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*Research Article*

# On the Upper Bounds of Eigenvalues for a Class of Systems of Ordinary Differential Equations with Higher Order

**Gao Jia, Li-Na Huang, and Wei Liu**

*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China*

Correspondence should be addressed to Gao Jia, gaojia79@yahoo.com.cn

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The estimate of the upper bounds of eigenvalues for a class of systems of ordinary differential equations with higher order is considered by using the calculus theory. Several results about the upper bound inequalities of the $(n+1)$th eigenvalue are obtained by the first $n$ eigenvalues. The estimate coefficients do not have any relation to the geometric measure of the domain. This kind of problem is interesting and significant both in theory of systems of differential equations and in applications to mechanics and physics.

## 1. Introduction

In many physical settings, such as the vibrations of the general homogeneous or nonhomogeneous string, rod and plate can yield the Sturm-Liouville eigenvalue problems or other eigenvalue problems. However, it is not easy to get the accurate values by the analytic method. Sometimes, it is necessary to consider the estimations of the eigenvalues. And since 1960s, the problems of the eigenvalue estimates had become one of the hotspots of the differential equations.

There are lots of achievements about the upper bounds of arbitrary eigenvalues for the differential equations and uniformly elliptic operators with higher orders [1–9]. However, there are few achievements associated with the estimates of the eigenvalues for systems of differential equations with higher order. In the following, we will obtain some inequalities concerning the eigenvalue $\lambda_{n+1}$ in terms of $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the systems of ordinary differential equations with higher order. In fact, the eigenvalue problems have great strong practical backgrounds and important theoretical values [10, 11].

Let $[a,b] \subset R^1$ be a bounded domain and $t \geq 2$ be an integer. The following eigenvalue problems are studied:

$$
\begin{gathered}
(-1)^t D^t\left(a_{11} D^t y_1 + a_{12} D^t y_2 + \cdots + a_{1n} D^t y_n\right) = \lambda s(x) y_1, \\
(-1)^t D^t\left(a_{21} D^t y_1 + a_{22} D^t y_2 + \cdots + a_{2n} D^t y_n\right) = \lambda s(x) y_2, \\
\vdots \\
(-1)^t D^t\left(a_{n1} D^t y_1 + a_{n2} D^t y_2 + \cdots + a_{nn} D^t y_n\right) = \lambda s(x) y_n, \\
D^k y_i(a) = D^k y_i(b) = 0 \quad (i = 1,2,\dots,n,\ k = 0,1,2,\dots,t-1),
\end{gathered}
\tag{1.1}
$$

where $D = d/dx$, $D^k = d^k/dx^k$, $a_{ij}(x)$ $(i,j = 1,2,\dots,n)$ and $s(x)$ satisfies the following conditions:

(1°) $a_{ij}(x) \in C^t[a,b]$, $a_{ij}(x) = a_{ji}(x), i,\ j = 1,2,\dots,n$;

(2°) for the arbitrary $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$, we have

$$
\mu_1 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \mu_2 |\xi|^2, \quad \forall x \in [a,b],
\tag{1.2}
$$

where $\mu_2 \geq \mu_1 > 0$, $\mu_1, \mu_2$ are both constants;

(3°) $s(x) \in C[a,b]$, and there are constants $\nu_1 \leq \nu_2$, such that $0 < \nu_1 \leq s(x) \leq \nu_2$.

According to the theories of the differential equations [11, 12], the eigenvalues of (1.1) are all positive real numbers, and they are discrete.

We change (1.1) to the form of matrix. Let

$$
\mathbf{y}^T = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad D^t \mathbf{y}^T = \begin{pmatrix} D^t y_1 \\ D^t y_2 \\ \vdots \\ D^t y_n \end{pmatrix}, \qquad \mathbf{A}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) \cdots & a_{2n}(x) \\ & \vdots & \\ a_{n1}(x) & a_{n2}(x) \cdots & a_{nn}(x) \end{pmatrix}.
\tag{1.3}
$$

By virtue of $a_{ij}(x) = a_{ji}(x)$, therefore $\mathbf{A}^T(x) = \mathbf{A}(x)$, (1.1) can be changed into the following form:

$$
(-1)^t D^t\left(\mathbf{A}(x) D^t \mathbf{y}^T\right) = \lambda s(x) \mathbf{y}^T,
\tag{1.4}
$$

$$
\mathbf{y}^{(k)}(a) = \mathbf{y}^{(k)}(b) = 0, \quad k = 0,1,2,\dots,t-1.
\tag{1.5}
$$

Obviously, (1.4)-(1.5) is equivalent to (1.1).

Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ are eigenvalues of (1.4)-(1.5), $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n, \ldots$ are the corresponding eigenfunctions and satisfy the following weighted orthogonal conditions:

$$\int_a^b s(x)\mathbf{y}_i\mathbf{y}_j^T dx = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, \ldots. \tag{1.6}$$

Multiplying $\mathbf{y}_i$ in sides of (1.4), by using (1.5) and integration by parts, we have

$$\lambda_i = \int_a^b D^t\mathbf{y}_i\mathbf{A}(x)D^t\mathbf{y}_i^T dx, \quad i = 1, 2, \ldots. \tag{1.7}$$

From (2°), we have

$$\int_a^b \left|D^t\mathbf{y}_i\right|^2 dx = \int_a^b D^t\mathbf{y}_i D^t\mathbf{y}_i^T dx \leq \frac{\lambda_i}{\mu_1}, \quad i = 1, 2, \ldots. \tag{1.8}$$

For fixed $n$, let

$$\mathbf{\Phi}_i = x\mathbf{y}_i - \sum_{j=1}^n b_{ij}\mathbf{y}_j, \quad i = 1, 2, \ldots, n, \tag{1.9}$$

where $b_{ij} = \int_a^b xs(x)\mathbf{y}_i\mathbf{y}_j^T dx$. Obviously, $b_{ij} = b_{ji}$, and $\mathbf{\Phi}_i$ are weighted orthogonal to $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$. Furthermore, $\mathbf{\Phi}_i(a) = \mathbf{\Phi}_i(b) = 0$, $i, j = 1, 2, \ldots, n$.

We can use the well-known Rayleigh theorem [11, 12] to obtain

$$\lambda_{n+1} \leq \frac{(-1)^t \int_a^b \mathbf{\Phi}_i D^t\left(\mathbf{A}(x)D^t\mathbf{\Phi}_i^T\right)dx}{\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx}. \tag{1.10}$$

It is easy to see that

$$\begin{aligned}
(-1)^t D^t\left(\mathbf{A}(x)D^t\mathbf{\Phi}_i^T\right) &= (-1)^t tD^t\left(\mathbf{A}(x)D^{t-1}\mathbf{y}_i^T\right) + (-1)^t tD^{t-1}\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right) \\
&\quad + (-1)^t xD^t\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right) - (-1)^t \sum_{j=1}^n b_{ij}D^t\left(\mathbf{A}(x)D^t\mathbf{y}_j^T\right) \\
&= (-1)^t tD^t\left(\mathbf{A}(x)D^{t-1}\mathbf{y}_i^T\right) + (-1)^t tD^{t-1}\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right) \\
&\quad + \lambda_i xs(x)\mathbf{y}_i^T - s(x)\sum_{j=1}^n \lambda_j b_{ij}\mathbf{y}_j^T.
\end{aligned} \tag{1.11}$$

We have

$$\begin{aligned}\int_a^b \mathbf{\Phi}_i(-1)^t D^t\left(\mathbf{A}(x)D^t\mathbf{\Phi}_i^T\right)dx &= \lambda_i\int_a^b xs(x)\mathbf{\Phi}_i\mathbf{y}_i^T dx + (-1)^t t\int_a^b \mathbf{\Phi}_i D^t\left(\mathbf{A}(x)D^{t-1}\mathbf{y}_i^T\right)dx \\ &\quad + (-1)^t t\int_a^b \mathbf{\Phi}_i D^{t-1}\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right)dx \\ &\quad - \int_a^b s(x)\mathbf{\Phi}_i\sum_{j=1}^n \lambda_j b_{ij}\mathbf{y}_j^T dx.\end{aligned} \tag{1.12}$$

