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Research Article

Forced Oscillations of Half-Linear Elliptic Equations via Picone-Type Inequality

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Picone-type inequality is established for a class of half-linear elliptic equations with forcing term, and oscillation results are derived on the basis of the Picone-type inequality. Our approach is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for ordinary half-linear differential equations.

1. Introduction

The p-Laplacian $\Delta_p v = \nabla \cdot (|\nabla v|^{p-2}\nabla v)$ arises from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaciology, and petroleum extraction (cf. Díaz [1]). It is important to study the qualitative behavior (e.g., oscillatory behavior) of solutions of p-Laplace equations with superlinear terms and forcing terms.

Forced oscillations of superlinear elliptic equations of the form

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + C(x)|v|^{\beta-1}v = f(x) \quad (\beta > \alpha > 0) \tag{1.1}$$

were studied by Jaroš et al. [2], and the more general quasilinear elliptic equation with first-order term

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + (\alpha+1)B(x) \cdot \left(|\nabla v|^{\alpha-1}\nabla v\right) + C(x)|v|^{\beta-1}v = f(x) \tag{1.2}$$

was investigated by Yoshida [3], where the dot $(\cdot)$ denotes the scalar product. There appears to be no known oscillation results for the case where $\alpha = \beta$. The techniques used in [2, 3] are not applicable to the case where $\alpha = \beta$.

The purpose of this paper is to establish a Picone-type inequality for the half-linear elliptic equation with the forcing term:

$$P[v] := \nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + (\alpha+1)B(x) \cdot \left(|\nabla v|^{\alpha-1}\nabla v\right) + C(x)|v|^{\alpha-1}v = f(x), \tag{1.3}$$

and to derive oscillation results on the basis of the Picone-type inequality. The approach used here is motivated by the paper [4] in which oscillation criteria for second-order nonlinear ordinary differential equations are studied. Our method is an adaptation of that used in [5]. Since the proofs of Theorems 2.2–3.3 are quite similar to those of [5, Theorems 1–4], we will omit them.

2. Picone-Type Inequality

Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that $\alpha > 0$ is a constant, $A(x) \in C(\overline{G};(0,\infty))$, $B(x) \in C(\overline{G};\mathbb{R}^n)$, $C(x) \in C(\overline{G};\mathbb{R})$, and $f(x) \in C(\overline{G};\mathbb{R})$.

The domain $\mathfrak{D}_P(G)$ of P is defined to be the set of all functions $v \in C^1(\overline{G};\mathbb{R})$ with the property that $A(x)|\nabla v|^{\alpha-1}\nabla v \in C^1(G;\mathbb{R}^n) \cap C(\overline{G};\mathbb{R}^n)$.

Lemma 2.1. *If $v \in \mathfrak{D}_P(G)$ and $|v| \geq k_0$ for some $k_0 > 0$, then the following Picone-type inequality holds for any $u \in C^1(G;\mathbb{R})$:*

$$\begin{aligned}
&-\nabla \cdot \left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)}\right) \\
&\quad \geq -A(x)\left|\nabla u - \frac{u}{A(x)}B(x)\right|^{\alpha+1} + \left(C(x) - k_0^{-\alpha}\left|f(x)\right|\right)|u|^{\alpha+1} \\
&\qquad + A(x)\left[\left|\nabla u - \frac{u}{A(x)}B(x)\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left(\nabla u - \frac{u}{A(x)}B(x)\right)\cdot\Phi\left(\frac{u}{v}\nabla v\right)\right] \\
&\qquad - \frac{u\varphi(u)}{\varphi(v)}\left(P[v] - f(x)\right),
\end{aligned} \tag{2.1}$$

where $\varphi(s) = |s|^{\alpha-1}s$ $(s \in \mathbb{R})$ and $\Phi(\xi) = |\xi|^{\alpha-1}\xi$ $(\xi \in \mathbb{R}^n)$.

