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Research Article

Oscillatory Nonautonomous Lucas Sequences

José M. Ferreira[1] and Sandra Pinelas[2]

[1] *Department of Mathematics, Instituto Superior Técnico, Avenue Rovisco Pais, 1049-001 Lisboa, Portugal*
[2] *Department of Mathematics, Universidade dos Açores, R. Mãe de Deus, 9500-321 Ponta Delgada, Portugal*

Correspondence should be addressed to Sandra Pinelas, spinelas@uac.pt

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The oscillatory behavior of the solutions of the second-order linear nonautonomous equation $x(n+1) = a(n)x(n) - b(n)x(n-1)$, $n \in \mathbb{N}_0$, where $a, b : \mathbb{N}_0 \to \mathbb{R}$, is studied. Under the assumption that the sequence $b(n)$ dominates somehow $a(n)$, the amplitude of the oscillations and the asymptotic behavior of its solutions are also analized.

1. Introduction

The aim of this note is to study the oscillatory behavior of the equation

$$x(n+1) = a(n)x(n) - b(n)x(n-1), \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b : \mathbb{N}_0 \to \mathbb{R}$. Equation (1.1) is the nonautonomous case of the so-called Lucas sequences, which are obtained through the recursive law:

$$x(n+1) = ax(n) - bx(n-1), \quad n \in \mathbb{N}_0, \tag{1.2}$$

which corresponds to have in (1.1) both sequences $a(n)$ and $b(n)$ constant and equal, respectively, to real numbers a and b. Lucas sequences are well known in number theory as an extension of the Fibonacci sequence (see [1, Chapter 2, Section IV]).

Several particular cases of (1.1) are considered in literature. This is the case of equation (see [2, Chapter 6])

$$p(n)x(n+1) + p(n-1)x(n-1) = q(n)x(n), \quad n \in \mathbb{N}_0, \tag{1.3}$$

with $q(n)$ being arbitrary and $p(n)$ being positive, corresponding to have in (1.1)

$$a(n) = \frac{q(n)}{p(n)}, \qquad b(n) = \frac{p(n-1)}{p(n)}. \tag{1.4}$$

The equation (see [3, 4] and references therein)

$$\Delta^2 x(n-1) + P(n)x(n) = 0, \quad n \in \mathbb{N}_0 \tag{1.5}$$

is also a particular case of (1.1). In fact, (1.5) can be written as

$$x(n+1) = (2 - P(n))x(n) - x(n-1), \quad n \in \mathbb{N}_0, \tag{1.6}$$

corresponding to have in (1.1), $a(n) = 2 - P(n)$ and $b(n)$ constant and equal to 1. Other particular cases of (1.1) will be referred along the text.

As usual, we will say that a solution $x(n)$ of (1.1) is nonoscillatory if it is either eventually positive or eventually negative. Otherwise $x(n)$ is called oscillatory. When all solutions of (1.1) are oscillatory, (1.1) is said oscillatory.

It is well known that (1.2) is oscillatory if and only if the polynomial $z^2 - az + b$ has no positive real roots. This is equivalent to have one of the following two cases:

$$\text{(i)}\ a \le 0, \quad b \ge 0; \qquad \text{(ii)}\ a > 0, \quad b > \frac{a^2}{4}. \tag{1.7}$$

In order to obtain (1.1) oscillatory, this specific case seems to justify that the sequence $b(n)$ be assumed positive. However, for $b(n) > 0$ for every n, notice that if one has either $a(n) \le 0$ eventually or $a(n)$ oscillatory, then $x(n)$ cannot be neither eventually positive nor eventually negative. That is, in such circumstances the equation is oscillatory. So hereafter we will assume that $a(n)$ and $b(n)$ are both positive sequences.

Through a direct manipulation of the terms of the solutions of (1.1), we show in the following sections a few results which seem, as far as we know, uncommon in literature. We state that if $b(n)$ dominates somehow the sequence $a(n)$, then all solutions of the equation exhibit a specific oscillatory and asymptotic behavior. Oscillations through the existence of periodic solutions will be studied in a sequel, but they cannot then happen under a similar relationship between $a(n)$ and $b(n)$.