In addition, using the fact that $\mathbf{\Phi}_i$ are weighted orthogonal to $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ and

$$\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx = \int_a^b xs(x)\mathbf{\Phi}_i\mathbf{y}_i^T dx, \tag{1.13}$$

we know that the last term of (1.12) is equal to zero. Thus, we have

$$\begin{aligned}\int_a^b \mathbf{\Phi}_i(-1)^t D^t\left(\mathbf{A}(x)D^t\mathbf{\Phi}_i^T\right)dx &= \lambda_i\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx \\ &\quad + (-1)^t t\int_a^b \mathbf{\Phi}_i D^t\left(\mathbf{A}(x)D^{t-1}\mathbf{y}_i^T\right)dx \\ &\quad + (-1)^t t\int_a^b \mathbf{\Phi}_i D^{t-1}\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right)dx.\end{aligned} \tag{1.14}$$

Set

$$\begin{aligned} I_i &= (-1)^t t\int_a^b \mathbf{\Phi}_i D^t\left(\mathbf{A}(x)D^{t-1}\mathbf{y}_i^T\right)dx, \qquad I = \sum_{i=1}^n I_i, \\ J_i &= (-1)^t t\int_a^b \mathbf{\Phi}_i D^{t-1}\left(\mathbf{A}(x)D^t\mathbf{y}_i^T\right)dx, \qquad J = \sum_{i=1}^n J_i.\end{aligned} \tag{1.15}$$

From (1.14), we have

$$\sum_{i=1}^n\int_a^b \mathbf{\Phi}_i(-1)^t D^t\left(\mathbf{A}(x)D^t\mathbf{\Phi}_i^T\right)dx = \sum_{i=1}^n\lambda_i\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx + I + J. \tag{1.16}$$

By using (1.10) and (1.16), one can give

$$\lambda_{n+1}\sum_{i=1}^n\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx \le \sum_{i=1}^n\lambda_i\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx + I + J. \tag{1.17}$$

Substituting $\lambda_n$ for $\lambda_i$ in (1.17), we get

$$(\lambda_{n+1} - \lambda_n) \sum_{i=1}^{n} \int_a^b s(x) |\boldsymbol{\Phi}_i|^2 dx \leq I + J. \tag{1.18}$$

In order to get the estimations of the eigenvalues, we only need to show the estimates about $I$, $J$, and $\sum_{i=1}^{n} \int_a^b s(x) |\boldsymbol{\Phi}_i|^2 dx$.

## 2. Lemmas

**Lemma 2.1.** *Suppose that the eigenfunctions* $\mathbf{y}_i$ *of* (1.4)-(1.5) *correspond to the eigenvalues* $\lambda_i$. *Then one has*

(1) $\int_a^b |D^p \mathbf{y}_i|^2 dx \leq \nu_1^{-1/(p+1)} (\int_a^b |D^{p+1} \mathbf{y}_i|^2 dx)^{p/(p+1)}$, $p = 1, 2, \ldots, t-1$;

(2) $\int_a^b |D\mathbf{y}_i|^2 dx \leq \nu_1^{-(1-(1/t))} (\lambda_i/\mu_1)^{1/t}$.

*Proof.* (1) By induction. If $p = 1$, using integration by parts and the Schwarz inequality, we have

$$\begin{aligned} \int_a^b |D\mathbf{y}_i|^2 dx &\leq \left| \int_a^b |D\mathbf{y}_i|^2 dx \right| = \left| \int_a^b D\mathbf{y}_i D\mathbf{y}_i^T dx \right| = \left| \int_a^b \mathbf{y}_i D^2 \mathbf{y}_i^T dx \right| \\ &\leq \left( \int_a^b |\mathbf{y}_i|^2 dx \right)^{1/2} \left( \int_a^b \left| D^2 \mathbf{y}_i^T \right|^2 dx \right)^{1/2} \leq \nu_1^{-1/2} \left( \int_a^b \left| D^2 \mathbf{y}_i^T \right|^2 dx \right)^{1/2}. \end{aligned} \tag{2.1}$$

Therefore, when $p = 1$, (1) is true.

