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Research Article

Oscillation and Asymptotic Behaviour of a Higher-Order Nonlinear Neutral-Type Functional Differential Equation with Oscillating Coefficients

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We will study oscillation of bounded solutions of higher-order nonlinear neutral delay differential equations of the following type: $[y(t)+p(t)f(y(\tau(t)))]^{(n)}+q(t)h(y(\sigma(t)))=0$, $t\geq t_0$, $t\in\mathbb{R}$, where $p\in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty}p(t)=0$, $q\in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t)$, $\sigma(t)\in C([t_0,\infty),\mathbb{R})$, $\tau(t)$, $\sigma(t)<t$, $\lim_{t\to\infty}\tau(t)$, $\sigma(t)=\infty$, and $f,h\in C(\mathbb{R},\mathbb{R})$. We obtain sufficient conditions for the oscillation of all solutions of this equation.

1. Introduction

In this paper, we are concerned with the oscillation of the solutions of a certain more general higher-order nonlinear neutral-type functional differential equation with an oscillating coefficient of the form

$$\left[y(t)+p(t)f\left(y(\tau(t))\right)\right]^{(n)}+q(t)h\left(y(\sigma(t))\right)=0,\quad t\geq t_0,\ t\in\mathbb{R},\tag{1.1}$$

where $p\in C([t_0,\infty),\mathbb{R})$ is oscillatory and $\lim_{t\to\infty}p(t)=0$, $q\in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t)$, $\sigma(t)\in C([t_0,\infty),\mathbb{R})$, $\tau(t)$, $\sigma(t)<t$, $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$, and $f,h\in C(\mathbb{R},\mathbb{R})$. As it is customary, a solution $y(t)$ is said to be oscillatory if $y(t)$ is not eventually positive or not eventually negative. Otherwise, the solution is called nonoscillatory. A differential equation

is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real-valued solutions y.

In [1, 2], several authors have investigated the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1.2}$$

where $q \in C([t_0, \infty), \mathbb{R}^+)$ and $\sigma(t) \in C([t_0, \infty), \mathbb{R})$. A classical result is that every solution of (1.2) oscillates if

$$\liminf_{t \to \infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e}. \tag{1.3}$$

In [3], Zein and Abu-Kaff have investigated the higher-order nonlinear delay differential equation

$$\left[x(t) + p(t)x(\tau(t))\right]^{(n)} + f(t, x(t), x(\sigma(t))) = s(t), \quad t \geq t_0, \; t \in \mathbb{R}, \tag{1.4}$$

where $p \in C([t_0, \infty), \mathbb{R})$, $\lim_{t \to \infty} p(t) = 0$, $\tau(t), \sigma(t) \in C([t_0, \infty), \mathbb{R})$, $\tau(t), \sigma(t) < t$, $\lim_{t \to \infty} \tau(t) = \infty$, $\lim_{t \to \infty} \sigma(t) = \infty$, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $yf(t, x, y) > 0$ for $xy > 0$, there exists an oscillatory function $r \in C^n(\mathbb{R}_+, \mathbb{R})$, such that $r^{(n)}(t) = s(t)$, $\lim_{t \to \infty} r(t) = 0$.

In [4], Bolat and Akin have investigated the higher-order nonlinear differential equation

$$\left[y(t) + p(t)y(\tau(t))\right]^{(n)} + \sum_{i=1}^{m} q_i(t) f_i\left(y(\sigma_i(t))\right) = s(t), \tag{1.5}$$