2. Oscillatory Behavior

The oscillatory behavior of (1.1) can be stated through the application of some results already existent in literature. This is the case of the following theorem based upon a result in [5].

Theorem 2.1. *If*

$$\frac{b(n)}{a(n-1)a(n)} > \frac{1}{4} \tag{2.1}$$

holds eventually and

$$\limsup_{n\to\infty}\frac{b(n)}{a(n-1)a(n)} > \frac{1}{4}, \tag{2.2}$$

then (1.1) *is oscillatory.*

Proof. By making the change of variable

$$y(n) = \frac{2^n x(n)}{\prod_{k=1}^{n-1} a(k)}, \tag{2.3}$$

one can see that (1.1) is equivalent to

$$\Delta^2 y(n-1) = \left(1 - \frac{4b(n)}{a(n-1)a(n)}\right) y(n-1), \quad n \in \mathbb{N}_0, \tag{2.4}$$

where by Δ we mean the usual difference operator.

Therefore applying [5, Theorem 1.3.1], one can conclude under (2.1) and (2.2) that every solution of (2.4) is oscillatory. Then, as all terms of the sequence $a(n)$ are positive, also $x(n)$ is oscillatory. □

Remark 2.2. In [6] the equation

$$x(t) = \sum_{j=1}^{p} \alpha_j(t) x(t-j), \tag{2.5}$$

where, for $j = 1, \dots, p$, $\alpha_j(t)$ are real continuous functions on $[0, +\infty[$, is studied. Through [6, Theorem 10], this equation is oscillatory if

$$\beta_j \leqslant \alpha_j(t) \leqslant \gamma_j \tag{2.6}$$

for $j = 1, \dots, p$, and

$$1 = \sum_{j=1}^{p} \gamma_j \exp(-\lambda j) \tag{2.7}$$

has no real roots. Since (1.1) is the discrete case of (2.5) with $p = 2$, if $a(n)$ and $b(n)$ are both bounded sequences, one can apply [6, Theorem 10]. As a matter of fact, if $a(n) \leq \alpha$ and $-b(n) \leq -\beta$, for every n, then through (1.7) one can conclude that (1.1) is oscillatory if $\beta > \alpha^2/4$. This condition implies (2.1) and (2.2).

Remark 2.3. In literature is largely considered the equation

$$x(n+1) - x(n) + b(n)x(n-\tau) = 0, \quad n \in \mathbb{N}_0, \tag{2.8}$$

(see [7] and references therein). For $\tau = 1$ one obtains the case of (1.1) where $a(n)$ is constant and equal to 1. In [8, Theorems 7.5.2 and 7.5.3] is stated that (1.1) in such case is oscillatory provided that one of the following two conditions is satisfied:

$$\liminf_{n\to\infty} b(n) > \frac{1}{4}, \qquad \limsup_{n\to\infty}(b(n) + b(n+1)) > 1. \tag{2.9}$$

Both these conditions imply the correspondent conditions (2.1) and (2.2).

Remark 2.4. In [2, Theorem 6.5.3] is stated that the particular case of (1.1) given by (1.3), with $q(n)$ also positive, is oscillattory if there exists $\varepsilon > 0$ such that

$$q(n)q(n+1) \leq (4-\epsilon)p^2(n) \tag{2.10}$$

holds eventually. On the other hand, one easily sees that (2.1) and (2.2) are, respectively, equivalent to have eventually

$$q(n-1)q(n) < 4p^2(n-1), \qquad \limsup_{n\to\infty}\frac{p^2(n-1)}{q(n-1)q(n)} > \frac{1}{4}. \tag{2.11}$$

Notice that (2.10) implies (2.11).

Remark 2.5. Through Theorem 2.1 one can state that (1.5) is oscillatory, if $P(n) < 2$, for every n,

$$\frac{P(n-1)P(n)}{2} < P(n-1) + P(n), \tag{2.12}$$

eventually, and

$$\liminf_{n\to\infty}\left(\frac{P(n-1)P(n)}{2} - P(n-1) - P(n)\right) < 0. \tag{2.13}$$

These conditions, with respect to those obtained in [3, 4], seem to be of different kind.

