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Research Article

$W^{2,2}$ A Priori Bounds for a Class of Elliptic Operators

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We obtain some $W^{2,2}$ a priori bounds for a class of uniformly elliptic second-order differential operators, both in a no-weighted and in a weighted case. We deduce a uniqueness and existence theorem for the related Dirichlet problem in some weighted Sobolev spaces on unbounded domains.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. The uniformly elliptic second-order linear differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{1.1}$$

with leading coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega), i, j = 1, \ldots, n$, and the associated Dirichlet problem

$$\begin{gathered} u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega), \\ Lu = f, \quad f \in L^2(\Omega), \end{gathered} \tag{1.2}$$

have been extensively studied under different hypotheses of discontinuity on the coefficients of L (we refer to [1] for a general survey on the subject). In particular, some $W^{2,2}$ bounds and the related existence and uniqueness results have been obtained.

Among the various hypotheses, in the framework of discontinuous coefficients, we are interested here in those of Miranda's type, having in mind the classical result of [2] where the leading coefficients have derivatives $(a_{ij})_{x_k} \in L^n(\Omega), n \geq 3$. First generalizations in this direction have been carried on, always considering a bounded and sufficiently regular set Ω, assuming that the derivatives belong to some wider spaces. In particular, in [3], the $(a_{ij})_{x_k}$ are in the weak-L^n space, while, in [4], they are supposed to be in an appropriate subspace of the classical Morrey space $L^{2p,n-2p}(\Omega)$, where $p \in]1, n/2[$. In [5], the leading coefficients are supposed to be close to functions whose derivatives are in $L^n(\Omega)$. A further extension, to a very general case, has been proved in [6, 7], supposing that the a_{ij} are in VMO, which means a kind of continuity in the average sense and not in the pointwise sense.

In this paper, we deal with unbounded domains and we impose hypotheses of Miranda's type on the leading coefficients, assuming that their derivatives $(a_{ij})_{x_k}$ belong to a suitable Morrey type space, which is a generalization to unbounded domains of the classical Morrey space. The existence of the derivatives is of crucial relevance in our analysis, since it allows us to rewrite the operator L in divergence form and puts us in position to use some known results concerning variational operators. A straightforward consequence of our argument is the following $W^{2,2}$-bound, having the only term $\|Lu\|_{L^2(\Omega)}$ in the right-hand side,

$$\|u\|_{W^{2,2}(\Omega)} \le c\|Lu\|_{L^2(\Omega)}, \quad \forall u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega), \tag{1.3}$$

where the dependence of the constant c is explicitly described (see Section 4). This kind of estimate often cannot be obtained when dealing with unbounded domains and clearly immediately takes to the uniqueness of the solution of problem (1.2).

In the framework of unbounded domains, under more regular boundary conditions, an analogous a priori bound can be found in [8], where different assumptions on the a_{ij} are taken into account. We quote here also the results of [9], where, in the spirit of [5], the leading coefficients are supposed to be close, in as specific sense, to functions whose derivatives are in spaces of Morrey type and have a suitable behavior at infinity.

The $W^{2,2}$-bound obtained in (1.3) allows us to extend our result to a weighted case. The relevance of Sobolev spaces with weight in the study of the theory of PDEs with prescribed boundary conditions on unbounded open subsets of $\mathbb{R}^n$ is well known. Indeed, in this framework, it is necessary to require not only conditions on the boundary of the set, but also conditions controlling the behaviour of the solution at infinity. In this order of ideas, we also consider the Dirichlet problem,

$$\begin{gathered} u \in W_s^{2,2}(\Omega) \cap \mathring{W}_s^{1,2}(\Omega), \\ Lu = f, \quad f \in L_s^2(\Omega), \end{gathered} \tag{1.4}$$

where $s \in \mathbb{R}$, $W_s^{2,2}(\Omega)$, $\mathring{W}_s^{1,2}(\Omega)$, and $L_s^2(\Omega)$ are weighted Sobolev spaces where the weight ρ^s is power of a function $\rho : \overline{\Omega} \to \mathbb{R}_+$, of class $C^2(\overline{\Omega})$, and such that

$$\begin{gathered} \sup_{x\in\Omega} \frac{\left|\partial^\alpha \rho(x)\right|}{\rho(x)} < +\infty, \quad \forall |\alpha| \le 2, \\ \lim_{|x|\to+\infty} \left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty, \qquad \lim_{|x|\to+\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0, \end{gathered} \tag{1.5}$$

see Sections 2 and 3 for more details. Also in this weighted case, we obtain the bound

$$\|u\|_{W_s^{2,2}(\Omega)} \le c\|Lu\|_{L_s^2(\Omega)}, \quad \forall u \in W_s^{2,2}(\Omega) \cap \mathring{W}_s^{1,2}(\Omega), \tag{1.6}$$

where the dependence of the constant c is again completely determined. From this a priori estimate, in Section 5, we deduce the solvability of problem (1.4).

Existence and uniqueness results for similar problems in the weighted case, but with different weight functions and different assumptions on the coefficients, have been proved in [10]. Recent results concerning a priori estimates for solutions of the Poisson and heat equations in weighted spaces can be found in [11], where weights of Kondrat'ev type are considered.

2. A Class of Weighted Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^n$, not necessarily bounded, $n \geq 2$. We want to introduce a class of weight functions defined on $\overline{\Omega}$.

To this aim, given $k \in \mathbb{N}_0$, we consider a function $\rho : \overline{\Omega} \to \mathbb{R}_+$ such that $\rho \in C^k(\overline{\Omega})$ and

$$\sup_{x\in\Omega} \frac{\left|\partial^\alpha \rho(x)\right|}{\rho(x)} < +\infty, \quad \forall |\alpha| \leq k. \tag{2.1}$$

As an example, we can think of the function

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R} . \tag{2.2}$$

In the following lemma, we show a property, needed in the sequel, concerning this class of weight functions.

