,*

(2) *$t \to G_N(t)$ is continuous from $[0,\infty[$ into $\mathcal{L}(H)$ for all N,*

(3) *for $N \to \infty$, $G_N(t)$ converges to $G(t)$ in $L^1(]0,T[,\mathcal{L}(H))$ for all $T > 0$,*

then the operator $\mathcal{G}$ is compact from $C^0([0,T],H)$ into $C^0([0,T],H)$ for all $T > 0$.

Proof. Regarding property (3) of Theorem 3.2, $\mathcal{G}$ is well defined on $C^0([0,T],H)$, and we have

$$\left\| \int_0^t G(s)f(t-s)ds - \int_0^t G_N(s)f(t-s)ds \right\| \leq \left(\int_0^T \|G(s) - G_N(s)\| ds \right) \|f\|_\infty. \tag{3.35}$$

This proves that $\mathcal{G}_N$ converges to $\mathcal{G}$ in $\mathcal{L}([0,T],H)$ using property (3) of Theorem 3.2.

To show that $\mathcal{G}$ is compact, it suffices to show that for all N, $\mathcal{G}_N$ is compact.

Let $\mathcal{B}$ be a bounded set of $C^0([0,T],H)$, $\mathcal{G}_N(\mathcal{B})$ is bounded in $C^0([0,T],H)$, using Ascoli result, it will be relatively compact if $\mathcal{G}_N(\mathcal{B})$ is equicontinuous and if for all t_0 in $[0,T]$, $\mathcal{G}_N(\mathcal{B})(t_0)$ is relatively compact in H.

But $\mathcal{G}_N(\mathcal{B})(t_0)$ being bounded and embedded in a subspace of finite dimension of H is relatively compact in H. Then, let us show the equicontinuity on t_0.

Let M and C_N such that

$$\|f\|_\infty \leq M, \quad \forall f \in \mathcal{B}, \qquad \|\mathcal{G}_N(t)\| \leq C_N \quad \forall t \in [0,T]. \tag{3.36}$$

For $0 \leq t_0 \leq t \leq T$, one has

$$\mathcal{G}_N(f)(t) - \mathcal{G}_N(f)(t_0) = \int_0^{t_0} (G_N(t-s) - G_N(t_0-s)) f(s) ds + \int_{t_0}^{t} G_N(t-s) f(s) ds. \tag{3.37}$$

We have

$$\begin{gathered} \left\| \int_{t_0}^{t} G_N(t-s) f(s) ds \right\| \leq M C_N |t-t_0|, \\ \left\| \int_0^{t_0} (G_N(t-s) - G_N(t_0-s)) f(s) ds \right\| \leq M \int_0^{t_0} \| G_N(t-s) - G_N(t_0-s) \| ds, \end{gathered} \tag{3.38}$$

which tend to 0 when $t \to t_0$ using property (2) of Theorem 3.2 and the continuity under the integral.

We apply Theorem 3.2 to the semigroup generated by the Laplacien, and we obtain

(1) $G_N(t)$ is of rank $N+1$ for all $t \geq 0$,

(2) $\|G_N(t) - G_N(t_0)\|^2 \leq \sum_{k=0}^{N} |e^{-t\lambda_k} - e^{-t_0\lambda_k}|^2$ which tends to 0 if $t \to t_0$,

(3) $\|G(t)u - G_N(t)u\|^2 \leq \sum_{k=N+1}^{\infty} e^{-2t\lambda_k} |u_k|^2 \leq e^{-2t\lambda_{N+1}} \|u\|^2$ if we take an increasing sequence (λ_k), then one gets

$$\|G(t) - G_N(t)\| \leq e^{-t\lambda_{N+1}}. \tag{3.39}$$

Thus, we have

$$\int_0^T \|G(t) - G_N(t)\| dt \leq \frac{1}{\lambda_{N+1}}, \tag{3.40}$$

which tends to 0 if $N \to \infty$.

Thus, $\mathcal{G}_i$ is compact for all i. We have

$$u_i^n(t) = G_i(t) u_i^0 + \mathcal{G}_i(g_i^n)(t), \tag{3.41}$$

where $(g_i^n)_{n\geq 0}$ is bounded in $C^0([0,T],H)$, then $(u_i^n)_{n\geq 0}$ belongs to a relatively compact set of $C^0([0,T],H)$.

Therefore, from the sequence $(u_i^n)_{n\geq 0}$, we can extract a subsequence $(u_i^m)_{m\geq 0}$ which converges uniformly to $u_i \in \mathcal{C}^0([0,T],H)$ for each i.