3. Oscillations

The results of the preceding section state that all solutions of (1.1) oscillate. In this section we will characterize the oscillations of such solutions.

We first notice that if two of three consecutive terms $x(k-1)$, $x(k)$, and $x(k+1)$ are zero, then necessarily $x(n) = 0$ for every $n > k+1$. This means that if the sequence $x(n)$ has two consecutive or alternate zeros, then it is eventually null. In the following we will exclude this trivial situation.

Lemma 3.1. (a) *For every n one has*

(i) $x(n) > 0 \wedge x(n+2) > 0 \Rightarrow x(n+1) > 0$,

(ii) $x(n) < 0 \wedge x(n+2) < 0 \Rightarrow x(n+1) < 0$.

(b) *If there exists a positive integer N such that*

$$b(n) > a(n)a(n-1) \quad \text{for every } n \geq N, \tag{3.1}$$

then

(iii) $x(n) > 0 \wedge x(n+2) > 0 \Rightarrow x(n+3) < 0$,

(iv) $x(n) < 0 \wedge x(n+3) < 0 \Rightarrow x(n+2) > 0$

for every $n \geq N$.

Proof. (a) From (1.1) one obtains

$$x(n+2) = a(n+1)x(n+1) - b(n+1)x(n). \tag{3.2}$$

So

$$x(n) > 0, \qquad x(n+2) > 0 \tag{3.3}$$

implies $x(n+1) > 0$, and if

$$x(n) < 0, \qquad x(n+2) < 0, \tag{3.4}$$

then $x(n+1) < 0$.

(b) Notice that $x(n)$ can be written as

$$x(n) = \frac{a(n+1)}{b(n+1)}x(n+1) - \frac{1}{b(n+1)}x(n+2). \tag{3.5}$$

Therefore by recurrence

$$x(n) = \frac{a(n+1)}{b(n+1)}\left(\frac{a(n+2)}{b(n+2)}x(n+2) - \frac{1}{b(n+2)}x(n+3)\right) - \frac{1}{b(n+1)}x(n+2), \tag{3.6}$$

and through a rearrangement

$$x(n) = \frac{1}{b(n+1)}\left(\frac{a(n+1)a(n+2)}{b(n+2)} - 1\right)x(n+2) - \frac{a(n+1)}{b(n+1)b(n+2)}x(n+3). \tag{3.7}$$

Thus if $x(n) > 0$, one has

$$x(n+3) < \left(a(n+2) - \frac{b(n+2)}{a(n+1)}\right)x(n+2), \tag{3.8}$$

and letting $n \geq N$ and $x(n+2) > 0$ one concludes that $x(n+3) < 0$, which shows (iii).

Analogously, $x(n) < 0$ and $n \geq N$ imply

$$\frac{1}{b(n+1)}\left(1 - \frac{a(n+1)a(n+2)}{b(n+2)}\right)x(n+2) > -\frac{a(n+1)}{b(n+1)b(n+2)}x(n+3), \tag{3.9}$$

and consequently one has $x(n+2) > 0$ if $x(n+3) < 0$. □

We notice that assumption (3.1) implies obviously (2.1) and (2.2), which means that under (3.1), equation (1.1) is oscillatory. However, under (3.1) one can further state some interesting characteristics regarding the oscillations of the solutions of (1.1).

For that purpose, let $k \in \mathbb{N}$. Denote by m_k^+ the smallest number of consecutive terms of $x(n)$, which, for $n > k$, are positive; by M_k^+ we mean the largest number of consecutive terms of $x(n)$, which, for $n > k$, are positive. Analogously, let m_k^- and M_k^- be, respectively, the smallest and largest numbers of consecutive terms of $x(n)$, which, for $n > k$, are negative.

Theorem 3.2. *Under* (3.1), *for every $k > N$ one has $m_k^-, m_k^+ \geq 2$ and $M_k^+, M_k^- \leq 3$.*

Proof. Assuming that there exists a $n > k$ such that

$$x(n) > 0, \qquad x(n+1) < 0, \qquad x(n+2) > 0, \tag{3.10}$$

one contradicts (i) of Lemma 3.1. In the same way,

$$x(n) < 0, \qquad x(n+1) > 0, \qquad x(n+2) < 0 \tag{3.11}$$

is in contradiction with Lemma 3.1(ii).