Lemma 2.1. *If assumption* (2.1) *is satisfied, then*

$$\sup_{x\in\Omega} \frac{\left|\partial^\alpha \rho^s(x)\right|}{\rho^s(x)} < +\infty \quad \forall s \in \mathbb{R}, \forall |\alpha| \leq k. \tag{2.3}$$

Proof. The proof is obtained by induction. From (2.1), we get

$$\left|\left(\rho^s\right)_{x_i}\right| = \left|s\rho^{s-1}\rho_{x_i}\right| \leq c_1\rho\rho^{s-1} = c_1\rho^s, \quad i = 1, \ldots, n, \tag{2.4}$$

with c_1 positive constant depending only on s. Thus (2.3) holds for $|\alpha| = 1$.

Now, let us assume that (2.3) holds for any β such that $|\beta| < |\alpha|$ and any $s \in \mathbb{R}$, and fix a β such that $|\beta| = |\alpha| - 1$. Then, using (2.1) and by the induction hypothesis written for $s - 1$, we have

$$\begin{aligned} \left|\partial^\alpha \rho^s\right| &= \left|\partial^\beta \left(\rho^s\right)_{x_i}\right| = \left|\partial^\beta\left(s\rho^{s-1}\rho_{x_i}\right)\right| \\ &\leq c_2 \sum_{\gamma\leq\beta} \left|\partial^{\beta-\gamma}\rho_{x_i}\partial^\gamma\rho^{s-1}\right| \leq c_3\rho\rho^{s-1} = c_3\rho^s, \quad \text{for } i = 1, \ldots, n, \end{aligned} \tag{2.5}$$

with c_3 positive constant depending only on s. Hence, (2.3) holds true also for α. □

Now, let us study some properties of a new class of weighted Sobolev spaces, with weight function of the above-mentioned type.

For $k \in \mathbb{N}_0$, $p \in [1,+\infty[$, and $s \in \mathbb{R}$, and given a weight function ρ satisfying (2.1), we define the space $W_s^{k,p}(\Omega)$ of distributions u on Ω such that

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \left\|\rho^s \partial^\alpha u\right\|_{L^p(\Omega)} < +\infty, \tag{2.6}$$

equipped with the norm given in (2.6). Moreover, we denote by $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ the closure of $C_\circ^\infty(\Omega)$ in $W_s^{k,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$.

Lemma 2.2. *Let $k \in \mathbb{N}_0$, $p \in [1,+\infty[$, and $s \in \mathbb{R}$. If assumption* (2.1) *is satisfied, then there exist two constants $c_1, c_2 \in \mathbb{R}_+$ such that*

$$c_1\|u\|_{W_s^{k,p}(\Omega)} \leq \left\|\rho^t u\right\|_{W_{s-t}^{k,p}(\Omega)} \leq c_2\|u\|_{W_s^{k,p}(\Omega)}, \quad \forall t \in \mathbb{R},\ \forall u \in W_s^{k,p}(\Omega), \tag{2.7}$$

with $c_1 = c_1(t)$ and $c_2 = c_2(t)$.

Proof. Observe that from (2.3), we have

$$\left|\partial^\alpha\left(\rho^t u\right)\right| \leq c_1 \sum_{\beta\leq\alpha}\left|\partial^{\alpha-\beta}\rho^t \partial^\beta u\right| \leq c_2\left|\rho^t \partial^\beta u\right|, \quad \forall|\alpha| \leq k, \tag{2.8}$$

with $c_2 \in \mathbb{R}_+$ depending only on t. This entails the inequality on the right-hand side of (2.7).

To get the left-hand side inequality, it is enough to show that

$$\left|\rho^t \partial^\alpha u\right| \leq c_3 \sum_{\beta\leq\alpha}\left|\partial^\beta\left(\rho^t u\right)\right|, \quad \forall|\alpha| \leq k, \tag{2.9}$$

with $c_3 \in \mathbb{R}_+$ depending only on t.

We will prove (2.9) by induction. From (2.3), one has

$$\left|\rho^t u_{x_i}\right| = \left|\left(\rho^t u\right)_{x_i} - \left(\rho^t\right)_{x_i} u\right| \leq c_4\left(\left(\rho^t u\right)_x + \rho^t|u|\right), \tag{2.10}$$

for $i = 1,\ldots,n$, with $c_4 \in \mathbb{R}_+$ depending only on t. Hence, (2.9) holds for $|\alpha| = 1$.

If (2.3) holds for any β such that $|\beta| < |\alpha|$, then, using again (2.3) and by the induction hypothesis, we have

$$\begin{aligned} \left|\rho^t \partial^\alpha u\right| &\leq \left|\partial^\alpha\left(\rho^t u\right)\right| + c_5 \sum_{\beta<\alpha}\left|\partial^{\alpha-\beta}\rho^t\right|\left|\partial^\beta u\right| \\ &\leq \left|\partial^\alpha\left(\rho^t u\right)\right| + c_6 \sum_{\beta<\alpha}\left|\rho^t \partial^\beta u\right| \leq c_7 \sum_{\beta\leq\alpha}\left|\partial^\beta\left(\rho^t u\right)\right|, \end{aligned} \tag{2.11}$$

with $c_7 \in \mathbb{R}_+$ depending only on t. □

Let us specify a density result.

Lemma 2.3. *Let $k \in \mathbb{N}_0$, $p \in [1,+\infty[$, and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (2.1) *is satisfied, then $\mathfrak{D}(\overline{\Omega})$ is dense in $W_s^{k,p}(\Omega)$.*

Proof. The proof follows by Lemma 2.2 in [12], since clearly both $\rho, \rho^{-1} \in L^\infty_{\text{loc}}(\overline{\Omega})$. □

This allows us to prove the following inclusion.