where $p(t)$, $q_i(t)$, $\tau(t)$, $s(t) \in C([t_0, \infty), \mathbb{R})$ for $i = 1, \ldots, m$, $p(t)$ and $s(t)$ are oscillating functions, $q_i(t) \geq 0$ for $i = 1, \ldots, m$, $\sigma_i(t) \in C^1([t_0, \infty), \mathbb{R})$, $\sigma_i'(t) > 0$, $\sigma_i(t) \leq t$, $\lim_{t \to \infty} \sigma_i(t) = \infty$ for $i = 1, \ldots, m$, $\lim_{t \to \infty} \tau(t) = \infty$, $f_i(u) \in C((\mathbb{R}, \mathbb{R}))$ is nondecreasing function, $uf(u) > 0$ for $u \neq 0$, and $i = 1, \ldots, m$. If n is odd, $\lim_{t \to \infty} p(t) = 0$, $\lim_{t \to \infty} r(t) = 0$, and $\int_{t_0}^{\infty} \nu^{n-1} q(\nu) d\nu = \infty$ for $i = 1, \ldots, m$, then every bounded solution of (1.5) is either oscillatory or tends to zero as $t \to \infty$. If n is even, $\lim_{t \to \infty} p(t) = 0$, and $\lim_{t \to \infty} r(t) = 0$, there exists a continuously differentiable function $\varphi(t)$

$$\begin{gathered}
\limsup_{t \to \infty} \int_{t_0}^{t} \varphi(\nu) \sum_{i=1}^{m} q_i(\nu) d\nu = \infty, \\
\limsup_{t \to \infty} \int_{t_0}^{t} \left[\frac{\left[\varphi'(\nu)\right]^2}{\varphi(\nu)\sigma_i'(\nu)\sigma_i^{n-2}(\nu)}\right] d\nu < \infty,
\end{gathered} \tag{1.6}$$

then every bounded solution of (1.5) is either oscillatory or tends to zero as $t \to \infty$.

Recently, many studies have been made on the oscillatory and asymptotic behaviour of solutions of higher-order neutral-type functional differential equations. Most of the known results which were studied are the cases when $f(u) = I(u)$, where I is the identity function; see, for example, [1–15] and references cited there in.

The purpose of this paper is to study oscillatory behaviour of solutions of (1.1). For the general theory of differential equations, one can refer to [5, 6, 12–14]. Many references to some applications of the differential equations can be found in [2].

In this paper, the function $z(t)$ is defined by

$$z(t) = y(t) + p(t) f\left(y(\tau(t))\right). \tag{1.7}$$

2. Some Auxiliary Lemmas

Lemma 2.1 (see [5]). *Let y be a positive and n-times differentiable function on $[t_0, +\infty)$. If $y^{(n)}(t)$ is of constant sign and not identically zero in any interval $[b, +\infty)$, then there exist a $t_1 \geq t_0$ and an integer l, $0 \leq l \leq n$ such that $n+l$ is even, if $y^{(n)}(t)$ is nonnegative, or $n+l$ odd, if $y^{(n)}(t)$ is nonpositive, and that, as $t \geq t_1$, if $l > 0$, $y^{(n)}(t) > 0$ for $k = 0, 1, 2, \ldots, l-1$, and if $l \leq n-1$, $(-1)^{k+1} y^{(n)}(t) > 0$ for $k = l, l+1, \ldots, n-1$.*

Lemma 2.2 (see [5]). *Let $y(t)$ be as in Lemma 2.1. In addition $\lim_{t\to\infty} y(t) \neq 0$ and $y^{(n-1)}(t) y^{(n)}(t) \leq 0$ for every $t \geq t_y$; then for every λ, $0 < \lambda < 1$, the following hold:*

$$y(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} y^{(n-1)}(t) \quad \text{for all large } t. \tag{2.1}$$

3. Main Results

Theorem 3.1. *Assume that n is even,*

(C_1) *there exists a function $H : \mathbb{R} \to \mathbb{R}$ such that H is continuous and nondecreasing and satisfies the inequality*

$$-H(-uv) \geq H(uv) \geq K H(u) H(v), \quad \text{for } u, v > 0, \tag{3.1}$$

where K is a positive constant, and

$$|h(u)| \geq |H(u)|, \quad \frac{H(u)}{u} \geq \gamma > 0, \quad H(u) > 0, \quad \text{for } u \neq 0, \tag{3.2}$$