If for $p = k$, (1) is true, that is,

$$\int_a^b \left| D^k \mathbf{y}_i \right|^2 dx \leq \nu_1^{-1/(k+1)} \left( \int_a^b \left| D^{k+1} \mathbf{y}_i \right|^2 dx \right)^{k/(k+1)}. \tag{2.2}$$

For $p = k+1$, using integration by parts, the Schwarz inequality and the result when $p = k$, one can give

$$\begin{aligned} \int_a^b \left| D^{k+1} \mathbf{y}_i \right|^2 dx &\leq \left| \int_a^b \left| D^{k+1} \mathbf{y}_i \right|^2 dx \right| = \left| \int_a^b D^k \mathbf{y}_i \cdot D^{k+2} \mathbf{y}_i^T dx \right| \\ &\leq \left( \int_a^b \left| D^k \mathbf{y}_i \right|^2 dx \right)^{1/2} \left( \int_a^b \left| D^{k+2} \mathbf{y}_i^T \right|^2 dx \right)^{1/2} \\ &\leq \nu_1^{-1/(2(k+1))} \left( \int_a^b \left| D^{k+2} \mathbf{y}_i \right|^2 dx \right)^{1/2} \left( \int_a^b \left| D^{k+1} \mathbf{y}_i \right|^2 dx \right)^{k/(2(k+1))}. \end{aligned} \tag{2.3}$$

By further calculating, one can give

$$\int_a^b \left|D^{k+1}\mathbf{y}_i\right|^2 dx \le \nu_1^{-1/((k+1)+1)}\left(\int_a^b \left|D^{(k+1)+1}\mathbf{y}_i\right|^2 dx\right)^{(k+1)/((k+1)+1)}. \tag{2.4}$$

Therefore, when $p = k+1$, (1) is true.

(2) Using (1) and the inductive method, we have

$$\begin{aligned}\int_a^b |D^p\mathbf{y}_i|^2 dx &\le \nu_1^{-1/(p+1)}\left(\int_a^b \left|D^{p+1}\mathbf{y}_i\right|^2 dx\right)^{p/(p+1)}\\ &\le \nu_1^{-2/(p+2)}\left(\int_a^b \left|D^{p+2}\mathbf{y}_i\right|^2 dx\right)^{p/(p+2)}\\ &\le \cdots \le \nu_1^{-(1-(p/t))}\left(\int_a^b \left|D^{t}\mathbf{y}_i\right|^2 dx\right)^{p/t}.\end{aligned} \tag{2.5}$$

From (1.8) and (2.5), we get

$$\int_a^b |D^p\mathbf{y}_i|^2 dx \le \nu_1^{-(1-(p/t))}\left(\int_a^b \left|D^{t}\mathbf{y}_i\right|^2 dx\right)^{p/t} \le \nu_1^{-(1-(p/t))}\left(\frac{\lambda_i}{\mu_1}\right)^{p/t}, \quad p = 1,2,\ldots,t. \tag{2.6}$$

Taking $p = 1$, we have

$$\int_a^b |D\mathbf{y}_i|^2 dx \le \nu_1^{-(1-(1/t))}\left(\frac{\lambda_i}{\mu_1}\right)^{1/t}. \tag{2.7}$$

So Lemma 2.1 is true. □

**Lemma 2.2.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of* (1.4)-(1.5). *Then one has*

$$I + J \le t(2t-1)\mu_1^{-(1-(1/t))}\nu_1^{-1/t}\mu_2\sum_{i=1}^{n}\lambda_i^{1-(1/t)}. \tag{2.8}$$

*Proof.* Since

$$\begin{aligned}
I_i &= (-1)^t t \int_a^b \mathbf{\Phi}_i D^t \left(\mathbf{A}(x) D^{t-1} \mathbf{y}_i^T\right) dx \\
&= (-1)^t t \int_a^b \left( x\mathbf{y}_i - \sum_{j=1}^n b_{ij} \mathbf{y}_j \right) D^t \left(\mathbf{A}(x) D^{t-1} \mathbf{y}_i^T\right) dx \\
&= (-1)^t t \int_a^b x\mathbf{y}_i D^t \left(\mathbf{A}(x) D^{t-1} \mathbf{y}_i^T\right) dx \\
&\quad - (-1)^t t \sum_{j=1}^n b_{ij} \int_a^b \mathbf{y}_j D^t \left(\mathbf{A}(x) D^{t-1} \mathbf{y}_i^T\right) dx \\
&= t^2 \int_a^b D^{t-1}\mathbf{y}_i \mathbf{A}(x) D^{t-1} \mathbf{y}_i^T dx + t \int_a^b x D^t \mathbf{y}_i \mathbf{A}(x) D^{t-1} \mathbf{y}_i^T dx \\
&\quad - t \sum_{j=1}^n b_{ij} \int_a^b D^t \mathbf{y}_j \left(\mathbf{A}(x) D^{t-1} \mathbf{y}_i^T\right) dx,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
J_i &= (-1)^t t \int_a^b \mathbf{\Phi}_i D^{t-1} \left(\mathbf{A}(x) D^t \mathbf{y}_i^T\right) dx \\
&= -t(t-1) \int_a^b D^{t-2}\mathbf{y}_i \mathbf{A}(x) D^t \mathbf{y}_i^T dx - t \int_a^b x D^{t-1} \mathbf{y}_i \mathbf{A}(x) D^t \mathbf{y}_i^T dx \\
&\quad + t \sum_{j=1}^n b_{ij} \int_a^b D^{t-1} \mathbf{y}_j \mathbf{A}(x) D^t \mathbf{y}_i^T dx,
\end{aligned} \tag{2.10}$$