Moreover, we can assume that u_i^m converges weakly to u_i in $L^2(]0,T[,V)$.

But $A_i \in \mathcal{L}(V,V')$ and (u_i^n) bounded in $L^2(]0,T[,V)$ imply

$$\int_0^T \|A_i u_i^n\|_{V'}^2 dt \leq \|A_i\|^2 \int_0^T \|u_i^n\|_V^2 dt. \tag{3.42}$$

Thus, the sequence $(A_i u_i^n)_{n\geq 0}$ is bounded in $L^2(]0,T[,V')$. We can assume that $A_i u_i^m$ converges weakly to $A_i u_i$ in $L^2(]0,T[,V')$.

Remark that for all $T < \infty$, we have

$$\mathcal{C}^0([0,T],H) \hookrightarrow L^2(]0,T[,V'). \tag{3.43}$$

□

Conclusion 2. One has

$$u_i^m \longrightarrow u_i \quad \text{in } \mathcal{C}^0([0,T],H). \tag{3.44}$$

Thus, $u_i \geq 0$ and $u_i(0) = u_{0i}$.

We check that

$$\begin{gathered} q_i\left(u^{m-1}\right) \longrightarrow q_i(u), \\ q_i\left(u^{m-1}\right)u_i^m \longrightarrow q_i(u)u_i \quad \text{in } \mathcal{C}^0([0,T],H) \quad \forall i. \end{gathered} \tag{3.45}$$

We have also

$$f_i\left(u^{m-1}\right) \longrightarrow f_i(u) \quad \text{in } \mathcal{C}^0([0,T],H) \cap L^2(]0,T[,V') \quad \forall i. \tag{3.46}$$

For all i, u_i^m is solution of

$$\left\langle \frac{\partial u_i^m}{\partial t}, v \right\rangle + \langle A_i u_i^m, v\rangle + \left(q_i\left(u^{m-1}\right)u_i^m, v\right) = \left\langle f_i\left(u^{m-1}\right), v\right\rangle \quad \forall v \in V. \tag{3.47}$$

We take $\phi \in \mathfrak{D}(]0,T[)$, and therefore $\phi v \in L^2(]0,T[,V)$,

$$\begin{aligned} \int_0^T \left\langle \frac{\partial u_i^m}{\partial t}, \phi v \right\rangle dt &+ \int_0^T \langle A_i u_i^m, \phi v\rangle dt + \int_0^T \left(q_i\left(u^{m-1}\right)u_i^m, \phi v\right) dt \\ &= \int_0^T \left\langle f_i\left(u^{m-1}\right), \phi v \right\rangle dt. \end{aligned} \tag{3.48}$$

The second term in the left side and the right side of the egality converges due to the weak convergence in $L^2(]0,T[,V')$. The third term in the left-hand side also passes to the limit, due to the convergence in $\mathcal{C}^0([0,T],H)$.

We deduce that $\partial u_i^m/\partial t$ converges weakly in $L^2(]0,T[,V')$.

But we have

$$u_i^m \longrightarrow u_i \quad \text{in } \mathcal{C}^0([0,T],H), \tag{3.49}$$

then

$$\frac{\partial u_i^m}{\partial t} \longrightarrow \frac{\partial u_i}{\partial t} \quad \text{in } \mathfrak{D}'(]0,T[,H). \tag{3.50}$$

Therefore, we obtain

$$\begin{gathered}\frac{\partial u_i^m}{\partial t} \longrightarrow \frac{\partial u_i}{\partial t} \text{ weakly in } L^2(]0,T[,V'),\\ \int_0^T \left\langle \frac{\partial u_i}{\partial t}, \phi v \right\rangle dt + \int_0^T \langle A_i u_i, \phi v\rangle dt + \int_0^T (q_i(u)u_i, \phi v)_H dt = \int_0^T \langle f_i(u), \phi v\rangle dt.\end{gathered} \tag{3.51}$$

This formulation being true for all ϕ, we have

$$\left\langle \frac{\partial u_i}{\partial t}, v \right\rangle + \langle A_i u_i, v\rangle + (q_i(u)u_i, v)_H = \langle f_i(u), v\rangle \quad \forall v \in V, \tag{3.52}$$

that is to say,

$$\begin{gathered}\frac{d}{dt}(u_i, v)_H + a(u_i, v) + (q_i(u)u_i, v)_H = \langle f_i(u), v\rangle \quad \forall v \in V, \ i,\\ \frac{\partial u_i}{\partial t} = f_i(u) - A_i u_i - q_i(u)u_i \in L^2(]0,T[,V') \quad \forall i.\end{gathered} \tag{3.53}$$