Lemma 2.4. *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$, and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (2.1) *is satisfied, then*

$$W^{k,p}(\Omega) \cap \overset{\circ}{W}{}^{k,p}(\Omega) \subset \overset{\circ}{W}{}^{k,p}_s(\Omega). \tag{2.12}$$

Proof. The density result stated in Lemma 2.3 being true, we can argue as in the proof of Lemma 2.1 of [10] to obtain the claimed inclusion. □

From this last lemma, we easily deduce that, if Ω has the segment property, also $C^k_o(\Omega) \subset \overset{\circ}{W}{}^{k,p}_s(\Omega)$.

Lemma 2.5. *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$, and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (2.1) *is satisfied, then the map*

$$u \longrightarrow \rho^s u \tag{2.13}$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$.

Proof. The first part of the proof easily follows from Lemma 2.2 with $t = s$. Let us show that $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$ if and only if $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$.

If $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$, there exists a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to u in $W^{k,p}_s(\Omega)$. Therefore, fixed $\varepsilon \in \mathbb{R}_+$, there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^s(\phi_h - u)\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{2.14}$$

Fix $h_1 > h_0$, clearly $\rho^s \phi_{h_1} \in \overset{\circ}{W}{}^{k,p}(\Omega)$, because of its compact support. Therefore, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s\phi_{h_1}$ in $W^{k,p}(\Omega)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|\psi_n - \rho^s\phi_{h_1}\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{2.15}$$

Putting together (2.14) and (2.15), we get

$$\|\psi_n - \rho^s u\|_{W^{k,p}(\Omega)} \le \|\psi_n - \rho^s\phi_{h_1}\|_{W^{k,p}(\Omega)} + \|\rho^s\phi_{h_1} - \rho^s u\|_{W^{k,p}(\Omega)} < \varepsilon, \tag{2.16}$$

for all $n > n_0$. Thus, $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$.

Vice versa, if we assume that $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$, we find a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s u$ in $W^{k,p}(\Omega)$. Hence, there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^{-s}\phi_h - u\|_{W^{k,p}_s(\Omega)} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{2.17}$$

Fix $h_1 > h_0$, since $\rho^{-s}\phi_{h_1} \in C_o^k(\Omega)$, which is contained in $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ by Lemma 2.4, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_o^\infty(\Omega)$ converging to $\rho^{-s}\phi_{h_1}$ in $W_s^{k,p}(\Omega)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\left\|\varphi_n - \rho^{-s}\phi_{h_1}\right\|_{W_s^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{2.18}$$

From (2.17) and (2.18), we get

$$\left\|\varphi_n - u\right\|_{W_s^{k,p}(\Omega)} \le \left\|\varphi_n - \rho^{-s}\phi_{h_1}\right\|_{W_s^{k,p}(\Omega)} + \left\|\rho^{-s}\phi_{h_1} - u\right\|_{W_s^{k,p}(\Omega)} < \varepsilon, \tag{2.19}$$

for all $n > n_0$, so that $u \in \overset{\circ}{W}{}_s^{k,p}(\Omega)$. □

3. Preliminary Results

From now on, we consider a weight $\rho : \overline{\Omega} \to \mathbb{R}_+$, $\rho \in C^2(\overline{\Omega})$, and such that (2.1) is satisfied (for $k = 2$). Moreover, we assume that

$$\lim_{|x|\to+\infty}\left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty, \qquad \lim_{|x|\to+\infty}\frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \tag{3.1}$$

An example of a function verifying our hypotheses is given by

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R}\setminus\{0\}. \tag{3.2}$$

We associate to ρ a function σ defined by

$$\begin{aligned} \sigma &= \rho \quad \text{if } \rho \longrightarrow +\infty, \text{ for } |x| \longrightarrow +\infty, \\ \sigma &= \frac{1}{\rho} \quad \text{if } \rho \longrightarrow 0, \text{ for } |x| \longrightarrow +\infty. \end{aligned} \tag{3.3}$$

Clearly σ verifies (2.1) and

$$\lim_{|x|\to+\infty}\sigma(x) = +\infty, \qquad \lim_{|x|\to+\infty}\frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{3.4}$$

Now, let us fix a cutoff function $f \in C_\circ^\infty(\overline{\mathbb{R}}_+)$ such that

$$0 \le f \le 1, \qquad f(t) = 1 \quad \text{if } t \in [0,1], \qquad f(t) = 0 \quad \text{if } t \in [2,+\infty[\,. \tag{3.5}$$

Then, set

$$\zeta_k : x \in \overline{\Omega} \longrightarrow f\left(\frac{\sigma(x)}{k}\right), \quad k \in \mathbb{N},$$
$$\Omega_k = \{x \in \Omega : \sigma(x) < k\}, \quad k \in \mathbb{N}\ . \tag{3.6}$$

By our definition, it follows that $\zeta_k \in C_\circ^\infty(\overline{\Omega})$ and

$$0 \le \zeta_k \le 1, \qquad \zeta_k = 1 \quad \text{on } \overline{\Omega_k}, \qquad \zeta_k = 0 \quad \text{on } \overline{\Omega \setminus \Omega_{2k}}, \quad k \in \mathbb{N}\ . \tag{3.7}$$

Finally, we introduce the sequence

$$\eta_k : x \in \overline{\Omega} \longrightarrow 2k\zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \quad k \in \mathbb{N}\ . \tag{3.8}$$