(C_2) $\lim_{t\to\infty} p(t) = 0$,

(C_3) $\int_{t_0}^{\infty} s^{n-1} q(s) ds = \infty$

and every solution of the first-order delay differential equation

$$w'(t) + q(t) K \gamma H\left(\frac{1}{2} \frac{\lambda}{(n-1)!} \sigma^{n-1}(t)\right) w(\sigma(t)) = 0 \tag{3.3}$$

is oscillatory. Then every bounded solution of (1.1) is either oscillates or tends to zero as $t \to \infty$.

Proof. Assume that (1.1) has a bounded nonoscillatory solution y. Without loss of generality, assume that y is eventually positive (the proof is similar when y is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Further, suppose that y does not tend to zero as $t \to \infty$. By (1.1) and (1.7), we have

$$z^{(n)}(t) = -q(t)h\big(y(\sigma(t))\big) \leq 0, \qquad t \geq t_1. \tag{3.4}$$

It follows that $z^{(\alpha)}(t)(\alpha = 0,1,2,\ldots,n-1)$ is strictly monotone and eventually of constant sign. Since y is bounded and does not tend to zero as $t \to \infty$, by virtue of (C_2), $\lim_{t\to\infty} p(t)f(y(\tau(t))) = 0$. Then we can find a $t_2 \geq t_1$ such that $z(t) = y(t) + p(t)f(y(\tau(t))) > 0$ eventually and $z(t)$ is also bounded for sufficiently large $t \geq t_2$. Because n is even and $(n+l)$ odd for $z^{(n)}(t) \leq 0$ and $z(t) > 0$ is bounded, by Lemma 2.1, since $l = 1$ (otherwise, $z(t)$ is not bounded), there exists a $t_3 \geq t_2$ such that for $t \geq t_3$

$$(-1)^{k+1} z^{(k)}(t) > 0 \quad (k = 1,2,\ldots,n-1). \tag{3.5}$$

In particular, since $z'(t) > 0$ for $t \geq t_3$, z is increasing. Since y is bounded, $\lim_{t\to\infty} p(t)f(y(\tau(t))) = 0$ by (C_2). Then, there exists a $t_4 \geq t_3$ by (1.7),

$$y(t) = z(t) - p(t)f\big(y(\tau(t))\big) \geq \frac{1}{2}z(t) > 0, \tag{3.6}$$

for $t \geq t_4$. We may find a $t_5 \geq t_4$ such that for $t \geq t_5$, we have

$$y(\sigma(t)) \geq \frac{1}{2}z(\sigma(t)) > 0. \tag{3.7}$$

From (3.4) and (3.7), we can obtain the result of

$$z^{(n)}(t) + q(t)h\left(\frac{1}{2}z(\sigma(t))\right) \leq 0, \tag{3.8}$$

for $t \geq t_5$. Since z is defined for $t \geq t_2$, and $z(t) > 0$ with $z^{(n)}(t) \leq 0$ for $t \geq t_2$ and not identically zero, applying directly Lemma 2.2 (second part, since z is positive and increasing), it follows from Lemma 2.2 that

$$y(\sigma(t)) \geq \frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)y^{(n-1)}(\sigma(t)). \tag{3.9}$$

Using (C_1) and (3.7), we find for $t \geq t_6 \geq t_5$,

$$\begin{aligned} h(y(\sigma(t))) &\geq H(y(\sigma(t))) \\ &\geq H\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)z^{(n-1)}(\sigma(t))\right) \\ &\geq KH\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)\right)H\left(z^{(n-1)}(\sigma(t))\right) \\ &\geq K\gamma H\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)\right)z^{(n-1)}(\sigma(t)). \end{aligned} \tag{3.10}$$

It follows from (3.4) and the above inequality that $z^{(n-1)}(t)$ is an eventually positive solution of

$$w'(t) + q(t)K\gamma H\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)\right)w(\sigma(t)) \leq 0. \tag{3.11}$$

By a well-known result (see [14, Theorem 3.1]), the differential equation

$$w'(t) + q(t)K\gamma H\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)\right)w(\sigma(t)) = 0, \quad t \geq t_7 \geq t_6 \tag{3.12}$$

has an eventually positive solution. This contradicts the fact that (1.1) is oscillatory, and the proof is completed. □

Thus, from Theorem 3.1 and [11, Theorem 2.3] (see also [11, Example 3.1]), we can obtain the following corollary.