we have

$$\begin{aligned}
I + J &= \sum_{i=1}^n (I_i + J_i) \\
&= \sum_{i=1}^n t \int_a^b \left( t D^{t-1}\mathbf{y}_i \mathbf{A}(x) D^{t-1} \mathbf{y}_i^T - (t-1) D^{t-2} \mathbf{y}_i \mathbf{A}(x) D^t \mathbf{y}_i^T \right) dx \\
&\quad - t \sum_{i,j=1}^n b_{ij} \int_a^b \left( D^t \mathbf{y}_j \mathbf{A}(x) D^{t-1} \mathbf{y}_i^T - D^{t-1} \mathbf{y}_j \mathbf{A}(x) D^t \mathbf{y}_i^T \right) dx.
\end{aligned} \tag{2.11}$$

By $a_{ij}(x) = a_{ji}(x)$, the last term of (2.11) is zero. Then we can get

$$I + J = \sum_{i=1}^n t \int_a^b \left( t D^{t-1}\mathbf{y}_i \mathbf{A}(x) D^{t-1} \mathbf{y}_i^T - (t-1) D^{t-2} \mathbf{y}_i \mathbf{A}(x) D^t \mathbf{y}_i^T \right) dx. \tag{2.12}$$

Using (2°), Lemma 2.1, (1) and (2.6), we have

$$\int_a^b D^{t-1}\mathbf{y}_i \mathbf{A}(x) D^{t-1}\mathbf{y}_i^T dx \le \mu_2 \int_a^b \left|D^{t-1}\mathbf{y}_i\right|^2 dx \le \mu_2 \nu_1^{-1/t}\left(\frac{\lambda_i}{\mu_1}\right)^{1-(1/t)}. \tag{2.13}$$

Using (2°), the Schwarz inequality, Lemma 2.1 (1), and (2.6), one can give

$$\begin{aligned}\left|-\int_a^b D^{t-2}\mathbf{y}_i \mathbf{A}(x) D^{t}\mathbf{y}_i^T dx\right| &\le \mu_2\left(\int_a^b \left|D^{t-2}\mathbf{y}_i\right|^2 dx\right)^{1/2}\left(\int_a^b \left|D^{t}\mathbf{y}_i^T\right|^2 dx\right)^{1/2} \\ &\le \mu_2 \nu_1^{-1/t}\left(\frac{\lambda_i}{\mu_1}\right)^{1-(1/t)}.\end{aligned} \tag{2.14}$$

Therefore, we obtain

$$I + J \le t(2t-1)\mu_1^{-(1-(1/t))}\nu_1^{-1/t}\mu_2 \sum_{i=1}^{n}\lambda_i^{1-(1/t)}. \tag{2.15}$$

□

**Lemma 2.3.** *If $\mathbf{\Phi}_i$ and $\lambda_i$ $(i = 1, 2, \dots, n)$ as above, then one has*

$$\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx \ge \frac{\mu_1^{1/t}\nu_1^{2-(1/t)} n^2}{4\nu_2^2}\left(\sum_{i=1}^{n}\lambda_i^{1/t}\right)^{-1}. \tag{2.16}$$