For any $k \in \mathbb{N}$, one has

$$\eta_k = \zeta_k(2k - \sigma) + \sigma \ge \sigma, \quad \text{in } \overline{\Omega_{2k}}, \tag{3.9}$$

$$\eta_k \le 2k + \sigma \le \left(\frac{2k}{\inf_{\overline{\Omega_{2k}}}\sigma} + 1\right)\sigma = (c_k + 1)\sigma, \quad \text{in } \overline{\Omega_{2k}}, \tag{3.10}$$

$$\eta_k = \sigma, \quad \text{in } \overline{\Omega \setminus \Omega_{2k}}, \tag{3.11}$$

where $c_k \in \mathbb{R}_+$ depends only on k. This entails that

$$\sigma \sim \eta_k, \quad \forall k \in \mathbb{N}. \tag{3.12}$$

Concerning the derivatives, easy calculations give that, for any $k \in \mathbb{N}$,

$$(\eta_k)_x = (\eta_k)_{xx} = 0, \quad \text{in } \overline{\Omega_k}, \tag{3.13}$$

$$(\eta_k)_x = \sigma_x, \qquad (\eta_k)_{xx} = \sigma_{xx}, \quad \text{in } \overline{\Omega \setminus \Omega_{2k}}, \tag{3.14}$$

$$(\eta_k)_x \le c_1\sigma_x, \qquad (\eta_k)_{xx} \le c_2\left(\frac{\sigma_x^2}{\sigma} + \sigma_{xx}\right), \quad \text{in } \overline{\Omega \setminus \Omega_{2k}}, \tag{3.15}$$

with c_1 and c_2 positive constants independent of x and k.

From (3.9), (3.11), (3.13), (3.14), and (3.15), we obtain, for any $k \in N$,

$$\frac{(\eta_k)_x}{\eta_k} \le c_1' \frac{\sigma_x}{\sigma}, \quad \text{in } \overline{\Omega},$$
$$\frac{(\eta_k)_{xx}}{\eta_k} \le c_2' \frac{\sigma_x^2 + \sigma\sigma_{xx}}{\sigma^2}, \quad \text{in } \overline{\Omega}, \tag{3.16}$$

where c_1' and c_2' are positive constants independent of x and k.

Combining (3.13) and (3.16), we have also, for any $k \in \mathbb{N}$,

$$\frac{(\eta_k)_x}{\eta_k} \leq c_1' \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma}, \text{ in } \overline{\Omega}, \tag{3.17}$$

$$\frac{(\eta_k)_{xx}}{\eta_k} \leq c_2' \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma\sigma_{xx}}{\sigma^2}, \text{ in } \overline{\Omega}. \tag{3.18}$$

We conclude this section proving the following lemma.

Lemma 3.1. *Let σ and $\Omega_k, k \in \mathbb{N}$, be defined by* (3.3) *and* (3.6)*, respectively. Then*

$$\lim_{k \to +\infty} \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{3.19}$$

Proof. Set

$$\begin{gathered} \varphi(x) = \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)}, \quad x \in \overline{\Omega}, \\ \psi_k = \sup_{\Omega \setminus \Omega_k} \varphi, \quad k \in \mathbb{N}. \end{gathered} \tag{3.20}$$

By the second relation in (3.4), the supremum of φ over $\overline{\Omega \setminus \Omega_k}$ is actually a maximum; thus, for every $k \in \mathbb{N}$, there exists $x_k \in \overline{\Omega \setminus \Omega_k}$ such that $\psi_k = \varphi(x_k)$.

To prove (3.19), we have to show that $\lim_{k \to +\infty} \psi_k = 0$.

We proceed by contradiction. Hence, let us assume that there exists $\varepsilon_0 > 0$ such that, for any $k \in \mathbb{N}$, there exists $n_k > k$ such that $\psi_{n_k} = \varphi(x_{n_k}) \geq \varepsilon_0$.

If the sequence $(x_{n_k})_{k \in \mathbb{N}}$ is bounded, there exists a subsequence $({x'}_{n_k})_{k \in \mathbb{N}}$ converging to a limit $x \in \overline{\Omega}$, and by the continuity of σ, $(\sigma({x'}_{n_k}))_{k \in \mathbb{N}}$ converges to $\sigma(x)$. On the other hand, ${x'}_{n_k} \in \overline{\Omega \setminus \Omega_k}$, thus $\sigma({x'}_{n_k}) \geq n_k$, which is in contrast with the fact that $(\sigma({x'}_{n_k}))_{k \in \mathbb{N}}$ is a convergent sequence.

Therefore, $(x_{n_k})_{k \in \mathbb{N}}$ is unbounded, so that there exists a subsequence $(x''_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \to +\infty} |x''_{n_k}| = +\infty$. Thus, by the second relation in (3.4), one has $\lim_{k \to +\infty} \varphi(x''_{n_k}) = 0$. This gives the contradiction since $\varphi(x''_{n_k}) \geq \varepsilon_0$. $\square$

4. A No Weighted A Priori Bound

We want to prove a $W^{2,2}$ bound for an uniformly elliptic second-order linear differential operator. Let us start recalling the definitions of the function spaces in which the coefficients of our operator will be chosen.

For any Lebesgue measurable subset G of $\mathbb{R}^n$, let $\Sigma(G)$ be the σ-algebra of all Lebesgue measurable subsets of G. Given $E \in \Sigma(G)$, we denote by $|E|$ the Lebesgue measure of E, by χ_E its characteristic function, and by $E(x, r)$ the intersection $E \cap B(x, r)$ $(x \in \mathbb{R}^n, r \in \mathbb{R}_+)$, where $B(x, r)$ is the open ball with center x and radius r.

For $n \geq 2$, $\lambda \in [0, n[$, $p \in [1, +\infty[$, and fixed t in $\mathbb{R}_+$, the space of Morrey type $M^{p,\lambda}(\Omega, t)$ is the set of all functions g in $L^p_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau \in]0,t] \\ x \in \Omega}} \tau^{-\lambda/p} \|g\|_{L^p(\Omega(x,\tau))} < +\infty, \tag{4.1}$$

endowed with the norm defined in (4.1). It is easily seen that, for any $t_1, t_2 \in \mathbb{R}_+$, a function g belongs to $M^{p,\lambda}(\Omega, t_1)$ if and only if it belongs to $M^{p,\lambda}(\Omega, t_2)$; moreover, the norms of g in these two spaces are equivalent. This allows us to restrict our attention to the space $M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega, 1)$.