Corollary 3.2. *If*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q(s)H\left(\frac{1}{2}\frac{\lambda}{(n-1)!}\sigma(t)^{n-1}\right)ds > \frac{1}{eK\gamma}, \tag{3.13}$$

then every bounded solution of (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Theorem 3.3. *Assume that n is odd and (C_2), (C_3) hold. Then, every bounded solution of* (1.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Assume that (1.1) has a bounded nonoscillatory solution y. Without loss of generality, assume that y is eventually positive (the proof is similar when y is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Further, we assume that $y(t)$ does not tend to zero as $t \to \infty$. By (1.1) and (1.7), we have for $t \geq t_1$

$$z^{(n)}(t) = -q(t)h(y(\sigma(t))) \leq 0. \tag{3.14}$$

That is, $z^{(n)}(t) \leq 0$. It follows that $z^{(\alpha)}(t)$ ($\alpha = 0, 1, 2, \ldots, n-1$) is strictly monotone and eventually of constant sign. Since $\lim_{t\to\infty}p(t) = 0$, there exists a $t_2 \geq t_1$, such that for $t \geq t_2$, we have $z(t) > 0$. Since y is bounded, by virtue of (C_2) and (1.7), there is a $t_3 \geq t_2$ such that z is also bounded, for $t \geq t_3$. Because n is odd and z is bounded, by Lemma 2.1, since $l = 0$ (otherwise, $z(t)$ is not bounded), there exists $t_4 \geq t_3$, such that for $t \geq t_4$, we have $(-1)^k z^{(k)}(t) > 0$ $(k = 1, 2, \ldots, n-1)$. In particular, since $z'(t) < 0$ for $t \geq t_4$, z is decreasing. Since z is bounded, we may write $\lim_{t\to\infty} z(t) = L$, $(-\infty < L < \infty)$. Assume that $0 \leq L < \infty$. Let $L > 0$. Then, there exist a constant $c > 0$ and a t_5 with $t_5 \geq t_4$, such that $z(t) > c > 0$ for $t \geq t_5$. Since y is bounded, $\lim_{t\to\infty} p(t) f(y(\tau(t))) = 0$ by (C_1). Therefore, there exists a constant $c_1 > 0$ and a t_6 with $t_6 \geq t_5$, such that $y(t) = z(t) - p(t) f(y(\tau(t))) > c_1 > 0$ for $t \geq t_6$. So, we may find t_7 with $t_7 \geq t_6$, such that $y(\sigma(t)) > c_1 > 0$ for $t \geq t_7$. From (3.14), we have

$$z^{(n)}(t) \leq -q(t) h(c_1) \quad (t \geq t_7). \tag{3.15}$$

If we multiply (3.15) by t^{n-1} and integrate from t_7 to t, then we obtain

$$F(t) - F(t_7) \leq -h(c_1) \int_{t_7}^{t} q(s) s^{n-1} ds, \tag{3.16}$$

where

$$F(t) = \int_{\gamma=2}^{t} (-1)^{\gamma} t^{n-1} z^{(n-\gamma-1)}(t+\gamma) dt. \tag{3.17}$$