*Proof.* By the definition of $\mathbf{\Phi}_i$, one has

$$\sum_{i=1}^{n}\int_a^b \mathbf{\Phi}_i D\mathbf{y}_i^T dx = \sum_{i=1}^{n}\int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx - \sum_{i,j=1}^{n} b_{ij}\int_a^b \mathbf{y}_j D\mathbf{y}_i^T dx. \tag{2.17}$$

Using $b_{ij} = b_{ji}$ and $\int_a^b \mathbf{y}_j D\mathbf{y}_i^T dx = -\int_a^b \mathbf{y}_i D\mathbf{y}_j^T dx$, it is easy to see that the last term of (2.17) is zero. Then we have

$$\sum_{i=1}^{n}\int_a^b \mathbf{\Phi}_i D\mathbf{y}_i^T dx = \sum_{i=1}^{n}\int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx. \tag{2.18}$$

Using integration by parts, one can give

$$\int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx = -\int_a^b |\mathbf{y}_i|^2 dx - \int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx, \tag{2.19}$$

$$\int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx = -\frac{1}{2}\int_a^b |\mathbf{y}_i|^2 dx. \tag{2.20}$$

By $1/\nu_2 \leq \int_a^b |\mathbf{y}_i|^2 dx \leq 1/\nu_1$, we have

$$\left| \int_a^b x\mathbf{y}_i D\mathbf{y}_i^T dx \right| = \frac{1}{2}\int_a^b |\mathbf{y}_i|^2 dx \geq \frac{1}{2\nu_2}. \tag{2.21}$$

From (2.18) and (2.21), we can get

$$\sum_{i=1}^{n} \left| \int_a^b \mathbf{\Phi}_i D\mathbf{y}_i^T dx \right| \geq \frac{n}{2\nu_2}. \tag{2.22}$$

Using the Schwarz inequality, Lemma 2.1 (2), and (3°), we have

$$\begin{aligned} \frac{n^2}{4\nu_2^2} &\leq \left(\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx\right)\left(\sum_{i=1}^{n}\int_a^b \frac{|D\mathbf{y}_i|^2}{s(x)} dx\right) \\ &\leq \left(\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx\right)\nu_1^{-(2-(1/t))}\mu_1^{-1/t}\sum_{i=1}^{n}\lambda_i^{1/t}. \end{aligned} \tag{2.23}$$

By further calculating, we can easily get Lemma 2.3. □

## 3. Main Results

**Theorem 3.1.** *If* $\lambda_i$ $(i = 1, 2, \ldots, n+1)$ *are the eigenvalues of* (1.4)-(1.5), *then*

$$(1) \quad \lambda_{n+1} \leq \lambda_n + \frac{4t(2t-1)\mu_2\nu_2^2}{\mu_1\nu_1^2 n^2}\sum_{i=1}^{n}\lambda_i^{1-(1/t)}\sum_{i=1}^{n}\lambda_i^{1/t}; \tag{3.1}$$

$$(2) \quad \lambda_{n+1} \leq \left(1 + \frac{4t(2t-1)\mu_2\nu_2^2}{\mu_1\nu_1^2}\right)\lambda_n. \tag{3.2}$$

*Proof.* From (1.18), we can get

$$(\lambda_{n+1} - \lambda_n) \leq (I + J)\left(\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2 dx\right)^{-1}. \tag{3.3}$$

Using Lemmas 2.2 and 2.3, we can easily get (3.1). In (3.1), Replacing $\lambda_i$ with $\lambda_n$, by further calculating, we can get (3.2). □

**Theorem 3.2.** *For* $n \geq 1$, *one has*

$$\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\lambda_{n+1} - \lambda_i} \geq \frac{\mu_1\nu_1^2 n^2}{4t(2t-1)\mu_2\nu_2^2}\left(\sum_{i=1}^{n}\lambda_i^{1-(1/t)}\right)^{-1}. \tag{3.4}$$

*Proof.* Choosing the parameter $\sigma > \lambda_n$, using (1.17), one can give

$$\lambda_{n+1}\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2dx \le \sigma\sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2dx + \sum_{i=1}^{n}\int_a^b(\lambda_i-\sigma)s(x)|\mathbf{\Phi}_i|^2dx + I + J. \tag{3.5}$$