We now introduce three subspaces of $M^{p,\lambda}(\Omega)$ needed in the sequel. The set $VM^{p,\lambda}(\Omega)$ is made up of the functions $g \in M^{p,\lambda}(\Omega)$ such that

$$\lim_{t \to 0} \|g\|_{M^{p,\lambda}(\Omega,t)} = 0, \tag{4.2}$$

while $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_\circ^{p,\lambda}(\Omega)$ denote the closures of $L^\infty(\Omega)$ and $C_\circ^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$, respectively. We point out that

$$M_\circ^{p,\lambda}(\Omega) \subset \widetilde{M}^{p,\lambda}(\Omega) \subset VM^{p,\lambda}(\Omega). \tag{4.3}$$

We put $M^p(\Omega) = M^{p,0}(\Omega)$, $VM^p(\Omega) = VM^{p,0}(\Omega)$, $\widetilde{M}^p(\Omega) = \widetilde{M}^{p,0}(\Omega)$, and $M_\circ^p(\Omega) = M_\circ^{p,0}(\Omega)$.

We want to define the moduli of continuity of functions belonging to $\widetilde{M}^{p,\lambda}(\Omega)$ or $M_\circ^{p,\lambda}(\Omega)$. To this aim, let us put, for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$,

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \le 1/h}} \|g\chi_E\|_{M^{p,\lambda}(\Omega)}. \tag{4.4}$$

Recall first that for a function $g \in M^{p,\lambda}(\Omega)$ the following characterization holds:

$$g \in \widetilde{M}^{p,\lambda}(\Omega) \Longleftrightarrow \lim_{h \to +\infty} F[g](h) = 0, \tag{4.5}$$

while

$$g \in M_\circ^{p,\lambda}(\Omega) \Longleftrightarrow \lim_{h \to +\infty} \left(F[g](h) + \|(1-\zeta_h)g\|_{M^{p,\lambda}(\Omega)} \right) = 0, \tag{4.6}$$

where ζ_h denotes a function of class $C_o^\infty(R^n)$ such that

$$0 \le \zeta_h \le 1, \quad \zeta_h|_{\overline{B(0,h)}} = 1, \quad \operatorname{supp} \zeta_h \subset B(0, 2h). \tag{4.7}$$

Thus, if g is a function in $\widetilde{M}^{p,\lambda}(\Omega)$, a *modulus of continuity* of g in $\widetilde{M}^{p,\lambda}(\Omega)$ is a map $\tilde{\sigma}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \le \tilde{\sigma}^{p,\lambda}[g](h), \qquad \lim_{h \to +\infty} \tilde{\sigma}^{p,\lambda}[g](h) = 0\ . \tag{4.8}$$

While, if g belongs to $M_o^{p,\lambda}(\Omega)$, a *modulus of continuity* of g in $M_o^{p,\lambda}(\Omega)$ is an application ${\sigma_o}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \|(1-\zeta_h)g\|_{M^{p,\lambda}(\Omega)} \le {\sigma_o}^{p,\lambda}[g](h), \qquad \lim_{h\to+\infty} {\sigma_o}^{p,\lambda}[g](h) = 0 \,. \tag{4.9}$$

If Ω has the property

$$|\Omega(x,r)| \ge Ar^n \quad \forall x \in \Omega\ ,\ \forall r \in\ ]0,1], \tag{4.10}$$

where A is a positive constant independent of x and r, it is possible to consider the space $BMO(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\overline{\Omega})$ such that

$$[g]_{BMO(\Omega,\tau)} = \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} ⨍_{\Omega(x,r)} \left| g - ⨍_{\Omega(x,r)} g\, dy \right| dy < +\infty \,, \tag{4.11}$$

where

$$⨍_{\Omega(x,r)} g\, dy = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g\, dy. \tag{4.12}$$

If $g \in BMO(\Omega) = BMO(\Omega,\tau_A)$, where

$$\tau_A = \sup\left\{ \tau \in \mathbb{R}_+ : \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} \frac{r^n}{|\Omega(x,r)|} \le \frac{1}{A} \right\}, \tag{4.13}$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,\tau)} \to 0$ for $\tau \to 0^+$.

If g belongs to $VMO(\Omega)$, a *modulus of continuity* of g in $VMO(\Omega)$ is function $\eta[g] :]0,1] \to \mathbb{R}_+$ such that

$$[g]_{BMO(\Omega,\tau)} \le \eta[g](\tau) \quad \forall \tau \in\ ]0,1], \quad \lim_{\tau\to 0^+} \eta[g](\tau) = 0 \,. \tag{4.14}$$

For more details on the above-defined function spaces, we refer to [8, 13–15].

Let us start proving a useful lemma.

Lemma 4.1. *If Ω has the uniform $C^{1,1}$-regularity property and*

$$g, g_x \in \begin{cases} VM^r(\Omega), & r > 2,\ \text{for } n = 2, \\ VM^{r,n-r}(\Omega), & r \in\ ]2,n],\ \text{for } n > 2, \end{cases} \tag{4.15}$$

then $g \in VMO(\Omega)$.

Proof. For $n > 2$, the result can be found in [16], combining Lemma 4.1 and the argument in the proof of Lemma 4.2.

Concerning $n = 2$, we firstly apply a known extension result, see [9, Corollary 2.2], stating that any function g such that $g, g_x \in VM^r(\Omega)$ admits an extension $p(g)$ such that $p(g), (p(g))_x \in VM^r(\mathbb{R}^2)$.