Proof. The following Picone identity holds for any $u \in C^1(G;\mathbb{R})$:

$$
\begin{aligned}
&-\nabla\cdot\left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)}\right)\\
&\quad= -A(x)\left|\nabla u-\frac{u}{A(x)}B(x)\right|^{\alpha+1}+C(x)|u|^{\alpha+1}\\
&\qquad+A(x)\left[\left|\nabla u-\frac{u}{A(x)}B(x)\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{u}{A(x)}B(x)\right)\cdot\Phi\left(\frac{u}{v}\nabla v\right)\right]\\
&\qquad-\frac{u\varphi(u)}{\varphi(v)}\left(P[v]-f(x)\right)-\frac{u\varphi(u)}{\varphi(v)}f(x)
\end{aligned}
\tag{2.2}
$$

(see, e.g., Yoshida [6, Theorem 1.1]). Since $|v| \ge k_0$, we obtain

$$\left|\varphi(v)\right| = |v|^{\alpha} \ge k_0^{\alpha}, \tag{2.3}$$

and therefore

$$\left|\frac{u\varphi(u)}{\varphi(v)}f(x)\right| \le |u|^{\alpha+1}k_0^{-\alpha}\left|f(x)\right|. \tag{2.4}$$

Combining (2.2) with (2.4) yields the desired inequality (2.1). □

Theorem 2.2. *Let $k_0 > 0$ be a constant. Assume that there exists a nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and*

$$M_G[u] := \int_G\left[A(x)\left|\nabla u-\frac{u}{A(x)}B(x)\right|^{\alpha+1}-\left(C(x)-k_0^{-\alpha}\left|f(x)\right|\right)|u|^{\alpha+1}\right]dx \le 0. \tag{2.5}$$

Then for every solution $v \in \mathfrak{D}_P(G)$ of (1.3)*, either v has a zero on $\overline{G}$ or*

$$|v(x_0)| < k_0 \quad \text{for some } x_0 \in G. \tag{2.6}$$

3. Oscillation Results

In this section we investigate forced oscillations of (1.3) in an exterior domain Ω in $\mathbb{R}^n$, that is, $\Omega \supset E_{r_0}$ for some $r_0 > 0$, where

$$E_r = \{x \in \mathbb{R}^n;\ |x| \ge r\} \quad (r > 0). \tag{3.1}$$

It is assumed that $\alpha > 0$ is a constant, $A(x) \in C(\Omega;(0,\infty))$, $B(x) \in C(\Omega;\mathbb{R}^n)$, $C(x) \in C(\Omega;\mathbb{R})$, and $f(x) \in C(\Omega;\mathbb{R})$.

The domain $\mathfrak{D}_P(\Omega)$ of P is defined to be the set of all functions $v \in C^1(\Omega;\mathbb{R})$ with the property that $A(x)|\nabla v|^{\alpha-1}\nabla v \in C^1(\Omega;\mathbb{R}^n)$.

A solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap E_r. \tag{3.2}$$

Theorem 3.1. *Assume that for any $k_0 > 0$ and any $r > r_0$ there exists a bounded domain $G \subset E_r$ such that* (2.5) *holds for some nontrivial $u \in C^1(\overline{G};\mathbb{R})$ satisfying $u = 0$ on ∂G. Then for every solution $v \in \mathfrak{D}_P(\Omega)$ of* (1.3)*, either v is oscillatory in Ω or*

$$\liminf_{|x|\to\infty} |v(x)| = 0. \tag{3.3}$$

Theorem 3.2. *Assume that for any $k_0 > 0$ and any $r > r_0$ there exists a bounded domain $G \subset E_r$ such that*

$$\widetilde{M}_G[u] := \int_G \left[2^\alpha A(x)|\nabla u|^{\alpha+1} - \left(C(x) - 2^\alpha A(x)^{-\alpha}|B(x)|^{\alpha+1} - k_0^{-\alpha}\left|f(x)\right|\right)|u|^{\alpha+1}\right]dx \le 0 \tag{3.4}$$

holds for some nontrivial $u \in C^1(\overline{G};\mathbb{R})$ satisfying $u = 0$ on ∂G. Then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3)*, either v is oscillatory in Ω or satisfies* (3.3).