Moreover if for some $n > k$ one has $x(n) = 0$, then both $x(n+1)$ and $x(n+2)$ are nonzero real numbers with equal sign and contrary to the sign of $x(n-1)$ and $x(n-2)$.

Thus $m_k^-, m_k^+ \geq 2$.

Suppose now that

$$x(n) > 0, \qquad x(n+1) > 0, \qquad x(n+2) > 0, \qquad x(n+3) > 0. \tag{3.12}$$

This is a situation which is in contradiction with (iii) of Lemma 3.1.

Analogously if

$$x(n) < 0, \qquad x(n+1) < 0, \qquad x(n+2) < 0, \qquad x(n+3) < 0, \tag{3.13}$$

we contradict Lemma 3.1(iv).

Hence $M_k^+, M_k^- \leq 3$. □

From this theorem one can conclude immediately the following statement.

Theorem 3.3. *Under* (3.1) *all solutions of* (1.1) *are oscillatory with at least two and no more than three consecutive terms of the same sign.*

Example 3.4. The particular case of (1.1), where $b(n)$ is constant and equal to $b > 0$, gives the equation

$$x(n+1) = a(n)x(n) - bx(n-1), \quad n \in \mathbb{N}_0. \tag{3.14}$$

Notice that by Theorem 2.1 this equation is oscillatory if $a(n)a(n-1) < 4b$ eventually and $\liminf_{n\to\infty} a(n)a(n-1) < 4b$. But if $a(n)a(n-1) < b$ eventually, then (3.14) is oscillatory having at least two and no more than three consecutive terms of the same sign. This is the case when $b \geq 4$ and

$$a(n) = \frac{\sqrt{n+1} + \sqrt{n-1}}{\sqrt{n}}, \tag{3.15}$$

since $a(n) \to 2$, as $n \to \infty$, in manner that $a(n) < 2$, for every n.

Example 3.5. Let now (1.1) with $a(n)$ constant and equal to $a > 0$:

$$x(n+1) = ax(n) - b(n)x(n-1), \quad n \in \mathbb{N}_0. \tag{3.16}$$

If $b(n) > a^2$ eventually, then all solutions of (3.16) oscillate with at least two and no more than three consecutive terms of the same sign. This is the case when $b(n) = n$ and a is any positive real number. However, the equation is oscillatory more generally if $b(n) > a^2/4$ and $\limsup_{n\to\infty} b(n) > a^2/4$.

Example 3.6. All solutions of (1.5) have this same type of oscillation, whenever $P(n) < 2$ for every n and eventually

$$(2 - P(n-1))(2 - P(n)) < 1. \tag{3.17}$$

This occurs, for example, if $1 < P(n) < 2$, for every n.

These examples show that the gain of Theorem 3.3 is not in concluding the oscillatory behavior of (1.1), but in showing the oscillation type that all solutions of the equation have.

4. Oscillation Amplitude

In this section we complement the oscillatory properties of $x(n)$ with the study of the amplitude with respect to the real line of its oscillations.

By a semicycle of $x(n)$ we will mean any maximal set of nonzero consecutive terms with the same sign. If

$$S = \{x(n), \ldots, x(n+k)\} \tag{4.1}$$

is a semicycle of $x(n)$, to the value

$$|S| = \max\{|x(n)|, \ldots, |x(n+k)|\} \tag{4.2}$$

we call the amplitude of S.

Under (3.1) we have shown that $x(n)$ is constituted by a sequence of semicycles alternately composed by positive and negative terms, each one having at least two and no more than three elements. Between semicycles is possible that might exist a null term of $x(n)$, but by Lemma 3.1(iii), a semicycle of three elements cannot precede a zero of $x(n)$.

If one denotes that sequence of semicycles is by S_k $(k = 1, 2, \ldots)$ in the following, we will analize the properties of the sequence $|S_k|$ of its amplitudes. For that purpose we will assume that for some integer positive N, one has

$$a(n+1)a(n) + 1 < b(n+1) \tag{4.3}$$

for every $n > N$. This condition implies obviously (3.1).