Then, we prove that for all $x_0 \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$, there exists a constant $c \in \mathbb{R}_+$ such that

$$\fint_{B(x_0,t)} \left| p(g) - \fint_{B(x_0,t)} p(g)dx \right| dx \le c\Big(t^{(r-2/r)} \|(p(g))_x\|_{L^r(B(x_0,t))}\Big). \tag{4.16}$$

Indeed, in view of the above considerations, if (4.16) holds true, one has that $p(g) \in VMO(\mathbb{R}^2)$, so $g \in VMO(\Omega)$.

Consider the function

$$g^* : z \in \mathbb{R}^2 \longrightarrow p(g)(x_0 + tz) \in \mathbb{R}. \tag{4.17}$$

By Poincaré-Wirtinger inequality and Hölder inequality, one gets

$$\begin{aligned}
&\fint_{B(x_0,t)} \left| p(g)(x) - \fint_{B(x_0,t)} p(g)(x)dx \right| dx \\
&\quad = \pi^{-1} \int_{B(0,1)} \left| g^*(z) - \fint_{B(0,1)} g^*(z)dz \right| dz \le c_1 \int_{B(0,1)} \left|(g^*)_z(z)\right| dz \\
&\quad = c_1 t^{-1} \int_{B(x_0,t)} \left|(p(g))_x(x)\right| dx \le c_1 t^{-1} |B(x_0,t)|^{(r-1/r)} \|(p(g))_x\|_{L^r(B(x_0,t))},
\end{aligned} \tag{4.18}$$

this gives (4.16). □

For reader's convenience, we recall here some results proved in [17], adapted to our needs.

Lemma 4.2. *If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^{r,\lambda}(\Omega)$, with $r > 2$ and $\lambda = 0$ if $n = 2$, and $r \in]2, n]$ and $\lambda = n - r$ if $n > 2$, then*

$$u \longrightarrow gu \tag{4.19}$$

is a bounded operator from $W^{1,2}(\Omega)$ to $L^2(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$, such that

$$\|gu\|_{L^2(\Omega)} \le c \ \|g\|_{M^{r,\lambda}(\Omega)} \|u\|_{W^{1,2}(\Omega)}, \tag{4.20}$$

with $c = c\ (\Omega, n, r)$.

Furthermore, if $g \in \widetilde{M}^{r,\lambda}(\Omega)$, then for any $\varepsilon > 0$ there exists a constant $c_\varepsilon \in \mathbb{R}_+$, such that

$$\|gu\|_{L^2(\Omega)} \le \varepsilon \ \|u\|_{W^{1,2}(\Omega)} + c_\varepsilon \|u\|_{L^2(\Omega)}, \tag{4.21}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, r, \widetilde{\sigma}^{r,\lambda}[g])$.

If $g \in M^{t,\mu}(\Omega)$, *with* $t \geq 2$ *and* $\mu > n - 2t$, *then the operator in* (4.19) *is bounded from* $W^{2,2}(\Omega)$ *to* $L^2(\Omega)$. *Moreover, there exists a constant* $c' \in \mathbb{R}_+$, *such that*

$$\|gu\|_{L^2(\Omega)} \leq c' \|g\|_{M^{t,\mu}(\Omega)} \|u\|_{W^{2,2}(\Omega)}, \tag{4.22}$$

with $c' = c'(\Omega, n, t, \mu)$.

Furthermore, if $g \in \widetilde{M}^{t,\mu}(\Omega)$, *then for any* $\varepsilon > 0$ *there exists a constant* $c'_\varepsilon \in \mathbb{R}_+$, *such that*

$$\|gu\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{W^{2,2}(\Omega)} + c'_\varepsilon \|u\|_{L^2(\Omega)}, \tag{4.23}$$

with $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, t, \mu, \widetilde{\sigma}^{t,\mu}[g])$.

Proof. The proof easily follows from Theorem 3.2 and Corollary 3.3 of [17]. □

From now on, we assume that Ω is an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$, with the uniform $C^{1,1}$-regularity property.

We consider the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{4.24}$$

with the following conditions on the coefficients:

$$\begin{gathered} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n, \\ \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \quad \text{a.e. in } \Omega,\ \forall \xi \in R^n, \end{gathered} \tag{h_1}$$

$$\begin{gathered} (a_{ij})_{x_j}, a_i \in M_o^{r,\lambda}(\Omega), \quad i,j = 1,\ldots,n, \\ \text{with } r > 2, \quad \lambda = 0 \text{ if } n = 2, \\ \text{with } r \in]2,n], \quad \lambda = n - r \text{ if } n > 2, \end{gathered} \tag{h_2}$$

$$\begin{gathered} a \in \widetilde{M}^{t,\mu}(\Omega), \quad \text{with } t \geq 2,\ \mu > n - 2t, \\ \operatorname*{ess\,inf}_{\Omega} a = a_0 > 0. \end{gathered} \tag{h_3}$$

We explicitly observe that under the assumptions (h_1)–(h_3) the operator $L : W^{2,2}(\Omega) \to L^2(\Omega)$ is bounded, as a consequence of Lemma 4.2.

We are now in position to prove the above-mentioned a priori estimate.

Theorem 4.3. *Let L be defined in* (4.24). *Under hypotheses* (h_1)–(h_3), *there exists a constant* $c \in \mathbb{R}_+$ *such that*

$$\|u\|_{W^{2,2}(\Omega)} \leq c\|Lu\|_{L^2(\Omega)}, \quad \forall u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \tag{4.25}$$

with $c = c(\Omega, n, \nu, r, t, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, {\sigma_o}^{r,\lambda}[(a_{ij})_{x_j}], {\sigma_o}^{r,\lambda}[a_i], \widetilde{\sigma}^{t,\mu}[a], a_0)$.