Let $\overline{\{Q(x)\}}(r)$ denote the spherical mean of $Q(x)$ over the sphere $S_r = \{x \in \mathbb{R}^n;\ |x| = r\}$. We define $p(r)$ and $q_{k_0}(r)$ by

$$\begin{aligned} p(r) &= \overline{\{2^\alpha A(x)\}}(r), \\ q_{k_0}(r) &= \overline{\left\{C(x) - 2^\alpha A(x)^{-\alpha}|B(x)|^{\alpha+1} - k_0^{-\alpha}\left|f(x)\right|\right\}}(r). \end{aligned} \tag{3.5}$$

Theorem 3.3. *If the half-linear ordinary differential equation*

$$\left(r^{n-1}p(r)\left|y'\right|^{\alpha-1}y'\right)' + r^{n-1}q_{k_0}(r)\left|y\right|^{\alpha-1}y = 0 \tag{3.6}$$

is oscillatory at $r = \infty$ for any $k_0 > 0$, then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3)*, either v is oscillatory in Ω or satisfies* (3.3).

Oscillation criteria for the half-linear differential equation (3.6) were obtained by numerous authors (see, e.g., Došlý and Řehák [7], Kusano and Naito [8], and Kusanoet al. [9]).

Now we derive the following Leighton-Wintner-type oscillation result.

Corollary 3.4. *If*

$$\int_{r_0}^{\infty}\left(\frac{1}{r^{n-1}p(r)}\right)^{1/\alpha}dr=\infty, \qquad \int^{\infty} r^{n-1}q_{k_0}(r)dr=\infty \tag{3.7}$$

for any $k_0>0$*, then for every solution* $v\in\mathfrak{D}_P(\Omega)$ *of* (1.3)*, either* v *is oscillatory in* Ω *or satisfies* (3.3).

Proof. The conclusion follows from the Leighton-Wintner oscillation criterion (see Došlý and Řehák [7, Theorem 1.2.9]). □

By combining Theorem 3.3 with the results of [8, 9], we obtain Hille-Nehari-type criteria for (1.3) (cf. Došlý and Řehák [7, Section 3.1], Kusano et al. [10], and Yoshida [11, Section 8.1]).

Corollary 3.5. *Assume that* $q_{k_0}(r)\geq 0$ *eventually and suppose that* $p(r)$ *satisfies*

$$\int_{r_0}^{\infty}\left(\frac{1}{r^{n-1}p(r)}\right)^{1/\alpha}dr=\infty, \tag{3.8}$$

and $q_{k_0}(r)$ *satisfies*

$$\liminf_{r\to\infty}(P(r))^{\alpha}\int_{r}^{\infty}s^{n-1}q_{k_0}(s)ds>\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{3.9}$$

for any $k_0>0$*, where*

$$P(r)=\int_{r_0}^{r}\left(\frac{1}{s^{n-1}p(s)}\right)^{1/\alpha}ds. \tag{3.10}$$

Then for every solution $v\in\mathfrak{D}_P(\Omega)$ *of* (1.3)*, either* v *is oscillatory in* Ω *or satisfies* (3.3).

Corollary 3.6. *Assume that* $q_{k_0}(r)\geq 0$ *eventually and suppose that* $p(r)$ *satisfies*

$$\int_{r_0}^{\infty}\left(\frac{1}{r^{n-1}p(r)}\right)^{1/\alpha}dr<\infty, \tag{3.11}$$

and $q_{k_0}(r)$ *satisfies either*

$$\int^{\infty}(\pi(r))^{\alpha+1}r^{n-1}q_{k_0}(r)dr=\infty \tag{3.12}$$

or

$$\liminf_{r\to\infty}\frac{1}{\pi(r)}\int_r^\infty(\pi(s))^{\alpha+1}s^{n-1}q_{k_0}(s)ds>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\tag{3.13}$$

for any $k_0>0$*, where*

$$\pi(r)=\int_r^\infty\left(\frac{1}{s^{n-1}p(s)}\right)^{1/\alpha}ds.\tag{3.14}$$

Then for every solution $v\in\mathfrak{D}_P(\Omega)$ *of* (1.3)*, either* v *is oscillatory in* Ω *or satisfies* (3.3).