Lemma 4.1. *Assume that* (4.3) *holds and that for every $n > N$*

$$b(n) > a(n). \tag{4.4}$$

If

$$x(n-1) > 0, \qquad x(n) > 0, \qquad x(n+1) > 0 \tag{4.5}$$

for $n > N$, then $x(n+2) < 0$ and $x(n+3) < 0$ such that

$$\max\{|x(n+2)|, |x(n+3)|\} > \max\{x(n-1), x(n), x(n+1)\}. \tag{4.6}$$

Proof. As $x(n+1) > 0$ we have

$$a(n)x(n) - b(n)x(n-1) > 0, \tag{4.7}$$

and so, by (4.4),

$$x(n) > \frac{b(n)}{a(n)}x(n-1) > x(n-1). \tag{4.8}$$

Now we will prove that $-x(n+2) > x(n)$ and $-x(n+3) > x(n+1)$. In fact, we have

$$\begin{aligned} x(n+2) &= a(n+1)x(n+1) - b(n+1)x(n) \\ &= a(n+1)(a(n)x(n) - b(n)x(n-1)) - b(n+1)x(n) \\ &= (a(n+1)a(n) - b(n+1))x(n) - a(n+1)b(n)x(n-1) \\ &< (a(n+1)a(n) - b(n+1))x(n) \end{aligned} \tag{4.9}$$

and by (4.3) we obtain

$$x(n+2) < -x(n), \tag{4.10}$$

for every $n > N$.

In the same way one can conclude that

$$x(n+3) < (a(n+2)a(n+1) - b(n+2))x(n+1) < -x(n+1). \tag{4.11}$$

This proves the lemma. □

Lemma 4.2. *Assume that* (4.3) *holds and that for every $n > N$*

$$a(n+1)b(n) > 1. \tag{4.12}$$

If

$$x(n-1) > 0, \qquad x(n) > 0, \qquad x(n+1) < 0, \qquad x(n+2) < 0 \tag{4.13}$$

for $n > N$, then

$$\max\{|x(n+1)|, |x(n+2)|\} > \max\{x(n-1), x(n)\}. \tag{4.14}$$

Proof. Noticing that

$$\begin{aligned} x(n+2) &= a(n+1)(a(n)x(n) - b(n)x(n-1)) - b(n+1)x(n) \\ &= (a(n+1)a(n) - b(n+1))x(n) - a(n+1)b(n)x(n-1), \end{aligned} \tag{4.15}$$

by (4.3) one has

$$\begin{gathered} x(n+2) < (a(n+1)a(n) - b(n+1))x(n) < -x(n), \\ x(n+2) < -a(n+1)b(n)x(n-1) < -x(n-1), \end{gathered} \tag{4.16}$$

which proves the lemma. □

Lemma 4.3. *Assume that* (4.3), (4.4), *and* (4.12) *hold. If*

$$x(n-1) > 0, \qquad x(n) > 0, \qquad x(n+1) = 0, \qquad x(n+2) < 0, \qquad x(n+3) < 0 \tag{4.17}$$

for $n > N$, *then*

$$\max\{|x(n+2)|, |x(n+3)|\} > \max\{x(n-1), x(n)\}. \tag{4.18}$$

Proof. First take into account that $x(n+1) = 0$ implies

$$a(n)x(n) = b(n)x(n-1) \Longleftrightarrow x(n) = \frac{b(n)}{a(n)}x(n-1) > x(n-1). \tag{4.19}$$

Notice now that

$$x(n+2) = -b(n+1)x(n) < -x(n) \tag{4.20}$$

since (4.3) implies $b(n) > 1$ for every $n > N$. On the other hand

$$\begin{aligned} x(n+3) &= a(n+2)x(n+2) \\ &= -a(n+2)b(n+1)x(n) \\ &< -x(n). \end{aligned} \tag{4.21}$$

This achieves the proof. □

Remark 4.4. Observe that $u(n) = -x(n)$ is also a solution of (1.1) which of course has the same oscillatory characteristics as $x(n)$. This fact enables to conclude similar lemmas by simple change of sign.