Since $(-1)^k z^{(k)}(t) > 0$, for $k = 1, 2, \ldots, n-1$ and $t \geq t_4$, we have $F(t) > 0$ for $t \geq t_7$. From (3.16), we have

$$-F(t_7) \leq -h(c_1) \int_{t_7}^{t} q(s) s^{n-1} ds. \tag{3.18}$$

By (C_3), we obtain

$$-F(t_7) \leq -h(c_1) \int_{t_7}^{t} q(s) s^{n-1} ds = -\infty, \tag{3.19}$$

as $t \to \infty$. This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} z(t) = 0$. Since y is bounded, by virtue of (C_2) and (1.7), we obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} p(t) f(y(\tau(t))) = 0. \tag{3.20}$$

Now, let us consider the case of $y(t) < 0$ for $t \geq t_1$. By (1.1) and (1.7),

$$z^{(n)}(t) = -q(t) h(y(\sigma(t))) \geq 0 \quad (t \geq t_1). \tag{3.21}$$

That is, $z^{(n)}(t) \geq 0$. It follow that $z^{(\alpha)}(t)$ $(\alpha = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $\lim_{t\to\infty} p(t) = 0$, there exists a $t_2 \geq t_1$, such that for $t \geq t_2$, we have $z(t) < 0$. Since $y(t)$ is bounded, by virtue of (C_2) and (1.7), there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded, for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then, $x^{(n)}(t) = -z^{(n)}(t)$. Therefore, $x(t) > 0$ and $x^{(n)}(t) \leq 0$ for $t \geq t_3$. From this, we observe that $x(t)$ is bounded. Because n is odd and x is bounded, by Lemma 2.1, since $l = 0$ (otherwise, x is not bounded), there exists a $t_4 \geq t_3$, such that $(-1)^k x^{(k)}(t) > 0$ for $k = 1, 2, \ldots, n-1$ and $t \geq t_4$. That is, $(-1)^k z^{(k)}(t) < 0$ for $k = 1, 2, \ldots, n-1$ and $t \geq t_4$. In particular, for $t \geq t_4$, we have $z'(t) > 0$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L, (-\infty < L \leq 0)$. As in the proof of $y(t) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts our assumption. Hence, the proof is completed. □

Example 3.4. We consider difference equation of the form

$$\left[y(t) + \frac{1}{t}\sin(t)\left(y^3(t-2) + y(t-2)\right)\right]^{(4)} + \frac{1}{t^2}y^3(t-3) = 0, \tag{3.22}$$

where $n = 4$, $\tau(t) = t-2$, $p(t) = (1/t)\sin(t)$, $q(t) = 1/t^2$, $\sigma(t) = t-3$, $h(y) = y^3$, and $f(y) = y^3+y$. By taking $H(u) = u$,

$$\liminf_{t\to\infty} \int_{t-3}^{t} \frac{1}{s^2}\frac{1}{2}\frac{1}{3!}\left(\frac{s-3}{2^3}\right)^3 ds > \frac{1}{e}, \tag{3.23}$$

we check that all the conditions of Theorem 3.1 are satisfied and that every bounded solution of (3.22) oscillates or tends to zero at infinity.

Example 3.5. We consider difference equation of the form

$$\left[y(t) + \cos t e^{-5t^2}\left[y^5(t-5) + 2y(t-5)\right]\right]^{(3)} + t^2 y^2(t-3) = 0, \quad t \geq 2, \tag{3.24}$$

where $n = 3$, $q(t) = t^2$, $\sigma(t) = t-3$, $\tau(t) = t-5$, and $p(t) = \cos t e^{-5t^2}$, $f(y) = y^5 - 2y$, $h(y) = y^2$. Hence, we have

$$\begin{gathered}\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{e^{5t^2}} \cos t = 0, \\ \int_{t_0}^{\infty} s^{n-1} q(s) ds = \int_{t_0}^{\infty} s^4 ds = \infty.\end{gathered} \tag{3.25}$$

Since Conditions (C_2) and (C_3) of Theorem 3.3 are satisfied, every bounded solution of (3.24) oscillates or tends to zero at infinity.

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