Remark 3.7. If the following hypotheses are satisfied:

$$\begin{gathered}C(x)-2^\alpha A(x)^{-\alpha}|B(x)|^{\alpha+1}>0\ \text{(eventually)},\\ \lim_{|x|\to\infty}\frac{|f(x)|}{C(x)-2^\alpha A(x)^{-\alpha}|B(x)|^{\alpha+1}}=0,\end{gathered}\tag{3.15}$$

then we observe that $q_{k_0}(r)>0$ eventually.

Example 3.8. We consider the half-linear elliptic equation

$$\nabla\cdot\left(A(x)|\nabla v|^{\alpha-1}\nabla v\right)+(\alpha+1)B(x)\cdot\left(|\nabla v|^{\alpha-1}\nabla v\right)+C(x)|v|^{\alpha-1}v=f(x),\quad x\in\Omega,\tag{3.16}$$

where $n=2$, $\Omega=E_1$, $A(x)=2|x|^{-1}$, $B(x)=2|x|^{-1-\alpha/(\alpha+1)}(\cos|x|,\sin|x|)$, $C(x)=|x|^{-1}(5/2+\sin|x|)$, and $f(x)=|x|^{-1}e^{-|x|}$. It is easy to verify that

$$\begin{gathered}\int_1^\infty\left(\frac{1}{rp(r)}\right)^{1/\alpha}dr=\infty,\\ q_{k_0}(r)=\frac{1}{r}\left(\frac{1}{2}+\sin r-k_0^{-\alpha}e^{-r}\right),\end{gathered}\tag{3.17}$$

and therefore

$$\int^\infty rq_{k_0}(r)dr=\int^\infty\left(\frac{1}{2}+\sin r-k_0^{-\alpha}e^{-r}\right)dr=\infty\tag{3.18}$$

for any $k_0>0$. Hence, from Corollary 3.4, we see that for every solution v of (3.16), either v is oscillatory in Ω or satisfies (3.3).

Example 3.9. We consider the half-linear elliptic equation

$$\nabla\cdot\left(A(x)|\nabla v|^{\alpha-1}\nabla v\right)+(\alpha+1)B(x)\cdot\left(|\nabla v|^{\alpha-1}\nabla v\right)+C(x)|v|^{\alpha-1}v=f(x),\quad x\in\Omega,\tag{3.19}$$

where $n = 2$, $\Omega = E_1$, $A(x) = 2|x|^{-1}$, $B(x) = |x|^{-\alpha/(\alpha+1)}(\sin|x|, \cos|x|)$, $C(x) = 3 + \cos|x|$, and $f(x) = e^{-|x|}\sin|x|$. It is easily checked that

$$\lim_{|x|\to\infty} \frac{|f(x)|}{C(x) - 2^{\alpha}A(x)^{-\alpha}|B(x)|^{\alpha+1}} = \lim_{|x|\to\infty} \frac{e^{-|x|}|\sin|x||}{2+\cos|x|} = 0, \tag{3.20}$$

and therefore $q_{k_0}(r) > 0$ eventually by Remark 3.7 . Furthermore, we observe that

$$\begin{aligned}
&\int_1^{\infty} \left(\frac{1}{rp(r)}\right)^{1/\alpha} dr = \infty, \\
&q_{k_0}(r) = 2 + \cos r - k_0^{-\alpha} e^{-r}|\sin r|, \\
&(P(r))^{\alpha} = 2^{-(\alpha+1)}(r-1)^{\alpha}, \\
&\int_r^{\infty} s q_{k_0}(s)ds = \int_r^{\infty} s\left(2 + \cos s - k_0^{-\alpha}e^{-s}|\sin s|\right)ds \\
&\qquad \ge \int_r^{\infty} s\left(1 - k_0^{-\alpha}e^{-s}\right)ds = \infty
\end{aligned} \tag{3.21}$$

for any $k_0 > 0$. Hence we obtain

$$\liminf_{r\to\infty} (P(r))^{\alpha} \int_r^{\infty} s^{n-1} q_{k_0}(s)ds = \infty. \tag{3.22}$$

It follows from Corollary 3.5 that for every solution v of (3.19), either v is oscillatory in Ω or satisfies (3.3).