Theorem 4.5. *Assuming that* (4.3), (4.4), *and* (4.12) *hold, then* $|S_k|$ *is eventually increasing.*

Proof. Let S_k and S_{k+1} be two consecutive semicycles. Several cases can be performed.

(1) Assume that

$$\begin{aligned} S_k &= \{x(n-1) > 0,\ x(n) > 0,\ x(n+1) > 0\}, \\ S_{k+1} &= \{x(n+2) < 0,\ x(n+3) < 0\}. \end{aligned} \tag{4.22}$$

Then by Lemma 4.1 one has $|S_{k+1}| > |S_k|$.

By Remark 4.4 the same holds if

$$\begin{aligned} S_k &= \{x(n-1) < 0,\ x(n) < 0,\ x(n+1) < 0\}, \\ S_{k+1} &= \{x(n+2) > 0,\ x(n+3) > 0\}. \end{aligned} \tag{4.23}$$

(2) If

$$S_k = \{x(n-1) > 0,\ x(n) > 0,\ x(n+1) > 0\},$$
$$S_{k+1} = \{x(n+2) < 0,\ x(n+3) < 0,\ x(n+4) < 0\}, \tag{4.24}$$

then still by (4.6) of Lemma 4.1 one obtains

$$|S_{k+1}| \geq \max\{|x(n+3)|, |x(n+2)|\} > |S_k|. \tag{4.25}$$

Again Remark 4.4 provides the same conclusion if

$$S_k = \{x(n-1) < 0,\ x(n) < 0,\ x(n+1) < 0\},$$
$$S_{k+1} = \{x(n+2) > 0,\ x(n+3) > 0,\ x(n+4) > 0\}. \tag{4.26}$$

(3) Let now

$$S_k = \{x(n-1) > 0,\ x(n) > 0\},$$
$$S_{k+1} = \{x(n+1) < 0,\ x(n+2) < 0\}. \tag{4.27}$$

The inequality (4.14) of Lemma 4.2 expresses directly that $|S_{k+1}| > |S_k|$. The use of Remark 4.4 enables to conclude that the same is verified when

$$S_k = \{x(n-1) < 0,\ x(n) < 0\},$$
$$S_{k+1} = \{x(n+1) > 0,\ x(n+2) > 0\}. \tag{4.28}$$

(4) If

$$S_k = \{x(n-1) > 0,\ x(n) > 0\},$$
$$S_{k+1} = \{x(n+1) < 0,\ x(n+2) < 0,\ x(n+3) < 0\}, \tag{4.29}$$

then by (4.14) one can obtain

$$|S_{k+1}| \geq \max\{|x(n+1)|, |x(n+2)|\} > |S_k|. \tag{4.30}$$

By Remark 4.4 one obtains the same for

$$S_k = \{x(n-1) < 0,\ x(n) < 0\},$$
$$S_{k+1} = \{x(n+1) > 0,\ x(n+2) > 0,\ x(n+3) > 0\}. \tag{4.31}$$

(5) In the case where

$$\begin{aligned} S_k &= \{x(n-1) > 0,\ x(n) > 0\}, \\ S_{k+1} &= \{x(n+2) < 0,\ x(n+3) < 0\} \end{aligned} \tag{4.32}$$

with $x(n+1) = 0$, one obtains $|S_{k+1}| > |S_k|$ directly from the inequality (4.18). The case where

$$\begin{aligned} S_k &= \{x(n-1) < 0,\ x(n) < 0\}, \\ S_{k+1} &= \{x(n+2) > 0,\ x(n+3) > 0\} \end{aligned} \tag{4.33}$$

can be obtained by the use of Remark 4.4.