For all i, one has

$$u_i^m(t) = G_i(t)u_{0i} + \int_0^t G_i(t-s)\left(-q_i\left(u^{m-1}\right)u_i^m + f_i\left(u_i^{m-1}\right)\right)(s)ds, \tag{3.54}$$

and as $q_i(u^{m-1})u_i^m$ and $f_i(u^{m-1})$ converge in $C^0([0,T],H)$, we have

$$u_i(t) = G_i(t)u_{0i} + \int_0^t G_i(t-s)(-q_i(u)u_i + f_i(u))(s)ds. \tag{R}$$

3.4. *Main Result*

Theorem 3.3. *If the initial condition satisfies* (3.1), *then system* ($\mathcal{S}$) *admits a unique nonnegative solution* $u_i \in W(0,T,V,V')$ *for all* i.

Moreover, for all i, u_i *satisfies relation* (3.62).

Proof. We have already shown existence of solution; thus, it remains to show uniqueness.

Let v be another solution of system ($\mathcal{S}$)

$$v_i \in W(0,T,V,V') \Longrightarrow v_i \in \mathcal{C}^0([0,T],H), \quad v_i \geq 0. \tag{3.55}$$

Consequently, we obtain

$$q_i(v)v_i + f_i(v) \in L^2(]0,T[,V'). \tag{3.56}$$

Thus, by Proposition 2.11, we have

$$v_i(t) = G_i(t)u_{0i} + \int_0^t G_i(t-s)\big(-q_i(v)v_i + f_i(v)\big)(s)ds. \tag{3.57}$$

By subtraction, we have

$$u_i(t) - v_i(t) = \int_0^t G_i(t-s)\big(-q_i(u)u_i + q_i(v)v_i + f_i(u) - f_i(v)\big)(s)ds, \tag{3.58}$$

with

$$q_i(u)u_i - q_i(v)v_i = q_i(u)(u_i - v_i) + \big(q_i(u) - q_i(v)\big)v_i. \tag{3.59}$$

u_i being positive, we have

$$\left\| \frac{u_j}{K \;\; + u_i} \right\| \leq \frac{1}{K}\|u_j\|_\infty, \tag{3.60}$$

where we have defined

$$\|u_j\|_\infty = \|u_j\|_{L^\infty(]0,T[,H)}. \tag{3.61}$$

If we define

$$\|u\|_\infty = \sum_{j=1}^{6}\|u_j\|_\infty, \tag{3.62}$$

there is $M_1 > 0$ such that

$$\|q_i(u)\|_\infty \leq M_1\|u\|_\infty \quad \forall u. \tag{3.63}$$

So the numerator of $q_i(u) - q_i(v)$ is the sum of terms of form $(u_k - v_k)v_j$ or $(u_j - v_j)u_k$, and we can find $M_2 > 0$ such that

$$|q_i(u) - q_i(v)|_H(s) \leq M_2\left(\sum_{j=1}^{6}|u_j(s) - v_j(s)|_H\right). \tag{3.64}$$

Also we can find $M_3 > 0$ such that

$$\left| f_i(u) - f_i(v) \right|_H (s) \leq M_3 \left(\sum_{j=1}^{6} \left| u_j(s) - v_j(s) \right|_H \right). \tag{3.65}$$

Summing and noting that $\|G_i(t-s)\| \leq N_i e^{\Omega_i T}$ with $N_i, \Omega_i > 0$, we can find M such that

$$\sum_{i=1}^{6} |u_i(t) - v_i(t)|_H \leq M \|u - v\|_\infty. \tag{3.66}$$

Replacing in (3.58), we obtain

$$\sum_{i=1}^{6} |u_i(t) - v_i(t)|_H \leq M^2 \|u - v\|_\infty \int_0^t s ds = M^2 \frac{t^2}{2} \|u - v\|_\infty. \tag{3.67}$$

By induction, we have

$$\sum_{j=1}^{6} \left| u_j(t) - v_j(t) \right|_H \leq \frac{M^n}{n!} T^n \|u - v\|_\infty,$$
$$\lim_{n \to +\infty} \frac{M^n}{n!} T^n \|u - v\|_\infty = 0. \tag{3.68}$$

□

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