By (2.22) and the Young inequality, we obtain

$$\frac{n}{2\nu_2} \le \frac{\delta}{2}\sum_{i=1}^{n}(\sigma-\lambda_i)\int_a^b s(x)|\mathbf{\Phi}_i|^2dx + \frac{1}{2\delta}\sum_{i=1}^{n}(\sigma-\lambda_i)^{-1}\int_a^b \frac{|D\mathbf{y}_i|^2}{s(x)}dx, \tag{3.6}$$

where $\delta > 0$ is a constant to be determined. Set

$$V = \sum_{i=1}^{n}\int_a^b s(x)|\mathbf{\Phi}_i|^2dx, \qquad T = \sum_{i=1}^{n}(\sigma-\lambda_i)\int_a^b s(x)|\mathbf{\Phi}_i|^2dx. \tag{3.7}$$

Using Lemma 2.1, (3.5), and (3.6), we can get the following results, respectively,

$$(\lambda_{n+1}-\sigma)V + T \le I + J, \tag{3.8}$$

$$\frac{n}{\nu_2} \le \delta T + \frac{1}{\delta}\mu_1^{-1/t}\nu_1^{-(2-(1/t))}\sum_{i=1}^{n}(\sigma-\lambda_i)^{-1}\lambda_i^{1/t}. \tag{3.9}$$

In order to get the minimum of the right of (3.9), we can take

$$\delta = T^{-1/2}\left(\mu_1^{-1/t}\nu_1^{-(2-(1/t))}\sum_{i=1}^{n}(\sigma-\lambda_i)^{-1}\lambda_i^{1/t}\right)^{1/2}. \tag{3.10}$$

By (3.9), and (3.10), we can easily get

$$T \ge \frac{\mu_1^{1/t}\nu_1^{2-(1/t)}n^2}{4\nu_2^2}\left(\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\sigma-\lambda_i}\right)^{-1}. \tag{3.11}$$

Using Lemma 2.2, (3.8), and (3.11), we have

$$(\lambda_{n+1}-\sigma)V + \frac{\mu_1^{1/t}\nu_1^{2-(1/t)}n^2}{4\nu_2^2}\left(\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\sigma-\lambda_i}\right)^{-1} \le t(2t-1)\mu_1^{-(1-(1/t))}\nu_1^{-1/t}\mu_2\sum_{i=1}^{n}\lambda_i^{1-(1/t)}, \tag{3.12}$$

that is,

$$(\lambda_{n+1}-\sigma)V \le t(2t-1)\mu_1^{-(1-(1/t))}\nu_1^{-1/t}\mu_2\sum_{i=1}^{n}\lambda_i^{1-(1/t)} - \frac{\mu_1^{1/t}\nu_1^{2-(1/t)}n^2}{4\nu_2^2}\left(\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\sigma-\lambda_i}\right)^{-1}. \tag{3.13}$$

Let the right term of (3.13) be $f(\sigma)$. It is easy to see that

$$\lim_{\sigma\to+\infty} f(\sigma) = -\infty,$$
$$\lim_{\sigma\to\lambda_n^+} f(\sigma) = t(2t-1)\mu_1^{-(1-(1/t))}\nu_1^{-1/t}\mu_2\sum_{i=1}^{n}\lambda_i^{1-(1/t)} > 0. \tag{3.14}$$

Hence, there is $\sigma_0 \in (\lambda_n, +\infty)$, such that

$$\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\sigma_0-\lambda_i} = \frac{\mu_1\nu_1^2 n^2}{4t(2t-1)\mu_2\nu_2^2}\left(\sum_{i=1}^{n}\lambda_i^{1-(1/t)}\right)^{-1}. \tag{3.15}$$

On the other hand, letting

$$g(\sigma) = \sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{\sigma-\lambda_i}, \tag{3.16}$$

we have

$$g'(\sigma) = -\sum_{i=1}^{n}\frac{\lambda_i^{1/t}}{(\sigma-\lambda_i)^2} \le 0. \tag{3.17}$$

It implies that $g(\sigma)$ is the monotone decreasing and continuous function, and its value range is $(0, +\infty)$. Therefore, there exits exactly one $\sigma_0$ to satisfy (3.15). From (3.13), we know that $\sigma_0 > \lambda_{n+1}$. Replacing $\sigma_0$ with $\lambda_{n+1}$ in (3.15), we can get the result. □

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