(6) Finally when $x(n+1) = 0$ and

$$\begin{aligned} S_k &= \{x(n-1) > 0,\ x(n) > 0\}, \\ S_{k+1} &= \{x(n+2) < 0,\ x(n+3) < 0,\ x(n+4) < 0\}, \end{aligned} \tag{4.34}$$

still through the inequality (4.18) of Lemma 4.3 we conclude that

$$|S_{k+1}| \geq \max\{|x(n+3)|, |x(n+2)|\} > |S_k|. \tag{4.35}$$

Again Remark 4.4 enables us to conclude the same for the case

$$\begin{aligned} S_k &= \{x(n-1) < 0,\ x(n) < 0\}, \\ S_{k+1} &= \{x(n+2) > 0,\ x(n+3) > 0,\ x(n+4) > 0\}. \end{aligned} \tag{4.36}$$

This completes the proof of the theorem. □

Through Theorem 4.5 one can state the following asymptotic behavior of $x(n)$.

Theorem 4.6. *Assume* (4.3) *and* (4.12). *If*

$$\limsup_{n\to\infty} \frac{1 + a(n)}{b(n)} < 1, \tag{4.37}$$

then

$$\limsup\ x(n) = +\infty, \qquad \liminf\ x(n) = -\infty. \tag{4.38}$$

Proof. Notice first that since

$$\frac{a(n)}{b(n)} < \frac{1 + a(n)}{b(n)}, \tag{4.39}$$

equation (4.37) implies (4.4).

Assuming that $x(n)$ is bounded, let

$$K = \sup\{|x(n)| : n \in \mathbb{N}\}. \tag{4.40}$$

From (1.1) we have

$$b(n)|x(n-1)| \le |x(n+1)| + a(n)|x(n)|, \tag{4.41}$$

and so

$$|x(n-1)| \le \frac{1 + a(n)}{b(n)} K \tag{4.42}$$

for every n.

On the other hand, taking arbitrarily $0 < \delta < 1$, there exists an integer $N > 0$ such that, for every $n > N$,

$$\frac{1 + a(n)}{b(n)} < \delta. \tag{4.43}$$

Thus by consequence

$$|x(n-1)| \le \delta K < K \tag{4.44}$$

for every $n > N$. This means that $K = \max\{|x(n)| : n = 1, \ldots, N-1\}$ which is in contradiction with Theorem 4.5.

Thus $x(n)$ is not a bounded sequence and again by Theorem 4.5 one can conclude the existence of two increasing sequences $n_\ell, n_k \in \mathbb{N}$ such that

$$x(n_\ell) \longrightarrow +\infty, \qquad x(n_k) \longrightarrow -\infty \tag{4.45}$$

as $k, \ell \to +\infty$, which achieves the proof. □

Example 4.7. Regarding (1.2), if $a > 0$ and

$$b > \max\left\{a + 1, a^2 + 1, \frac{1}{a}\right\}, \tag{4.46}$$

then all its solutions exhibit the properties stated in Theorems 3.3 and 4.6. For (2.8) the same holds whenever

$$\liminf_{n\to\infty} b(n) > 2. \tag{4.47}$$

With respect to (1.3) the same behavior is verified for all its solutions provided that one has, eventually,

$$\begin{gathered} q(n)q(n+1) + p(n)p(n+1) < p(n-1)p(n+1), \\ p(n)p(n+1) < q(n+1)p(n-1), \\ \limsup_{n\to\infty} \frac{p(n)q(n)}{p(n-1)} < 1. \end{gathered} \tag{4.48}$$

Example 4.8. Let us consider the Hermite polynomial equation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x) \tag{4.49}$$

with

$$H_{-1}(x) = 1, \qquad H_0(x) = x. \tag{4.50}$$

For every $x > 0$, all the assumptions (4.3), (4.12), and (4.37) are satisfied. Then we can complement the oscillatory result stated in [5, Theorem 1.3.5], by saying that $H_n(x)$ is an oscillatory sequence with at least two and no more than three consecutive terms of the same sign, admitting one subsequence going to $+\infty$ and another tending to $-\infty$. In a different framework the same situation can be seen for other kinds of orthogonal polynomials in [9, 10].

Remark 4.9. Finally we observe that for having the asymptotic behavior stated in Theorem 4.6, we are implicitly asking that the sequence $b(n)$ dominates in some manner $a(n)$. That kind of domination is not verified in (1.5). As a matter of fact, the required assumption that $P(n) < 2$, for every n, does not allow that the corresponding conditions (4.3) and (4.37) be verified.

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