Example 3.10. We consider the half-linear elliptic equation

$$\nabla\cdot\left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + (\alpha+1)B(x)\cdot\left(|\nabla v|^{\alpha-1}\nabla v\right) + C(x)|v|^{\alpha-1}v = f(x), \quad x\in\Omega, \tag{3.23}$$

where $n = 2$, $\Omega = E_1$, $A(x) = |x|^{-1}e^{|x|}$, $B(x) = |x|^{-\alpha/(\alpha+1)}e^{|x|}(\cos|x|, \sin|x|)$, $C(x) = e^{2|x|}$, and $f(x)$ is a bounded function. It is easy to see that

$$\begin{aligned}
&C(x) - 2^{\alpha}A(x)^{-\alpha}|B(x)|^{\alpha+1} = e^{2|x|} - 2^{\alpha}e^{|x|}, \\
&\lim_{|x|\to\infty} \frac{|f(x)|}{C(x) - 2^{\alpha}A(x)^{-\alpha}|B(x)|^{\alpha+1}} = \lim_{|x|\to\infty} \frac{|f(x)|}{e^{2|x|} - 2^{\alpha}e^{|x|}} = 0,
\end{aligned} \tag{3.24}$$

and hence $q_{k_0}(r) > 0$ eventually by Remark 3.7 . Since $f(x)$ is bounded, there exists a constant $M > 0$ such that $|f(x)| \le M$. Moreover, we see that

$$\begin{aligned}
&\int_1^\infty \left(\frac{1}{rp(r)}\right)^{1/\alpha} dr = \int_1^\infty \left(\frac{1}{4e^r}\right)^{1/2} dr < \infty, \\
&\pi(r) = \alpha 2^{-2/\alpha} e^{-r/\alpha}, \\
&q_{k_0}(r) = e^{2r} - 2^\alpha e^r - k_0^{-\alpha} \overline{\{|f(x)|\}}(r) \ge e^{2r} - 2^\alpha e^r - k_0^{-\alpha} M.
\end{aligned} \tag{3.25}$$

If $\alpha > 1$, then

$$\int^\infty (\pi(r))^{\alpha+1} r q_{k_0}(r) dr \ge \frac{\alpha^{\alpha+1}}{2^{2(\alpha+1)/\alpha}} \int^\infty r\left(e^{((\alpha-1)/\alpha)r} - 2^\alpha e^{-r/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)r}\right) dr = \infty, \tag{3.26}$$

and if $0 < \alpha < 1$, then

$$\begin{aligned}
&\liminf_{r\to\infty} \frac{1}{\pi(r)} \int_r^\infty (\pi(s))^{\alpha+1} s q_{k_0}(s) ds \\
&\quad \ge \liminf_{r\to\infty} \frac{\alpha^\alpha}{4} e^{r/\alpha} \int_r^\infty s\left(e^{((\alpha-1)/\alpha)s} - 2^\alpha e^{-s/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)s}\right) ds \\
&\quad \ge \liminf_{r\to\infty} \frac{\alpha^\alpha}{4} e^{r/\alpha} \int_r^\infty \left(e^{((\alpha-1)/\alpha)s} - 2^\alpha e^{-s/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)s}\right) ds \\
&\quad = \liminf_{r\to\infty} \frac{\alpha^\alpha}{4} \left(\frac{\alpha}{1-\alpha} e^r - 2^\alpha \alpha - k_0^{-\alpha} M \frac{\alpha}{\alpha+1} e^{-r}\right) = \infty
\end{aligned} \tag{3.27}$$

for any $k_0 > 0$. Corollary 3.6 implies that for every solution v of (3.23), either v is oscillatory in Ω or satisfies (3.3).

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