Proof. Let us put

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.26}$$

and fix $u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Lemma 4.1 being true, Lemma 3.1 of [18] (for $n = 2$) and Theorem 5.1 of [17] (for $n > 2$) apply, so that there exists a constant $c_1 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,2}(\Omega)} \leq c_1 \left(\|L_0 u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right), \tag{4.27}$$

with $c_1 = c_1(\Omega, n, \nu, \|a_{ij}\|_{L^\infty(\Omega)}, {\sigma_o}^{r,\lambda}[(a_{ij})_{x_j}])$. Therefore,

$$\|u\|_{W^{2,2}(\Omega)} \leq c_1 \left(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \sum_{i=1}^{n} \|a_i u_{x_i}\|_{L^2(\Omega)} + \|au\|_{L^2(\Omega)} \right). \tag{4.28}$$

On the other hand, from Lemma 4.2, one has

$$\begin{gathered} \|a_i u_{x_i}\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{W^{2,2}(\Omega)} + c_\varepsilon \|u_{x_i}\|_{L^2(\Omega)}, \\ \|au\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{W^{2,2}(\Omega)} + c'_\varepsilon \|u\|_{L^2(\Omega)}, \end{gathered} \tag{4.29}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, r, {\sigma_o}^{r,\lambda}[a_i])$ and $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, t, \mu, \widetilde{\sigma}^{t,\mu}[a])$.

Furthermore, classical interpolation results give that there exists a constant $K \in \mathbb{R}_+$ such that

$$\|u_x\|_{L^2(\Omega)} \leq K\varepsilon \|u\|_{W^{2,2}(\Omega)} + \frac{K}{\varepsilon} \|u\|_{L^2(\Omega)}, \tag{4.30}$$

with $K = K(\Omega)$. Combining (4.28), (4.29) and (4.30) we conclude that there exists $c_2 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,2}(\Omega)} \leq c_2 \left(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right), \tag{4.31}$$

with $c_2 = c_2(\Omega, n, \nu, r, t, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, {\sigma_o}^{r,\lambda}[(a_{ij})_{x_j}], {\sigma_o}^{r,\lambda}[a_i], \widetilde{\sigma}^{t,\mu}[a])$.

To show (4.25), it remains to estimate $\|u\|_{L^2(\Omega)}$. To this aim let us rewrite our operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(a_{ij}\right)_{x_j} + a_i \right) u_{x_i} + au, \tag{4.32}$$

in order to adapt to our framework some known results concerning operators in variational form. Following along the lines, the proofs of Theorem 4.3 of [19] (for $n = 2$) and of Theorem 4.2 of [13] (for $n > 2$), with opportune modifications—we explicitly observe that the continuity of the bilinear form associated to (4.32) in our case is an immediate consequence of Lemma 4.2—we obtain that

$$\|u\|_{L^2(\Omega)} \leq c_3 \|Lu\|_{L^2(\Omega)}, \tag{4.33}$$

with $c_3 = c_3(n, \nu, r, \sigma_o{}^{r,\lambda}[(a_{ij})_{x_j}], \sigma_o{}^{r,\lambda}[a_i], a_0)$. Putting together (4.31) and (4.33), we obtain (4.25). □

5. Uniqueness and Existence Results

This section is devoted to the proof of the solvability of a Dirichlet problem for a class of second-order linear elliptic equations in the weighted space $W^{2,2}_s(\Omega)$. The $W^{2,2}$-bound obtained in Theorem 4.3 allows us to show the following a priori estimate in the weighted case.

Theorem 5.1. *Let L be defined in* (4.24)*. Under hypotheses* (h_1)–(h_3)*, there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,2}_s(\Omega)} \leq c \|Lu\|_{L^2_s(\Omega)}, \quad \forall u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega), \tag{5.1}$$

with $c = c(\Omega, n, s, \nu, r, t, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, \|a_i\|_{M^{r,\lambda}(\Omega)}, \sigma_o{}^{r,\lambda}[(a_{ij})_{x_j}], \sigma_o{}^{r,\lambda}[a_i], \tilde{\sigma}^{t,\mu}[a], a_0)$.

Proof. Fix $u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega)$. In the sequel, for sake of simplicity, we will write $\eta_k = \eta$, for a fixed $k \in \mathbb{N}$. Observe that η satisfies (2.1), as a consequence of (3.16), so that Lemma 2.5 applies giving that $\eta^s u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Therefore, in view of Theorem 4.3, there exists $c_0 \in \mathbb{R}_+$, such that

$$\|\eta^s u\|_{W^{2,2}(\Omega)} \leq c_0 \|L(\eta^s u)\|_{L^2(\Omega)}, \tag{5.2}$$

with $c_0 = c_0(\Omega, n, \nu, r, t, \mu, \|a_{ij}\|_{L^\infty(\Omega)}, \sigma_o{}^{r,\lambda}[(a_{ij})_{x_j}], \sigma_o{}^{r,\lambda}[a_i], \tilde{\sigma}^{t,\mu}[a], a_0)$. Easy computations give

$$\begin{aligned} L(\eta^s u) = \eta^s Lu - \; & s \sum_{i,j=1}^{n} a_{ij} \left((s-1)\eta^{s-2} \eta_{x_i} \eta_{x_j} u + \eta^{s-1} \eta_{x_i x_j} u + 2\eta^{s-1} \eta_{x_i} u_{x_j} \right) \\ & + \; s \sum_{i=1}^{n} a_i \eta^{s-1} \eta_{x_i} u. \end{aligned} \tag{5.3}$$

Putting together (5.2) and (5.3), we deduce that

$$\begin{aligned} \|\eta^s u\|_{W^{2,2}(\Omega)} \leq c_1 \Bigg(& \|\eta^s Lu\|_{L^2(\Omega)} + \sum_{i,j=1}^{n} \left(\left\| \eta^{s-2} \eta_{x_i} \eta_{x_j} u \right\|_{L^2(\Omega)} + \left\| \eta^{s-1} \eta_{x_i x_j} u \right\|_{L^2(\Omega)} \right. \\ & \left. + \left\| \eta^{s-1} \eta_{x_i} u_{x_j} \right\|_{L^2(\Omega)} \right) + \sum_{i=1}^{n} \left\| a_i \eta^{s-1} \eta_{x_i} u \right\|_{L^2(\Omega)} \Bigg), \end{aligned} \tag{5.4}$$

where $c_1 \in \mathbb{R}_+$ depends on the same parameters as c_0 and on s.

On the other hand, from Lemma 4.2 and (3.17), we get

$$\left\| a_i \eta^{s-1} \eta_{x_i} u \right\|_{L^2(\Omega)} \le c_2 \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \| a_i \|_{M^{r,\lambda}(\Omega)} \| \eta^s u \|_{W^{1,2}(\Omega)}, \tag{5.5}$$

with $c_2 = c_2(\Omega, n, r)$.

Combining (3.17), (3.18), (5.4), and (5.5), with simple calculations we obtain the bound

$$\| \eta^s u \|_{W^{2,2}(\Omega)} \le c_3 \left[\| \eta^s L u \|_{L^2(\Omega)} + \left(\sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} + \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \right) \| \eta^s u \|_{W^{2,2}(\Omega)} \right], \tag{5.6}$$

where c_3 depends on the same parameters as c_1 and on $\| a_i \|_{M^{r,\lambda}(\Omega)}$.

By Lemma 3.1, it follows that there exists $k_o \in \mathbb{N}$ such that

$$\left(\sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} + \sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x}{\sigma} \right) \le \frac{1}{2c_3}. \tag{5.7}$$

Now, if we still denote by η the function η_{k_o}, from (5.6) and (5.7), we deduce that

$$\| \eta^s u \|_{W^{2,2}(\Omega)} \le 2c_3 \| \eta^s L u \|_{L^2(\Omega)}. \tag{5.8}$$

Then, by Lemma 2.2 and by (3.12), written for $k = k_o$, we have

$$\sum_{|\alpha| \le 2} \| \sigma^s \partial^\alpha u \|_{L^2(\Omega)} \le c_4 \| \sigma^s L u \|_{L^2(\Omega)}, \tag{5.9}$$

with c_4 depending on the same parameters as c_3 and on k_o.

This last estimate being true for every $s \in \mathbb{R}$, we also have

$$\sum_{|\alpha| \le 2} \| \sigma^{-s} \partial^\alpha u \|_{L^2(\Omega)} \le c_5 \| \sigma^{-s} L u \|_{L^2(\Omega)}. \tag{5.10}$$

The bounds in (5.9) and (5.10) together with the definition (3.3) of σ give estimate (5.1). □

Lemma 5.2. *The Dirichlet problem*

$$\begin{gathered} u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \\ -\Delta u + b u = f, \quad f \in L_s^2(\Omega), \end{gathered} \tag{5.11}$$

where

$$b = 1 + \left| -s(s+1)\sum_{i=1}^{n}\frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{n}\frac{\sigma_{x_i x_i}}{\sigma} \right|, \tag{5.12}$$

is uniquely solvable.

Proof. Observe that u is a solution of problem (5.11) if and only if $w = \sigma^s u$ is a solution of the problem

$$\begin{gathered} w \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \\ -\Delta(\sigma^{-s}w) + b\sigma^{-s}w = f, \quad f \in L_s^2(\Omega). \end{gathered} \tag{5.13}$$

Clearly, for any $i \in \{1,\ldots,n\}$,

$$\frac{\partial^2}{\partial x_i^2}(\sigma^{-s}w) = \sigma^{-s}w_{x_i x_i} - 2s\sigma^{-s-1}\sigma_{x_i}w_{x_i} + s(s+1)\sigma^{-s-2}\sigma_{x_i}^2 w - s\sigma^{-s-1}\sigma_{x_i x_i}w, \tag{5.14}$$

then (5.13) is equivalent to the problem

$$\begin{gathered} w \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \\ -\Delta w + \sum_{i=1}^{n}\alpha_i w_{x_i} + \alpha w = g, \quad g \in L^2(\Omega), \end{gathered} \tag{5.15}$$

where

$$\alpha_i = 2s\frac{\sigma_{x_i}}{\sigma}, \quad i = 1,\ldots,n, \qquad \alpha = b - s(s+1)\sum_{i=1}^{n}\frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{n}\frac{\sigma_{x_i x_i}}{\sigma}, \quad g = \sigma^s f. \tag{5.16}$$

Using Theorem 5.2 in [18] (for $n = 2$), Theorem 4.3 of [20] (for $n > 2$), (1.6) of [8], and the hypotheses on σ, we obtain that (5.15) is uniquely solvable and then problem (5.11) is uniquely solvable too. □

Theorem 5.3. *Let L be defined in* (4.24). *Under hypotheses* (h_1)–(h_3), *the problem*

$$\begin{gathered} u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \\ Lu = f, \quad f \in L_s^2(\Omega), \end{gathered} \tag{5.17}$$

is uniquely solvable.

Proof. For each $\tau \in [0,1]$, we put

$$L_\tau = \tau(L) + (1-\tau)(-\Delta + b). \tag{5.18}$$

In view of Theorem 5.1,

$$\|u\|_{W_s^{2,2}(\Omega)} \leq c\|L_\tau u\|_{L_s^p(\Omega)}, \quad \forall u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega),\ \forall \tau \in [0,1]. \tag{5.19}$$

Thus, taking into account the result of Lemma 5.2 and using the method of continuity along a parameter (see, e.g., Theorem 5.2 of [21]), we obtain the claimed result. □

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